Promoting Grade 10 Learners' Algebraic Reasoning Through Folding Back: An Autoethnographic Perspective

by

MALOSE ALSIN TLHAKO

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DECLARATION

Name and Surname: MALOSE ALSIN TLHAKO

Student number: 24730467

Degree: Master of Education in Mathematics Education

Research title: Promoting Grade 10 Learners' Algebraic Reasoning Through Folding Back: An Autoethnographic Perspective

Supervisor: Dr Koena Mabotja

I, the undersigned, declare that I am the author of the above dissertation, and that all the sources used or quoted have been fully indicated and acknowledged through complete references.

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I further declare that I have not previously submitted this dissertation, or part of it, for examination at UNISA for another qualification or at any other higher education institution.

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First and foremost, I thank God for His mercy that has never failed me. He carried me through from the very beginning until the end (Deuteronomy 1:31):

- Secondly, this dissertation would not have been made possible without the acknowledged individuals who were directly or indirectly involved in its completion:
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DEDICATION

I dedicate this work to my late mother, Maphuthi Jane Tlhako. Her unwavering love and guidance inspired me to remain persistent and never give up. This dissertation reflects the strong foundation she built within me. Her absence is deeply felt at this moment of completing this study.

This is for you, Ma. I miss you dearly. May your soul continue to rest in peace.

ABSTRACT

The purpose of this study was to promote Grade 10 learners' algebraic reasoning through the use of folding back as a teaching frame. Algebraic reasoning is a sine qua non for mathematical understanding and, as a result, no field of mathematics can escape its application. In this regard, it is regarded as a prerequisite for STEM careers. Nevertheless, the literature highlights learners' challenges, errors, and limited understanding in the application of algebraic reasoning across various areas of mathematical knowledge. These challenges are often attributed to traditional teaching strategies characterised by teacher-centred approaches and limited opportunities for learners to actively participate in the construction of mathematical knowledge. Accordingly, this qualitative study employed an autoethnographic research design to explore how teaching framed by the notion of folding back can promote Grade 10 learners' algebraic reasoning. The participants were 12 Grade 10 mathematics learners purposively sampled from a class of 22 learners who were exposed to four weeks of exploratory teaching. In line with folding back studies, data were collected through learning activities, video recordings, document analysis, and semi-structured interviews. Polkinghorne's narrative analysis was utilised to explore how Grade 10 learners' algebraic reasoning was promoted through folding back. The interpretations used to produce compelling narratives were informed by the theoretical framework of folding back, as outlined by Martin (2008). Consequently, information-rich interactions drawn from learners' algebraic reasoning activities, where folding back was observed, were selected for analysis. Among other findings, the study revealed that learners improved their algebraic reasoning skills by evaluating their solutions within a folding back learning environment. In addition, the results showed that learners developed the ability to justify their thinking as they folded back across various layers of understanding. Furthermore, the study found that teacher and peer interventions provided Grade 10 learners with opportunities to explain, justify and refine their algebraic reasoning while the action of folding back allowed them to make sense of solutions in real-life financial mathematics context. However, the study also found that the implementation of folding back may be hindered by limited teacher availability, learners' reluctance to engage in peer questioning, and difficulties in reaching a shared reasoning pathway. The findings therefore suggests that folding back is a valuable teaching approach for promoting Grade

10 learners' algebraic reasoning. In light of this, the study recommends that mathematics teachers adopt probing questions and create collaborative small-group interactions. In addition, future research should explore folding back in other mathematical topics and research the role of home language in folding back classrooms, as well as quantitatively measure its effect on learners' achievement in mathematics.

Keywords: Algebraic reasoning, folding back, Grade 10 mathematics learners, autoethnographic

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CHAPTER 1 INTRODUCTION AND BACKGROUND

"*There is a stage in the curriculum when the introduction of algebra may make simple things hard, but not teaching algebra will soon render it impossible to make hard things simple*" (Tall & Thomas,1991, p. 128).

1.1 INTRODUCTION

This autoethnographic study aligns with the above epigraph from Tall and Thomas' (1991) work on promoting versatile thinking in algebra. My understanding of their assertion is that algebra serves as a sine qua non for mathematical understanding and, as a result, no field of mathematics can escape its application. Thus, the emphasis should be placed on developing learners' proficiency in algebra. Accordingly, various studies indicate that proficiency in algebra is a strong predictor of overall mathematics achievement (Gasim, 2018; Spiller, 2023). A study by Gasim (2018) reveals that lower performance in algebra topics negatively impacts overall mathematics achievement, as many mathematics topics depend on algebraic thinking. In addition, Spiller (2023), as well as Luneta and Legesse (2023), note that algebraic skills, such as handling variables, expressions, and equations, enhance the comprehension of geometric concepts. Consequently, learners who find algebra challenging are likely to experience difficulties in geometry as well, given that both areas require the manipulation of variables and equations.

Furthermore, the connection between algebra and other mathematics topics resonates with the mathematics curriculum in South Africa, which emphasises that "a general focus of the algebra content area is for the learner to achieve efficient manipulative skills in the use of algebra" (Department of Basic Education (DBE), 2011a, p. 10). The curriculum contends that learners should be able to use algebraic skills to analyse "situations in a variety of contexts in order to make sense of them" (DBE, 2011a, p. 10) and to solve mathematical problems. In this regard, it can be argued that algebraic skills are transferable across various content areas in mathematics. One such skill relevant to this study is algebraic reasoning, which refers to the cognitive ability to identify, generalise, and manipulate patterns or relationships using symbols, variables, and operations to solve problems (Ahmadah, 2020; Indraswari et al., 2018; Rahmawati et al., 2019). Algebraic manipulation is premised on the meaningful use of symbols, whereby learners should be able to select variables and construct expressions and equations within context (National Council of Teachers of Mathematics (NCTM), 2016).

Notwithstanding the benefits of algebraic reasoning, my experience of teaching mathematics in the Further Education and Training (FET) phase indicates that Grade 10 learners often encounter challenges when applying algebra across a variety of contexts. For example, in analytical geometry, where learners are required to use a known formula to derive an equation to solve for an unknown variable, many struggle to align the given information with the appropriate equation. In some instances, even when learners correctly match the given information, they misuse algebraic rules, resulting in incorrect solutions. This inability to derive and manipulate equations from known mathematical knowledge is not limited to analytical geometry, but extends to mathematical concepts such as word problems. Learners struggle to analyse word problems that require algebraic manipulation, as they experience difficulty translating verbal information into algebraic expressions or equations. Furthermore, this challenge appears to extend to mathematical topics such as financial mathematics and trigonometry, both two-dimensional and three-dimensional, which also require the application of algebraic reasoning. These challenges are not unique to my learners, as research points to a global concern regarding learners' struggles with algebraic reasoning (Basir et al., 2022; Makonye, 2015; Mathaba & Bayaga, 2019; Pournara, 2020; Yildiz & Durmaz, 2021). Empirical evidence from Setiawan (2022) indicates that learners who can accurately solve basic questions at a formal level often demonstrate limited understanding of underlying algebraic concepts, as they struggle when faced with slightly modified problem scenarios. In addition, Selowa and Dhlamini (2023) highlight learners' difficulties in expressing algebraic thinking pictorially, symbolically, and verbally.

These challenges are further reflected in recent National Senior Certificate (NSC) diagnostic reports, which indicate that many Grade 12 learners exhibit poor algebraic reasoning skills. This has been attributed to deficiencies in fundamental mathematical competencies that should have been developed in the lower grades (DBE, 2024). Learners struggle to interpret mathematical information, often leading to incorrect use of formulae due to reliance on rote problem-solving methods (DBE, 2024). Similarly, research indicates that many learners lack a strong foundation in algebraic thinking and adopt formal or routine approaches, which result in errors during algebraic procedures (Kontorovich, 2020; Veloso et al., 2021; Yildiz & Durmaz, 2021). Consequently, researchers such as Selowa and Dhlamini (2023) recommend the need for teachers to foster algebraic thinking in the early grades. In response, this autoethnographic study focuses on the Grade 10 mathematics classroom, which serves as an entry grade into the FET phase. The algebraic knowledge developed in Grade 10 forms prior knowledge that enables learners to establish conceptual connections in Grades 11 and 12. It can therefore be argued that underdeveloped foundational algebraic knowledge negatively affects learners' comprehension of algebra in subsequent grades.

Novikasari (2020) highlights that teachers in introductory grades often rely on rote memorisation rather than problem-based or reflective approaches that foster reasoning in the early stages of algebra. When such instructional strategies dominate classrooms, learners are confined to learning procedures and proofs without developing a solid understanding of fundamental concepts (Chirove & Ogbonnaya, 2021). This deficiency leaves learners ill-equipped to apply their knowledge in practical situations (Kobandaha et al., 2019). Consequently, there is a need to create effective learning environments that promote learners' algebraic reasoning. In this regard, researchers recommend reflective teaching approaches that prioritise learner engagement, emphasise dialogue among learners rather than teacher-led interaction, and adopt problem-based learning strategies (Audina et al., 2023; Barana, 2021; Chuene et al., 2023; Novikasari, 2020). Such learning environments encourage learners to develop problem-solving strategies, thereby enhancing their understanding of reasoning processes (Nadeak & Naibaho, 2020). Earlier work by Tall and Thomas (1991) revealed that the use of computers enhances learners' versatile thinking skills in algebra. Subsequent studies have employed technology-informed approaches to enhance learners' algebraic reasoning and conceptual understanding (Chimoni et al., 2023; Decker-Woodrow et al., 2023; Li & Zulnaidi, 2019; Marange, 2021; Papadopoulos, 2019). While these approaches have proven effective, this autoethnographic study seeks to explore the role of a theory-informed teaching approach in promoting algebraic reasoning among Grade 10 learners. One such approach is based on the folding back framework developed by Martin (2008).

Folding back, as discussed in detail in Chapter 2, is defined as non-linear movement across identified layers of understanding that occurs when learners encounter a problem without an immediate solution. In response, learners return to prior understandings and ideas to rework and reconceptualise mathematical concepts (Martin, 2008; Pirie & Kieren, 1989, 1994). This reflective process promotes metacognition, enabling learners to monitor and evaluate their thinking and engage more deeply with mathematical ideas. Through this process, learners can identify misconceptions, reflect on prior knowledge, and connect different mathematical concepts, thereby fostering the development of robust mathematical reasoning skills (Chuene et al., 2023). Such skills are essential for the advancement of mathematics education in South Africa.

Previous studies by Chuene et al. (2023) and Mabotja et al. (2018) found that learners' ability to transition between different levels of thinking through folding back positively enhances their understanding of geometry and their reasoning skills. Similarly, Hähkiöniemi et al. (2022) argue that when learners are confronted with challenging mathematical situations, revisiting earlier concepts supports the development of deeper understanding necessary to overcome such challenges (Yao & Manouchehri, 2020a). A study by Karimah (2024) further supports this view, demonstrating that folding back across different cognitive layers enables learners to exhibit deeper reasoning, resulting in improved conceptual understanding. Although previous research has explored the benefits of folding back in enhancing mathematical understanding and reasoning (Mabotja et al., 2018), there remains a paucity of studies examining the use of folding back to enhance algebraic reasoning among Grade 10 learners, particularly within a problem-based learning environment in South Africa. Existing literature has not sufficiently addressed the application of folding back in the context of algebraic reasoning. Consequently, this autoethnographic study seeks to present my experiences of promoting Grade 10 learners' algebraic reasoning through folding back. The study therefore aims to contribute valuable insights into how folding back can be utilised to promote algebraic reasoning among Grade 10 learners and to address this critical gap in the literature within a South African classroom context.

1.2 RESEARCH PROBLEM

A general focus of the algebra content area is for learners to develop efficient manipulative skills in its application (DBE, 2011c; Sibgatullin et al., 2022; Yang & Deng, 2024). My understanding of this description is that algebraic skills are transferable across various content areas in mathematics. Thus, learners should be able to use algebraic skills to analyse “situations in a variety of contexts in order to make sense of them” (DBE, 2011c, p. 10) and to solve mathematical problems. Consequently, algebraic reasoning is regarded as a foundation for advanced mathematical learning, which suggests that cultivating learners’ algebraic thinking skills should be a primary objective of mathematics instruction (Maphutha, 2023). Nevertheless, a well-documented trend of learners’ challenges related to algebraic reasoning persists (Kobandaha et al., 2019; Mutodi & Mosimege, 2021). These challenges are often attributed to traditional teaching approaches that fail to engage learners with fundamental concepts within learner-centred environments (Novikasari, 2020; Selowa & Dhlamini, 2023). Although attempts have been made to enhance learners’ understanding of algebraic skills (Al-Mutairi & Marzouq, 2025; Eriksson & Sumpter, 2021), there remains a scarcity of research focusing specifically on the promotion of learners’ algebraic reasoning, particularly within the South African context. Consequently, this autoethnographic study sought to promote Grade 10 learners’ algebraic reasoning through teaching framed by the notion of folding back.

1.3 PURPOSE OF THE STUDY

The purpose of this autoethnographic study was to promote Grade 10 learners’ algebraic reasoning through teaching framed by the notion of folding back.

1.4 RESEARCH QUESTIONS

The following main research question guided the study:

1.4.1 How does Folding Back promote Grade 10 learners' algebraic reasoning?

1.4.2 What are the barriers that hinder the effective implementation of folding back in promoting algebraic reasoning?

1.5 OBJECTIVES OF THE STUDY

1.5.1 To explore how folding back promote grade 10 algebraic reasoning.

1.5.2 To identify the barriers that could hinder effective implementation of folding back in promoting algebraic reasoning.

1.6 RESEARCH METHODOLOGY

To address the above research questions, I adopted an interpretivist autoethnographic research design, which is defined as a "research method that uses personal experience ('auto') to describe and interpret ('graphy') cultural texts, experiences, beliefs, and practices ('ethno')" (Adams et al., 2017, p. 1). Accordingly, autoethnography is often written in the first person and departs from conventional academic writing that is characterised by a third-person, passive voice and a neutral, objective tone (Cohen et al., 2018). It is within this description that I locate the justification for the frequent use of the first-person voice in this autoethnographic study. Thus, I narrate stories of my experiences of using folding back to frame the promotion of Grade 10 learners' algebraic reasoning within their mathematics classroom as a cultural context (Gorichanaz, 2021). As I am directly immersed in my Grade 10 learners' learning experiences in my role as their mathematics teacher, I "can talk about these issues in ways different from others who have limited experiences with these topics" (Adams et al., 2017, p. 3).

The participants in this study were 12 Grade 10 learners at a public high school in Thabazimbi, Limpopo Province. They were purposively sampled from a class of 22 learners who were exposed to exploratory teaching framed by the notion of folding back during fieldwork. The rationale for involving the entire class was to minimise potential selection bias, as noted by Cohen et al. (2018). However, for reporting purposes, a purposive sample of 12 learners who participated in the learning activities was used as data sources. From this group, four learners were invited to participate in semi-structured interviews. Data were collected through learning activities, video recordings, documentation of learners' written work, and semi-structured interviews. Polkinghorne's

(1995) narrative analysis and analysis of narratives were employed to analyse the data. The analysis focused on the “interpretations and extrapolations of latent meanings so that the conversations could be coherently understood” (Mabotja et al., 2018, p. 3). The interpretation of interactions was grounded in Martin’s (2008) folding back framework.

Rigour and quality were ensured through the criteria of credibility, dependability, confirmability, and transferability. Ethical principles, including permission, confidentiality, and informed consent, were duly observed. A comprehensive account of the methodology is presented later in Chapter 3.

1.7 SIGNIFICANCE OF THE STUDY

The study aims to explore how algebraic reasoning skills can be promoted in a problem-based learning environment through the folding back phenomenon. It is anticipated that the findings will provide educators with valuable insights into creating an optimal learning environment that encourages Grade 10 learners to reflect on their learning and to develop their algebraic reasoning skills. This autoethnographic research seeks to contribute to the existing body of knowledge on algebraic reasoning by addressing a gap in the literature concerning the role of folding back within problem-based learning in the development of these skills.

1.8 STRUCTURE OF THE STUDY

This research report is structured into five chapters. Chapter One provides an overview of the background to the study and outlines the purpose of the research, as well as the research questions that guide the inquiry. Chapter Two presents the theoretical framework underpinning the study and includes a review of relevant literature. Chapter Three details the research methodology and provides justification for the selection of an interpretivist research paradigm and an autoethnographic research design. The methods of data collection, considerations of quality criteria, and ethical considerations are also addressed. Chapter Four presents an analysis of the interactions between the researcher-teacher and the learners, based on the collected data. Finally, Chapter Five discusses the findings in relation to the two research questions that guided the study and offers concluding recommendations.

1.9 CHAPTER SUMMARY

In this chapter, I presented the background of the study and outlined its purpose, research questions, methodology, and significance. I also described the setting in which the study was conducted. The chapter concludes with an overview of the structure of the dissertation.

CHAPTER 2 LITERATURE REVIEW AND THEORETICAL FRAMEWORK

2.1 INTRODUCTION

In the previous chapter, I outlined the background and the research problem that this autoethnographic study sought to address. In this chapter, I present the literature review, which explores how folding back within a problem-based learning environment can promote the algebraic reasoning of Grade 10 learners. The chapter is divided into two main sections, namely the theoretical framework and the literature review. Although the sequence in which the theoretical framework and the literature review are presented is often debated, in this chapter the theoretical framework is presented first, followed by the literature review. This decision is informed by the exploratory nature of the study and its grounding in a theory-based teaching approach. Accordingly, "the theoretical framework provides a grounding base, or an anchor, for the literature review" (Grant & Osanloo, 2014, p. 12). I therefore consider this approach to be fit for purpose within this autoethnographic study (Cohen et al., 2018).

On the one hand, the theoretical framework focuses on the elaboration and justification of Martin's (2008) folding back framework, which is rooted in Pirie and Kieren's theory of the dynamic growth of mathematical understanding. In addition, attention is drawn to several studies that have employed folding back in mathematics education research. On the other hand, the literature review addresses the following areas: an overview of algebra in South Africa's mathematics curriculum, algebraic reasoning, learners' challenges in algebra, problem-based learning and algebraic reasoning, and the role of folding back in mathematics education.

2.2 THEORETICAL FRAMEWORK

This section provides a description of Martin's (2008) folding back theoretical framework and explains why it is well suited to this study. Folding back has its roots in the Pirie and Kieren (1994) theory of mathematical understanding. Accordingly, the section is organised into three main parts. First, a description of the Pirie and Kieren theory is presented. Second, the folding back framework, as elaborated by Martin (2008), is discussed, with its key elements described in relation to how they are utilised in this autoethnographic study to promote Grade 10 learners' algebraic reasoning. Lastly, the role of folding back in mathematics education is examined, drawing on selected studies that have employed the folding back framework.

2.2.1 Dynamical growth of mathematical understanding

The Pirie-Kieren theory is influenced by von Glasersfeld's (1987) "constructivist definition of understanding as a continuing process of organising one's knowledge structures" (Pirie & Kieren, 1994, p. 166). The evolution of their theory started in their first publication in 1989 (Pirie & Kieren, 1989a), which was followed by conference presentations (Pirie & Kieren, 1989b, 1990). Their publications largely focused on the notion of mathematical understanding as a recursive process. Hence, Pirie and Kieren perceive the growth of mathematical understanding as a whole, dynamic, levelled, but nonlinear, transcendently recursive process (Martin, 2008; Pirie & Kieren, 1989; Yao & Manouchehri, 2020a). They use the recursive notion to explain that they "do not see the growth of understanding as a monodirectional process" (Pirie & Kieren, 1994, pp. 171-172). Martin and Towers (2016) further opine that the "recursive structure and character of the theory embed more localised ways of thinking mathematically (and specific ways of acting)" (p. 282). Thus, the theory's alignment with constructivist principles highlights the importance of organising knowledge structures and reinforces the idea that learning is an active and ongoing process. This focus on constructing understanding rather than merely acquiring knowledge encourages learners to engage deeply with mathematical concepts, fostering critical thinking and problem-solving skills.

The theory is characterised by eight-nested circles of the layers of understanding, which describe the growth of understanding of learners in mathematics (Figure 2.1). The layers are named primitive knowing, image-making, image-having, property noticing, formalising, observing, structuring, and inventising.

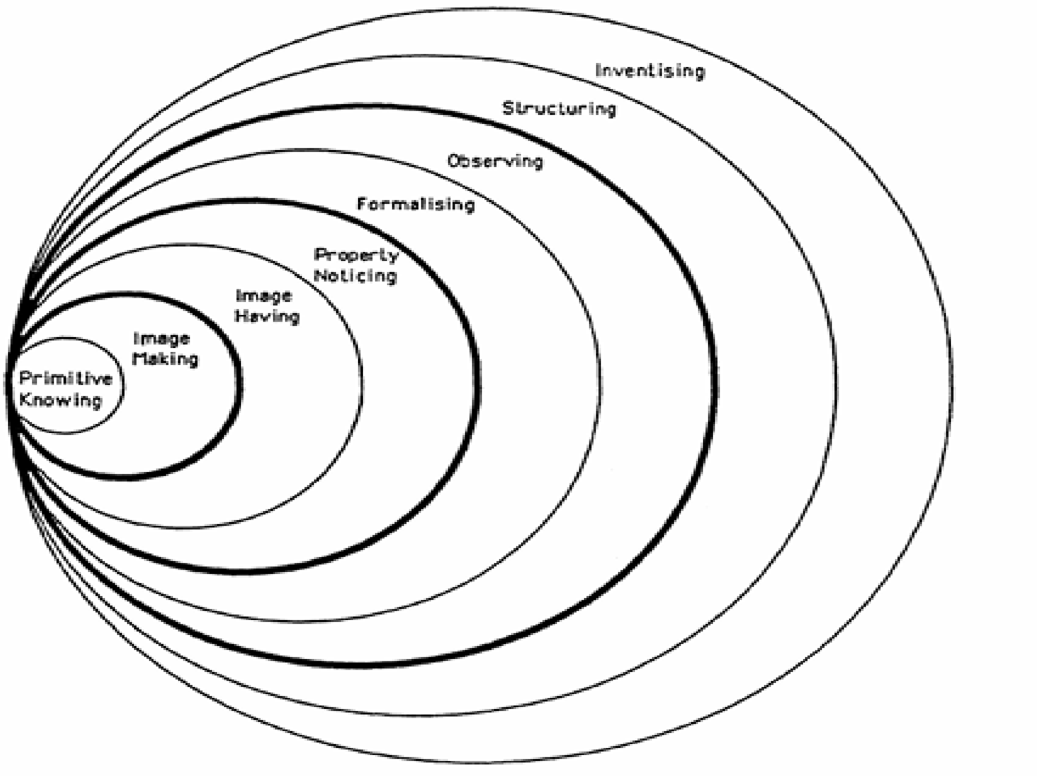

Figure 2.1: The Pirie–Kieren model for the dynamical growth of mathematical understanding (Pirie & Kieren, 1994, p. 167).

The nested two-dimensional diagrammatic model intends to demonstrate that each layer contains all previous layers and is included in all subsequent layers, emphasising the embedded nature of the growth of mathematical understanding. They use the model as an effort to virtually describe that they “see growth as represented by back and forth movement between levels, and it is thus that we characterise understanding as a dynamic and organising process” (Pirie & Kieren, 1994, p. 172). The non-linear movement suggests “a continual movement between different layers or ways of thinking, with no implication of a linear, ladder-like system” (Martin & Towers, 2016, p. 282). Thus, the Pirie-Kieren model offers valuable insight for “observing and describing the process through which learners’ mathematical knowledge is organised and reorganised, and for how learners think about their understandings” (Martin & Towers, 2016, p. 282). The Pirie-Kieren Theory offers a valuable opportunity for mapping out the growth of understanding of a learner in learning a particular mathematical concept (Chuene et al., 2024; Mabotja et al., 2018; Yao, 2020; Yao & Manouchehri, 2020a, 2020b). The acknowledgement of non-linear movement between different levels portrays the complexity of how learners interact with mathematical concepts, recognising that understanding is not a straightforward, step-by-step journey. Therefore, the model serves as a descriptive tool for educators and researchers to map out learners' growth of mathematical understanding and observe and analyse the cognitive processes involved in learning.

The model grows outwards from the primitive layer to the inventising layer; nevertheless, it does not suggest that the growth of mathematics is hierarchical (Mabotja, 2017; Martin & Towers, 2016). Although Pirie and Kieren (1994) acknowledge some hierarchy within the model, they highlight that the layers should not be perceived as a representation of low-level or higher-level mathematical understanding. In this regard, they underscored that "Just as the term 'primitive knowing' does not imply low-level mathematics, so there is no intention to link the outer levels necessarily with 'better' or 'high-level' mathematics" (Pirie & Kieren, 1994, p. 172). The contention is that a non-hierarchical perspective is crucial for recognising that different layers of understanding do not equate to better or inferior learners' mathematical capabilities.

Although Pirie-Kieren's Theory describes eight layers, for this study, I focus on the first five layers. The rationale for focusing on only 5 layers is that the remaining 3 layers are considered too advanced for developing algebraic reasoning in Grade 10.

2.2.1.1 Primitive knowing layer

The understanding that learners bring knowledge from their past experiences to their current learning environment is akin to a *primitive knowing* layer of knowledge (Pirie & Kieren, 1989). This level of awareness is generally considered a foundation for developing mathematical understanding (Mabotja et al., 2018; Martin, 2008; Putri, 2024). In the context of this study, the knowledge that learners bring into their learning environment is crucial not only for their algebraic conceptual development but also for their algebraic reasoning in contextual mathematical problems.

2.2.1.2 Image-making layer

The layer of *image-making* requires the learner to distinguish between the fundamental knowledge influenced by the learning environment to meet the learning needs of the specific mathematical concept (Pirie & Kieren, 1989). The learner is engaging in activities aimed at helping him or her to develop particular representations of the mathematical concept to get an idea (Chuene et al., 2023; Martin & Towers, 2016). This layer is closely connected to foundational understanding, as an individual's knowledge is shaped by the knowledge they bring to the classroom and can be expressed in language or action (Putri, 2024; Yao & Manouchehri, 2020b). In this study, problem-based learning activities engaged learners in the algebraic knowledge that they bring to the classroom. This act assisted in shaping images from the foundational knowledge that learners bring to the classroom into a cohesive understanding necessary to enhance their algebraic reasoning skills.

2.2.1.3 Image-having layer

The image-having process involves learners utilising mental constructs to grasp mathematical ideas rather than relying solely on physical activities (Putri, 2024). Consequently, in *image-having*, learners are not required to engage in specific learning activities, unlike in *image-making* (Martin, 2008; Martin & Towers, 2016). Instead, consider how the mathematical images that learners developed in the previous layer can be harnessed to assist them in identifying general properties (Jannah, 2023; Yao & Manouchehri, 2020b). Recognising these general properties can help learners make sense of the algebraic reasoning activities, as it is assumed that they already possess an understanding of algebraic thinking at this level.

2.2.1.4 Property noticing layer

According to Hähkiöniemi et al. (2022), *property noticing* is characterised by learners' abilities to identify the differences and similarities between various images. Successfully identifying these differences and similarities requires learners to engage in more advanced levels of sophistication relative to their understanding of the topic (Pirie & Kieren, 1994; Martin, 2008). Learners analyse what they can articulate regarding the learning activities and reflect on their comprehension. Similarly, learners engaged in the provided algebraic thinking activities can begin to reflect on the images created in image-making and identify attributes and features that may enhance their algebraic reasoning.

2.2.1.5 Formalising layer

Lastly, *formalisation* occurs when the learner intentionally reflects on and actively engages with the properties (Yao & Manouchehri, 2020b). This level involves learners recognising the knowledge they have developed and forming mathematical definitions based on the observed properties and generalisations of concepts (Chuene et al., 2023; Martin, 2008). Similarly, learners' awareness of their constructed knowledge enhances their reasoning skills when engaging in algebraic reasoning activities.

In addition to the eight layers, the theory includes three distinct features: "Don't need" boundaries, folding back, and the complementarities of acting and expressing. It is in the notion of folding back that the reported study found its relevance.

2.3 FOLDING BACK

Pirie and Kieran (1994) defined the notion of folding back as an essential feature of their theory as follows:

> This is the activity, vital to growth of understanding, which reveals the non-unidirectional nature of coming to understand mathematics. When faced with a problem or question at any level, which is not immediately solvable, one needs to fold back to an inner level in order to extend one's current, inadequate understanding. This returned-to, inner level activity, however, is not identical to the original inner level actions; it is now informed and shaped by outer level interests and understanding. (p. 173).

The description above suggests that we view the non-linear movement across different layers of understanding as a folding-back process. In addition, folding back is triggered by mathematical activities that learners often struggle with. Therefore, learners who successfully solve mathematics activities appear to have limited opportunities for folding back. The challenges learners experience in their learning endeavours could possibly be due to their current underdeveloped understanding. Hence, Kieren et al. (1999) argued that

> no matter what level or how sophisticated the understanding of a person whenever they find their mental and physical actions and their situation incoherent or comprehendible, they are prompted to fold back to an inner level of activity in order to extend their current action capabilities and action spaces. (p. 218).

The suggestion is that learners fold back to organise and reorganise their initial inadequate understanding in order to build deeper mathematical understanding, which will help them solve mathematical activities successfully (Chuene et al., 2023; Hähkiöniemi et al., 2023; Patmaniar et al., 2021; Pirie & Kieran, 1994; Rahayuningsih et al., 2022). The contention is that learners fold back to “reconstruct and elaborate on an inner level of understanding to support the next level” (Chuene et al., 2023, p.2). Thus, the goal of folding back is to help learners build nuanced mathematical understanding. Karimah et al. (2024) provide evidence that students often encounter difficulties when attempting to solve complex mathematical problems, particularly if they lack a solid foundation in foundational concepts such as algebra. In essence, “Different students will move in different ways and at different speeds through the levels, folding back again and again to enable them to build broader, but also more sophisticated or deeper understanding” (Pierie & Kieren, 1994, p. 173). The implication of folding back in the current study is that learners will fold back to various layers of understanding to develop their knowledge and understanding of algebra to apply algebraic reasoning in solving contextual problems.

Although the concept of folding back originates from Pirie-Kieren’s theory, this autoethnographic study is situated within Martin’s (2008) folding back framework. Martin (2008) is primarily influenced by the fact that “the definition offered by Pirie and Kieren remained essentially undeveloped and unelaborated in their work” (p. 64). In this regard, Mabotja (2017) emphasised that the Pirie-Kieren theory explains what folding back is, whereas Martin's (2008) framework elaborates how and why it occurs. Martin’s (2008) folding back framework provides comprehension elaboration on the folding phenomenon, as shown in Figure 2.2 on the next page. It is in such a comprehensive elaboration that this study seeks to promote Grade 10 learners’ algebraic reasoning. Thus, the folding back framework is used as a lens to guide my instructional decisions in promoting Grade 10 learners’ algebraic reasoning, rather than being a specific teaching strategy itself.

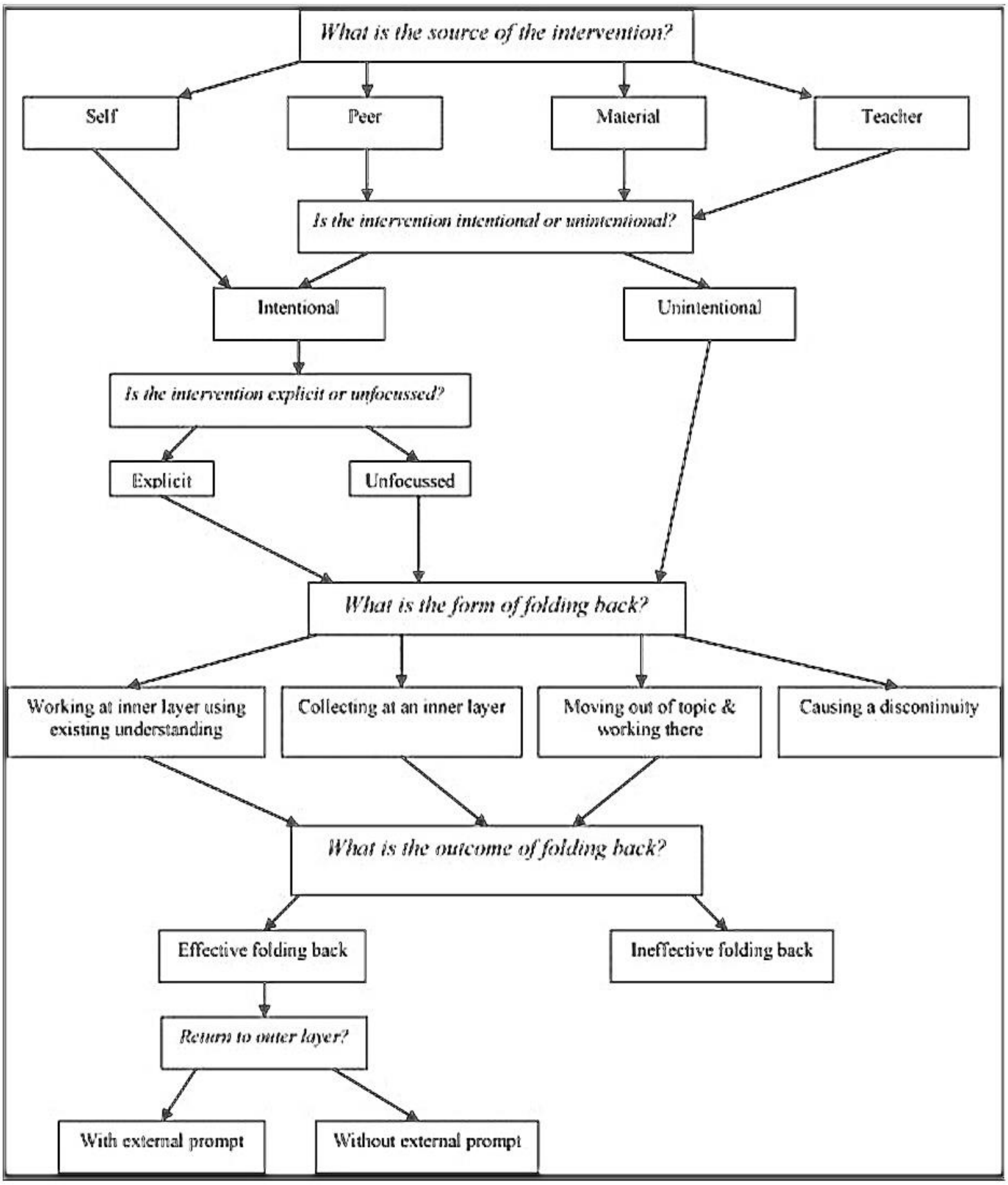

Figure 2.2: The framework for folding back (Martin, 2008, p. 72)

Martin's (2008) framework for folding back is premised upon three essential components: the source of intervention, the form of folding back, and an outcome. These components are described as follows:

2.3.1 The source of folding back

Martin (2008) defines the source as mathematical actions that prompt learners to fold back between layers of understanding. In addition, Martin's framework identifies four important sources of folding back, namely self-invoked intervention, peer intervention by another learner, material intervention, and teacher intervention. These interventions can either be intentional or unintentional. On the one hand, intentional intervention entails prompting the learner to reflect on concepts they might revisit or reconsider based on their prior knowledge, offering explicit directions on what to concentrate on (Chuene et al., 2023; Yao & Manouchehri, 2022b). For example, the teacher may, through probing questions, guide learners to a particular mathematical idea. Häkkinen et al. (2022) emphasise that folding back is most effective when teachers create structured opportunities for students to verbalise misconceptions, test conjectures, and iteratively refine their reasoning. On the other hand, unintentional intervention happens when the learner instinctively revisits earlier understandings without a deliberate aim (Heimbuch & Bodemer, 2017; Martin, 2008). Self-invoked as a source of folding back acknowledges "that some learners do possess a natural ability to fold back in response to becoming self-aware of the limitations of their present understandings" (Martin, 2008, p. 83). The contention is that, as a learner works on a mathematical problem, he or she can fold back without external stimulus.

Martin further differentiates between focused and unfocused intentional intervention. Focused intentional intervention guides the learner toward the specific concept they need to revisit (Martin, 2008). In contrast, unfocused intervention builds on a learner's prior knowledge without guiding them to fold back on the specific concept. Instead, they encourage the learner to be self-aware of the limitations of his or her understandings without prescribing a solution" (Martin, 2008, p. 83). The emphasis is on using instructional actions that provide a subtle and less directive stimulus for learners (Mabotja et al., 2018; Martin, 2008; Yao & Manouchehri, 2022b). Although other studies like Mabotja (2017) have only considered three of the four sources of intervention mentioned by Martin (2008), this study considered all four: self, peer, teacher, and material intervention. The problem-based classroom setup influences this decision. As such, all four sources of intervention are of paramount importance and supported (Imron et al., 2022; Yang et al., 2022).

2.3.2 The form of folding back

The form of folding back is either prompted or dictated by the outcome of the source of intervention and is classified by Martin (2008) into four subcategories: engaging at an inner layer with existing understanding (leveraging prior knowledge in mathematics rather than adhering to a formal mathematical approach), collecting at an inner layer (the enhancement of existing knowledge by learners through the creation of new relevant understandings), diverging from the topic and working there (learners revisiting their primitive knowing to facilitate a conceptual grasp of the current mathematical concept they will encounter), and inducing a discontinuity. According to Yao and Manouchehri (2022a), when categorising the forms of folding back, it is important to consider "the intent of a folding back action and the layer of understanding upon which the action is intended to act" (p. 254). This study also categorised the Grade 10 learners' responses to the intervention source into four distinct subcategories, aiming to provide meaning to their actions in the classroom.

2.3.3 The outcome of folding back

The outcome of folding back occurs once a learner has revisited the previous layers and can be classified as either effective or ineffective (Martin, 2008). Effective folding back occurs when learners can apply their expanded understanding to overcome the initial obstacle upon returning to the original problem (Martin et al., 2012; Patmaniar et al., 2021). On the other hand, ineffective folding back happens when learners are not able to return to the outer layer and successfully solve the problem at hand. Thus, folding back may not always lead to learners developing a deeper mathematical understanding; there are instances where learners may struggle to apply knowledge from their foundational understanding to more complex problem-solving situations (Chuene, 2023; Mabotja et al., 2018).

2.4 THE ROLE OF FOLDING BACK IN MATHEMATICS EDUCATION

Over the past decades, research on the role of folding back in mathematics education has been on an upward trajectory. Among others, the number of dissertations, peer-reviewed journal articles, and book chapters on the notion of folding back affirms its place in mathematics education. As a result, in this section, I discuss how other researchers used folding back to develop learners' mathematics understanding and reasoning abilities across various mathematics topics. The discussion draws insight into research aspects such as purpose, methodology, findings, and recommendations.

As earlier mentioned, folding back has its roots in the Pirie-Kieren Theory. However, Martin's (1991, 2008) work sought to shed light on the nature of folding back and its framework. Martin's (1991) earlier doctoral work, supervised by Professor Susan Pirie (one of the developers of Pirie-Kieren Theory), focused on the nature of folding back and its implication for both mathematics teachers and learners; it is in his later publication that he "examines in detail…the process of folding back and develops a theoretical work of categories and sub-categories that fully describe the phenomenon" (Martin, 2008, p. 64). The study employed grounded theory with an emphasis on elaborating an existing theory rather than generating a new theory. The participants in the study were seven small groups of students from the first year of secondary school to postgraduate teacher education students. Video recordings, learners' written work, and observational notes were used as part of the data collection tools. It was concluded that the folding back acts as a necessary mechanism for growth in learners, and effective folding back is a powerful facilitator of a developing mathematical understanding" (p. 83). It is in Martin's work that this autoethnographic study finds a theoretical lens to frame instructional decisions and an analytical framework to analyse interactions.

Mabotja's (2017) master's study focused on developing Grade 10 learners' geometric reasoning through the lens of Martin's (2008) folding back framework. Teaching experiment methodology, which involved teaching experiments and a series of episodes. The participants in the study were 7 learners sampled from a Grade 10 mathematics classroom consisting of 55 learners. As with Martin's (2008) study, Mabotja's (2017) study used video recordings, observations, and learning activities as data collection instruments. Extracts of learners' interactions in the classroom and outside the classroom were analysed using Polkinghorne's (1995) narrative analysis. The findings revealed that "learners operating at different layers of mathematical understanding are able to share their geometry knowledge with their peers" (Mabotja, 2017, p. v). The findings were consistent with Mabotja et al. (2018), which revealed that folding back offers an opportunity to trace learners' geometric reasoning. My understanding of Mabotja's (2017) study is that folding back provides rich opportunities for learners to learn from each other and enhance their questioning and reasoning abilities. In this current study, unlike Mabotja's study that used teaching experiment methodology with grain size (7 participants), I employ autoethnographic research, which "uses personal experience to describe and interpret" (Adams et al., 2017, p. 1) how the use of folding back promotes Grade 10 algebraic reasoning in the contextual financial mathematics context.

In another study, Chuene et al. (2023) explored the type of talk that supports the process of folding for growth in understanding geometry. They argued that the success of folding back in mathematics learning environments is linked to exploratory talk. In addition, two groups of learners were sources of data. In analysing the data that was collected by video recordings, Polkinghorne's (1995) narrative analysis was used. It was further augmented by Pirie and Kieren's notion of folding back and Wegerif and Mercer's types of talk to synthesise the stories. Their findings revealed that exploratory talk, where learners build on each other's ideas to develop their understanding of geometry concepts, promotes folding back. My understanding of Chuene et al.'s (2023) study is that not every kind of talk that learners engage in while learning mathematics can result in effective folding back.

Patmaniar et al. (2021) conducted a descriptive-exploratory study, which sought to examine learners' level of solving mathematics problem-solving through the folding back phenomenon. The participants were purposively sampled – 33 (male and female) high school learners – and grouped based on their gender. Task-based interviews were used as a data collection method. Their findings revealed that, despite differences in the level of learners' understanding in terms of gender, both were able to carry out the activities by explaining information from a mathematical problem, having an overview of a particular topic, abstracting mathematical concepts, and understanding the mathematical ideas in accordance with a given problem, amongst others (Patmaniar et al., 2021, p. 507). Thus, the findings of this study provide insight that folding back happens regardless of learners' gender.

Yao and Manouchehri (2022a) conducted a study that sought to report on folding back as students engaged in mathematical activities on mathematical generalisations about properties of geometric transformations within a dynamic geometry environment. As with Mabotja et al. (2018), Yao and Manouchehri (2022a) utilised a teaching experiment methodology to "directly interact with students so as to experience firsthand student thinking" (p. 248). The participants were five rising eighth graders and two rising ninth graders, who participated in a summer camp taught by the authors, where they explored properties of geometric transformations with Geometer's Sketchpad (GSP). The results of their analysis yielded a theoretical model that describes students' reflections in the development of mathematical generalisations in a technologically rich learning environment. Although their model (see Yao & Manouchehri, 2022a, for a detailed description) has some features of Martin's folding back (source, form, and outcome), the source of folding back included students, a teacher, and technology. In their epilogue, they alluded to the fact that teachers' instructional moves are likely to foster the learners' construction or refinement of generalisations in their mathematical learning endeavours. Although in this autoethnographic study, I am not using their proposed framework, their findings on teacher intervention shed light in terms of guiding how I interacted with Grade 10 learners in promoting algebraic reasoning.

Hähkiöniemi et al. (2022) conducted a study that explored the interplay between the teacher and digital learning environment in supporting folding back. The participants in the study were second-, fourth-, and sixth-grade learners, who were divided into numerous groups under the guidance of 12 student teachers. Each group was allocated a laptop as they worked on learning activities. Data was collected through recording the screen of each laptop using screen capture software, audio recording, and a video camera. The findings of their study revealed the synergy between teacher intervention and the digital environment in promoting folding back. On the contrary, their results also revealed non-synergistic guidance, "where a teacher contradicts the guidance from the environment [and] can productively support folding back" (Hähkiöniemi et al., 2022, p. 476). As with Yao and Manouchehri (2022a), the results of Hähkiöniemi et al. (2022) put emphasis on the importance of the teachers' role in promoting folding back in technology-based learning environments. While the current autoethnographic study does not focus on technology integration, the results on teachers' actions are necessary for understanding the teachers' role as a source of the folding back. Their research underscores that effective folding back requires students to recognise the limitations in their understanding and emphasises the importance of adaptive guidance from teachers. This support is crucial for fostering an environment where students can engage in reflective practices, thereby demonstrating the teacher's potential to enhance learners' algebraic reasoning.

In conclusion, most studies that explored the role of folding back in mathematics education mainly adopted qualitative research methodologies. For example, the use of teaching experiment methodologies is prevalent amongst the studies on folding back (e.g., Chuene et al., 2023; Mabotja, 2017; Mabotja et al., 2018; Yao & Manouchehri, 2022a). Moreover, data collection methods, such as mathematics learning activities, observations, and video recordings, are commonly used. The choice of these data collection methods is consistent with Pirie's (1996) assertion that "folding back can only be observed by close attention to the activities and talk of students as they work at a mathematical problem" (p.4). This description could possibly be a reason for small sample sizes, where the interactions between the learners and the researcher-teachers are observed and recorded in the folding-back learning environment. In other words, it is not practically possible to pay close attention to learners' interactions and thought processes while working on mathematics learning activities in overcrowded mathematics classrooms.

Although some of the studies (Hähkiöniemi et al., 2022; Yao & Manouchehri, 2022a) did not specifically use Martin's framework, they agree that the teacher intervention, as a source of folding back, plays an essential role in facilitating the back-and-forth non-linear movement across various layers of understanding. The teacher interventions, as reported, provide insight into how this autoethnographic study can promote algebraic reasoning. In addition, the studies also provide insight into possible methodologies that can be adopted to foreground the folding-back phenomenon. As earlier mentioned, qualitative research designs are commonly used; however, there is a paucity of studies that focus on folding back within autoethnography research designs to report on teachers' personal experiences. Thus, my study seeks to contribute to the methodological gap in folding-back studies. Moreover, as earlier mentioned, there is limited research that focuses on the role of folding back on learners' algebraic reasoning, particularly within the South African Grade 10 context. In addition, empirical research is scarce with a focus on the challenges concerning the use of focusing back in mathematics classrooms. As a result, this autoethnographic study seeks to promote Grade 10 learners' algebraic reasoning through folding back and challenges related to the use of folding back. The rationale for focusing on grade-level algebraic reasoning is accounted for in the next section of this chapter, which focuses on the literature review.

2.5 OVERVIEW OF ALGEBRA IN SOUTH AFRICA'S MATHEMATICS CURRICULUM

The mathematics curriculum in South Africa is structured to systematically enhance learners' understanding of mathematics across various schooling phases. These phases consist of the foundation phase, which includes Grades 1-3 (DBE, 2011b); the intermediate phase, encompassing Grades 4-6 (DBE, 2011c); the senior phase, covering Grades 7-9 (DBE, 2011a); and the FET phase, which comprises Grades 10-12 (DBE, 2011d). As a result, different mathematical topics are presented to learners as they progress through these phases. Mokgonyane et al. (2017) state that in the foundation phase, learners are introduced to basic arithmetic and problem-solving, which establishes a foundation for more abstract thinking in later phases. Given its wide-ranging importance, algebra is introduced early in education (Sun, 2023). Acar (2019) indicates that a strong positive correlation exists between the number sense of secondary school students and their levels of algebraic thinking. By focusing on algebraic thinking in foundational grades, learners may develop the ability to generalise (Moretti et al., 2021). In addition, early exposure to algebraic concepts facilitates a smoother transition from arithmetic to algebra, reducing the difficulties students encounter when formal algebra instruction begins in middle school (Leong et al., 2017). In this study, I align with Tall and Thomas (1991), who advocated for algebra in the mathematics curriculum by stating that “there is a stage in the curriculum when the introduction of algebra may make simple things hard, but not teaching algebra will soon render it impossible to make hard things simple” (p. 128).

The curriculum progression through the intermediate and senior phases to the FET phase highlights purposeful advancement where foundational concepts are intricately linked with more complex topics, as demonstrated in a study by Pournara (2016). This structure highlights the continuity in learning, where foundational skills bolster the comprehension of more advanced topics, which is essential for developing a robust understanding of mathematical concepts (Mutodi & Mosimege, 2021). Furthermore, this progressive structure aims to correspond with cognitive development stages; however, research indicates a disconnect between curriculum expectations and learners' preparedness, especially during the transition from Grade 9 into the FET phase, where algebra takes on a more abstract and symbolically complex form (Moloi et al., 2022).

The introduction and expansion of algebra, especially during the FET phase, is central to this progression. Consequently, the mathematical knowledge that students gain in the early grades underpins their achievement in subsequent grades. My understanding is that the learners' knowledge they acquire in the previous phases of schooling serves as an essential foundation in the FET phase. As a result, failing to master crucial content in fundamental areas, such as algebra, during the introductory FET grade (Grade 10) can negatively impact students' overall mathematics performance in Grades 11 or 12 (Clotfelter et al., 2019). Hence, various NCS diagnostic reports indicate that Grade 12 learners consistently struggle with algebra-related tasks, which they attribute to underdeveloped knowledge in the foundational phases.

In this autoethnographic study, the focus is on Grade 10, which is regarded as an entry grade for the FET phase. Although I provide a specific knowledge area that underpins algebra in Grade 10, this autoethnographic study is not premised upon improving such content but rather upon promoting learners' algebraic reasoning skills in contextual problems. Thus, teaching framed on the notion of folding back is not intended to improve learners' understanding of algebraic topics such as equations. Instead, the emphasis is on promoting grade learners' ability to use algebra in contextual problems, in this case, financial mathematics contexts. A general focus of the algebra content area, as highlighted by DBE (2011a), "is for the learner to achieve efficient manipulative skills in the use of algebra" (p.10). My understanding of this description is that algebraic skills are transferable across various content areas in mathematics. Thus, learners should be able to use algebraic skills to analyse "situations in a variety of contexts to make sense of them" (DBE, 2011a, p. 10) and solve mathematical problems. I am of the view that the algebraic reasoning skills learners develop in Grade 10 serve as a foundation for the following grades in the FET phase. As outlined in the CAPS curriculum, learners in this important grade should develop a nuanced understanding of algebra. Moreover, concentrating on Grade 10 is justified, as the early stages of understanding algebraic concepts can pose challenges (Faradilla et al., 2024).

According to DBE (2011d), Grade 10 learners are expected to understand that real numbers can be irrational or rational and must develop fluency to perform algebraic operations with both types of numbers when simplifying algebraic expressions and solving equations. In addition, the expressions learners are required to simplify include fractions with denominators of cubes and those that require them to use the laws of exponents for rational exponents. Furthermore, they must be able to manipulate these expressions through factorisation and expansion. The factorisation and expansion include factorising the difference/sums of two cubes and trinomials and expanding through multiplying a binomial by a trinomial.

Similarly, using mathematical knowledge from the simplification of algebraic expressions, learners are expected to solve a variety of equations. These equations include linear equations, quadratic equations (using factorisation), and literal equations, whereby they change the subject of a formula. Moreover, learners are expected to be able to solve exponential equations either in the form of a system of linear equations or as a word problem and to solve linear inequalities. The integration of algebra in word problems exposes learners to mathematics modelled in real-life contexts, allowing them to see its essence beyond the classroom.

According to Kurz and Lee (2023), the ability to perform these mathematical manipulations lays a strong foundation for concepts such as geometry, trigonometry, number patterns, analytical geometry, functions, and financial mathematics that require enhanced algebraic reasoning skills in Grades 11 and 12. This perspective is reinforced by Pratiwi et al. (2018), who highlighted the essential role of algebra in problem-solving, bridging the gap between arithmetic and more advanced mathematics. It is therefore claimed that algebra serves as a base for concepts assessed in both papers 1 and 2 in these grades in the FET. This is supported by the CAPS document, highlighting that algebra weighs 35% in Grade 10, which is greater than any other content area, 30% in Grade 11, and 17% in Grade 12, as its significance is amplified when considered in conjunction with other added mathematical topics.

2.6 REASONING IN MATHEMATICS

Mathematical reasoning is generally viewed as a cognitive process that enhances learners' mathematical understanding and is premised on logical deductions to arrive at valid mathematical conclusions (Sari et al., 2022; Rahmawati et al., 2018; Stylianides et al., 2016). As a cognitive process, it provides learners with the opportunity to analyse new situations in mathematics, construct logical assumptions related to mathematical topics, articulate their thought processes through mathematical statements, and arrive at conclusions for effective problem-solving (Wilujeng, 2024). According to Mabotja (2017), mathematical reasoning refers to learners' ability to logically justify the mathematical constructs or concepts they develop through their learning experiences in a supportive classroom environment. These descriptions position mathematical reasoning as a foundation for logical thought.

To cultivate a more profound understanding of mathematical concepts, learners need to engage in mathematical reasoning. However, numerous studies emphasise that the challenges learners encounter, such as comprehending problem instructions and applying rules during problem-solving, often stem from ineffective mathematical reasoning (Basir et al., 2022; Norhidayah, 2023; Patmaniar et al., 2021; Yildiz & Durmaz, 2021). Insights from these studies indicate that educators, through their pedagogical methods in mathematical classrooms, should prioritise the development of reasoning skills that promote meaningful engagement with mathematical concepts. This aligns with the specific aims of mathematics as outlined in the curriculum, which underscore that "Teaching should not be limited to "how" but rather feature the "when" and "why" of problem types. Learning procedures and proofs without a sufficient understanding of why they are important will leave learners ill-equipped to use their knowledge in later life" (DBE, 2011d, p. 8). This specific aim highlights not only learners' ability to justify their thought process but also the necessity for learners to apply mathematical reasoning beyond the mathematics classrooms. In addition, strong mathematical reasoning skills are highly valued in the workplace. It allows employees to effectively analyse data, make well-informed decisions, and efficiently solve problems (Hafizi & Kamarudin, 2020). Hence, Palinussa et al. (2021) recognise the significant impact of mathematical reasoning in various scientific fields and everyday situations. Mathematical reasoning further supports innovation in fields such as finance, data science, and technology, where analytical thinking and problem-solving are continuously in demand. In addition, there is a clear connection between strong mathematical reasoning skills and academic success, especially when learners move from secondary to tertiary education (Di Martino et al., 2022). Therefore, the teaching of mathematics should strive to enhance learners' reasoning skills by situating mathematical problems in familiar contexts (Ardiniawan et al., 2023; Brodie, 2009; Ningsi et al., 2024).

Furthermore, learners should participate in problem-solving activities that necessitate justifying and explaining their thought processes to develop strong mathematical reasoning abilities. In this context, Askew (2020) posits that reasoning ought to be regarded as a cognitive habit, cultivated progressively through sustained engagement with mathematical tasks that necessitate critical analysis and justification of solutions. As such, mathematics education should emphasise deep conceptual understanding rather than rote memorisation, utilising rigorous reasoning practices. In addition, such activities should promote the use of collaborative learning, wherein peer interactions and diverse viewpoints can improve learners' capacity to reason through complex mathematical situations (Chuene et al., 2023; Mabotja et al., 2018; Segerby & Chronaki, 2018).

Literature highlights numerous forms of mathematical reasoning, including Abductive Reasoning, Analogical Reasoning, Deductive Reasoning, and Inductive Reasoning (Hull, 2017; Lestari et al., 202). The various types of reasoning hold significant importance across the diverse educational levels of learners as they progress, as each one addresses specific areas within the discipline (Lestari et al., 2022). However, this section is confined to deductive and inductive reasoning, as discussed in the next two sub-sections.

2.6.1 Deductive Reasoning

Sternberg et al. (2021) define deductive reasoning as a process that depends on established rules and known premises to reach definitive conclusions. This characterisation is supported by the definition provided by Quezada (2023), which describes deductive reasoning as a logical process in which general premises or statements are used to formulate valid conclusions. Morsanyi et al. (2018) agree with the definition offered by Quezada (2023), emphasising in their study that the defining feature of deductive reasoning is its capacity to yield conclusions that must be true, assuming the initial premises are accurate. Sternberg et al. (2021) indicate that it typically begins with broad statements or axioms and progresses toward specific instances or outcomes through a series of logical steps.

Consequently, researchers such as Morsanyi et al. (2018) consider this form of reasoning to be critically significant in mathematics, especially in formal proofs, where the legitimacy of conclusions relies entirely on the premises given. Wong (2017) emphasises its significance beyond mere mathematical proofs by highlighting how deductive reasoning improves students' overall problem-solving skills across different mathematical contexts. For example, a study conducted by Józsa et al. (2024) indicated that deductive reasoning promoted students' comprehension of mathematical concepts, aided in the derivation of new insights, and improved their overall academic performance in mathematics. The reported findings suggest that deductive reasoning not only facilitates the development of learners' critical thinking and problem-solving skills but also plays a crucial role in the acquisition of mathematical knowledge.

A study conducted by Datsogianni et al. (2020) revealed that even young learners possess the ability to make valid deductions in specific contexts, highlighting the importance of deductive reasoning in promoting scientific reasoning and hypothesis generation. Similarly, Chasanah (2019) links deductive reasoning to students' cognitive development. As learners progress through various educational stages, their ability to engage in deductive reasoning develops. In this context, Józsa et al. (2024) investigated the progression of deductive reasoning abilities in children aged 4 to 8 years. Their results demonstrated a significant upward trend in deductive reasoning abilities within these age groups, consistent with established theories of cognitive development that indicate reasoning skills progress with age and experience (Józsa et al., 2024). This study, drawing from the same analogy, focuses on a lower FET grade (grade 10). It asserts that fostering deductive reasoning skills associated with algebraic reasoning at an early stage establishes the groundwork for more intricate reasoning processes that students will face in their subsequent education. According to Pitta-Pantazi et al. (2025), deductive reasoning strengthens learners’ ability to make use of established mathematical rules to manipulate algebraic expressions and further logically justify their solutions.

However, this primary reliance on established premises that must be universally accepted as accurate is limiting in algebra, as learners are to solely rely on these established rules when approaching algebraic problems. Therefore, in a situation where a learner incorrectly applies a law/rule, it means that every deductive step they follow after that would also be flawed, leading to an erroneous conclusion (Miyazaki et al., 2017). Consequently, in these instances, the conclusions reached, no matter how logically coherent they might appear, will still be erroneous, restricting learners from exploring, questioning, or adapting mathematical concepts in unfamiliar or complex situations. These limitations imply that, although deductive reasoning is valuable, it is not sufficient on its own to support learners' algebraic development. This creates a need for an additional form of reasoning that compensates for its shortfalls, what Hull (2017) refers to as inductive reasoning.

2.6.2 Inductive Reasoning

Inductive reasoning, as described by Dantlgraber et al. (2019), primarily involves the capacity to identify connections and to derive and apply overarching rules from specific observations. It equips learners with the ability to understand mathematical rules through exploration, observation and questioning. In a mathematical context, inductive reasoning commonly involves analysing various examples to draw a more general conclusion. Consequently, this reasoning process is typically characterised by the capacity to generalise from specific instances available, integrating the awareness that the conclusions, although probably accurate, are not guaranteed to apply universally across all situations (Ndemo et al., 2018).

In contrast to deductive reasoning, which ensures conclusions when the premises are accurate, inductive reasoning generally presents probabilistic conclusions that require additional testing and validation, as noted by Hayes and Heit (2018). This open-ended nature, described by Payadnya (2019), allows for the possibility of discovery and innovative ideas, a view echoed by Jeannotte and Kieran (2017), who note that inductive reasoning enables individuals to form patterns from observed data, potentially leading to innovative ideas and advancements in understanding. Furthermore, Zalaghi and Khazaei (2016) highlight that inductive reasoning is often preferred over deductive reasoning in early mathematical exploration because it begins with forming ideas from specific instances. In contrast, deductive reasoning starts with general principles to reach specific conclusions.

In mathematics education, inductive reasoning, much like deductive reasoning, aids students in cultivating problem-solving skills and mathematical thinking by prompting them to investigate and recognise patterns (Ansari et al., 2020). A study conducted by Sosa and Aparicio (2021) highlights the crucial function of inductive reasoning in the processes of knowledge generation, problem-solving, and the formulation of generalisations within the realm of mathematics education. The skill set identified by Dantlgraber et al. (2019) is essential for promoting critical thinking and creativity in students, especially as they encounter increasingly complex mathematical concepts, where recognising patterns and trends becomes crucial. For example, when students utilise their comprehension of mathematical patterns in new contexts, they successfully contextualise their learning. Consequently, this adaptability, as noted by Malambo et al. (2023), is crucial in a rapidly evolving world where effective problem-solving often requires the application of knowledge in unfamiliar settings. A study aimed at analysing students' mathematical reasoning abilities, with a focus on their performance in solving mathematical problems, reported the significance of inductive reasoning in problem-solving within mathematics education (Rahmawati et al., 2018). Additionally, a study conducted by Walidah and Susanti (2021) examining how high school students engage in argumentation while proving mathematical induction, based on their mathematical abilities, revealed that students employ inductive reasoning as a fundamental tool for justifying their conclusions throughout the proof process.

Furthermore, Kamsurya and Ngadino (2024) explored students' inductive reasoning skills, particularly in addressing fractional problems using a contextual approach. Their research emphasises the significant role of inductive reasoning in enhancing students' mathematical reasoning skills, enabling them to formulate general rules derived from specific arithmetic examples. While inductive reasoning is crucial for assisting learners in recognising patterns, constructing arguments, and developing generalisation skills, it does have its inherent limitations. Ndemo et al. (2018) highlight that inductive reasoning lacks the certainty needed for conclusions derived from specific observations to be universally applicable. The absence of a guarantee poses a significant challenge for learners, particularly when they are required to shift their reasoning from tangible arithmetic examples to more abstract algebraic contexts.

GENÇ and Erbaş (2020) further assert that such transitions can impede learners from forming valid generalisations, as previous inductive experiences may not sufficiently equip them for the challenges of abstract reasoning, such as algebraic reasoning. This argument highlights the importance of intentionally strengthening learners' algebraic reasoning, which can be viewed as situated at the intersection of deductive and inductive reasoning. Algebraic reasoning therefore functions as a cognitive bridge, drawing on inductive reasoning to support learners in recognising patterns and on deductive reasoning to justify and verify generalisations, abstract relationships and symbolic manipulations, skills that are crucial for deeper mathematical understanding and effective problem-solving.

2.7 ALGEBRAIC REASONING

Algebraic reasoning generally refers to the cognitive ability to identify, generalise, and manipulate patterns or relationships using symbols, variables, and operations to solve problems (Ahmadah, 2020; Indraswari et al., 2018; Rahmawati et al., 2019). At its core, the emphasis is on the examination of mathematical symbols and the principles governing the manipulation of such symbols, which denote quantities and their interrelations in equations (Indraswari et al., 2018; Kaput, 2017; Lee, 2024; Setiawan, 2022). This algebraic manipulation is premised upon the meaningful use of symbols, where learners should be able to select variables and construct expressions and equations in context (National Council of Teachers of Mathematics (NCTM), 2016). Engaging in such processes allows learners to generalise concepts in specific contexts, reinforce their understanding through discourse, and express them formally (Rahmawati et al., 2019). In doing so, learners should recognise the solution steps as a logical understanding of the relationship (NCTM, 2016). Hence, learners' "ability to solve problems related to learning and how to state generalisations about numbers, quantities, relations, and functions is part of algebraic reasoning" (Basir et al. 2022, p. 96).

2.8 IMPORTANCE OF ALGEBRAIC REASONING

Algebraic reasoning is essential in mathematics education, acting as a cornerstone for students' comprehension and use of mathematical concepts throughout their academic experience (Garzón & Bautista, 2018). Walick and Burns (2017) assert that algebraic thinking enhances an individual’s comprehension and generalisation of fundamental skills, informs the acquisition of new knowledge, directs problem solving in novel and abstract contexts, and broadens algebraic learning beyond mere procedural recall. Sibgatullin et al. (2022) indicate that these skills encompass manipulating symbols, recognising patterns, and generalising mathematical concepts, all of which are essential as learners progress in their mathematical education. Acosta and Alsina (2020) and Somasundram et al. (2019) further highlight that students’ capacity to recognise patterns and generalise mathematical relationships fosters the development of the foundational elements essential for algebraic thinking.

Furthermore, algebraic thinking extends beyond the mastery of equations; it encompasses mental activities such as contemplating, discussing, and discovering solutions to mathematical challenges. It fosters a mindset that appreciates functional relationships, which are essential in mathematics (Sibgatullin et al., 2022). Tafari et al. (2024) emphasise that algebraic reasoning equips learners with the skills required to address more complex mathematical problems, which is vital for academic achievement. Further research indicates that algebraic reasoning enhances the ability to generalise mathematical concepts and formulate arguments, which are skills necessary for addressing real-world challenges (Ahmadah, 2020; Walick & Burns, 2017). The highlights that algebraic reasoning is far more than just about procedural manipulation but enables learners to engage meaningfully with mathematical ideas necessary to develop their higher-order thinking.

Akkan (2016) and Yang and Deng (2024) underscore that algebraic thinking fosters learners’ ability to think abstractly and make logical inferences. The contention is that algebra promotes critical thinking in learners regarding the relationships between quantities, a skill that is essential in numerous mathematical scenarios and everyday situations (Kusuma et al., 2018; Wilujeng, 2024). In addition, algebraic reasoning underpins essential problem-solving skills required for addressing real-world challenges and higher-level mathematical enquiries (Burgos et al., 2024). Furthermore, Burgos et al. (2024) and Kaput (2017) further emphasise that meaningful engagement in algebraic thinking requires constructing the meanings of algebraic relations and drawing generalisations connected to real-life contexts. In other words, algebra enables individuals to interpret the world more effectively by translating encountered situations into mathematical language, thereby explaining and predicting events in daily life. The contention is that “contextual problems help strengthen algebraic thinking by linking abstract algebraic concepts to real-life situations, making them more comprehensible for students” (Al-Mutairi & Marzouq, 2025, p.3).

In addition, algebraic reasoning forms the foundation for students aiming to pursue careers in STEM fields (Grønmo, 2018; Veith et al., 2023). Algebraic reasoning skills are crucial in improving learners’ abilities in scientific fields, where mathematical applications are typical (Turşucu et al., 2020). In financial planning and other areas, calculating investments, interest on loans, or creating budgets necessitates using algebra to identify optimal strategies and forecast future outcomes (Juraev & Bozorov, 2024). The connection between algebra and various fields emphasises the importance of possessing fundamental algebraic thinking skills, as they are considered a natural method for simplifying everyday problems (Fitrianna & Dahlan, 2022). These studies support that developing algebraic reasoning skills holds long-term implications for learners’ academic and career trajectories. Research also demonstrates that developing algebraic reasoning early equips learners with essential reasoning abilities required for success in later mathematics and higher education (Barboza et al., 2020; Li & Zulnaidi, 2019).

Algebraic reasoning is considered a gateway for advanced mathematical learning (Stein et al., 2011, p. 454), suggesting that cultivating students' algebraic thinking skills should be a primary objective of mathematics instruction (Maphutha, 2023). Hence, algebra serves as the core of the high school mathematics curriculum (Gabuya et al., 2025). I concur with this viewpoint, as almost every subject in the high school mathematics curriculum requires a fundamental understanding of algebra. Hence, it acts as a cohesive instrument that links arithmetic with geometry, trigonometry, and functions (Maphutha, 2023). Similarly, Mabotja's (2017) study revealed that learners utilise algebraic reasoning when solving geometric problems. Relevant to the current study, algebraic reasoning plays a critical role in financial mathematics concepts such as interest calculation and investments (Al-Mutairi & Marzouq, 2025). Thus, in this autoethnographic study, I focus on learners’ algebraic reasoning skills application in the Grade 10 financial mathematics content area. The rationale for focussing on financial mathematics does not necessarily stem from learners' challenges in this topic, but rather from its contextual nature. Researchers such as Al-Mutairi and Marzouq (2025) and Laswadi (2023) argue that contextual problems provide a fertile learning environment for learners to apply their algebraic reasoning skills in real-life situations.

Al-Mutairi and Marzouq (2025) propose four algebraic thinking processes that are critical in contextual problems involving financial mathematics. Firstly, understanding the problem involves repeatedly reading the text and extracting both known and unknown information. Secondly, implementing the solution involves identifying the meaning of symbols and applying abstract symbols to define variables. This process also involves applying algebraic properties to arrive at the solution and using functional reasoning to identify relationships between variables. Thirdly, evaluating the solution involves verifying, simplifying, and manipulating symbols to ensure its accuracy. Lastly, expanding the solution emphasises the importance of generalising the pattern of the simple interest formula.

2.9 LEARNER'S CHALLENGES IN ALGEBRAIC REASONING

Despite the importance of algebraic reasoning, literature shows that learners experience challenges or difficulties in this cognitive mathematical skill. As a result, I elaborate on some of the challenges as documented in the literature. These include learners' misconceptions and common algebraic errors, mathematical anxiety, and traditional teaching approaches as a source of algebraic reasoning challenges.

2.9.1 Misconceptions and Common Algebraic Errors

Misconceptions refer to the misapplication of rules and over- or under-generalisation (Bowers, 2021; Drews, 2005). Such misapplication implies an erroneous formulation of a mathematical concept, attributable to flawed reasoning and knowledge (Mgidi, 2024). Learners' misconceptions impede their development of mathematical understanding of the concepts (Baidoo, 2019; Im & Jiterndra, 2020). Researchers such as Luneta (2008) and Umar and Galadima (2025) opined that misconceptions often manifest through errors. Mathematical errors are regarded as mistakes, blunders, or deviations from precision (Matuku, 2017; Luneta & Makonye, 2010). Young and Booth (2020) emphasise that these errors in understanding can result in incorrect algebraic manipulations, which further propagate confusion as students advance through the curriculum. The transition from arithmetic to algebra is often hindered by misconceptions and errors that impede learners' comprehension and reasoning skills (Gasim, 2018). This crucial shift entails a higher level of abstraction in their problem-solving approaches and requires an understanding of algebraic operations, which are fundamental to algebraic reasoning (Kusuma et al., 2018). Various errors are identified in the literature; these include, among others, conceptual, procedural, or computational errors as discussed in the following subsections (Agustyaningrum et al., 2018; Baidoo, 2019; Chauraya & Mashingaidze, 2017).

2.9.1.1 Conceptual errors

According to Delastri and Lolang (2023), conceptual errors refer to errors that arise from an incorrect understanding of a mathematical concept or principle. Their study further highlights that such errors occur when learners misunderstand the underlying concept or rely on reasoning that is not mathematically valid. Sehole et al. (2023) support this view, noting that learners who memorise mathematical procedures without fully understanding them often fail to translate problem statements into correct algebraic expressions, an indication of a gap in conceptual understanding. Many students struggle to connect algebraic symbols to their conceptual meanings, thus becoming trapped in the mechanism of memorising procedures without understanding the deeper concepts (Lin et al., 2024). Chauraya and Mashingaidze (2017) observed that learners often struggle to transition from working with numerical values to engaging with variables represented by letters. Khalid et al. (2020) attribute this difficulty to learners transferring arithmetic-based thinking to an algebraic context, while Wahyuni et al. (2023) add that many learners, especially those in the lower grades, often assign specific numeric values to variables without realising that these symbols can represent a range of numbers. The findings of AL-Rababaha et al. (2020) support this notion, indicating that errors in algebra frequently arise from students' limited comprehension of abstract concepts or the manipulative nature of algebraic symbols, which are often viewed as mere labels instead of functional components of mathematical expressions. Young and Booth (2020) emphasise that these transition errors can result in incorrect algebraic manipulations, which can further propagate confusion as students advance through the curriculum.

In addition, researchers such as Ardiansari et al. (2022) and Putri et al. (2024) highlight misconceptions regarding the equal sign and equivalence. In their studies, they report that the majority of learners often misinterpret the equal sign as a prompt to compute rather than a symbol to indicate equivalence. In another study, Mutodi et al. (2023) revealed that learners solved inequalities as if they were equations and treated an inequality sign as an equal sign, which they allude to stemming from insufficiently grasping fundamental principles. Therefore, it is through these misconceptions that Agung et al. (2021) emphasised their ability to confuse students as they worked to manipulate algebraic expressions and solve equations, resulting in a persistent lack of conceptual understanding.

Weldeana et al. (2023) completely tie learners' misconceptions to their inability to integrate new algebraic ideas into suitable cognitive schemas. According to their study, they highlight that when learners initially incorporate algebraic concepts into unsuitable schemas, these misconceptions can become firmly established, leading to persistent poor algebraic reasoning despite the availability of correct information. For instance, in approaching a word problem, students frequently understand the problem itself, but they do not fully grasp the information contained in the question, which impedes their capacity to formulate additional solutions (Pomalato et al., 2020). In some instances, they emphasise the need to translate students' incomplete understanding of the problem into authentic mathematics. As such, their study believes that misconceptions are not merely trivial mistakes; they stem from cognitive frameworks that shape how individuals perceive and engage with mathematical concepts. These studies seem to agree that learners' conceptual errors frequently arise from misunderstandings of the meaning of symbols (including variables), equivalence, and equations, which are essential concepts to be understood for success in algebra and other concepts that require algebraic thinking.

2.9.1.2 Procedural errors

Procedural errors are mistakes that learners make as a result of incorrectly applying rules, whether in the use of formulae or in the simplification steps followed when solving a problem (Mgidi, 2022; Sehole et al., 2023). This view is supported by Chauraya and Mashingaidze (2017), who state that in algebra such errors typically arise from the incorrect application of algebraic operations or rules. For example, Makonye and Stepwell (2016) observe that learners often simplify expressions incorrectly by transferring operational rules from arithmetic to algebra without recognising the structural differences between the two, a pattern similarly demonstrated in the work of Pomalato et al. (2020).

I have also encountered these errors first-hand in my own classroom. Some learners frequently attempt to add or subtract terms that are not like terms. In other cases, when simplifying expressions with brackets, learners misapply the distributive rule as a result of vague or incorrect generalisations of the distributive property. Research findings echo these challenges, with several studies reporting that learners struggle with procedural manipulation of algebraic expressions, factorisation, simplification, division of terms, and the misuse of the lowest common denominator (LCD) (Agung et al., 2021; Chauraya & Mashingaidze, 2017; Chirove & Ogbonnaya, 2021; Putri et al., 2024).

Mgidi (2022) further reports that learners misapply exponential laws and make procedural mistakes when dividing terms or solving for x, reinforcing the view that these errors stem from weaknesses in procedural fluency. Ndemo and Ndemo (2023) argue that such errors arise from a lack of foundational operational knowledge, which leads learners to apply procedures mechanically without understanding their purpose. They further explain that learners may employ rules that are not applicable to the problem at hand, a view supported by Yao et al. (2021), who contend that many learners lack the operational skills necessary for accurate algebraic manipulation. Collectively, these studies suggest that underdeveloped procedural knowledge, the overgeneralisation of mathematical rules, and reliance on memorised procedures contribute more significantly to procedural errors than the mere incorrect application of a rule.

2.9.1.3 Computational errors

Agustyaningrum et al. (2018) define computational errors as calculation mistakes learners make when manipulating numbers involving the four primary operations (division, multiplication, addition, and subtraction). This term is supported by Mathaba and Bayaga (2021), who report that such errors occur because learners misuse operational signs, which directly leads to incorrect calculations. In algebraic contexts, these errors become particularly evident when learners manipulate algebraic expressions (Baybayon & Lapinid, 2024). For instance, Mathaba and Bayaga (2021) found that some learners multiplied variables instead of simplifying through division. Similarly, Mathaba et al. (2024) noted that learners also struggle to expand the algebraic expressions. Nelson and Powell (2018) also documented computation errors in fraction addition, where learners added all digits from the numerators and denominators (e.g., ½ + ½ = 6, by adding (1 + 2 + 1 + 2). Across these examples, learners inconsistently applied operational rules, which disrupted their algorithmic procedures. Nelson and Powell (2018) attribute such misuse and misinterpretation of operational signs to poor recall of arithmetic facts and a lack of secure operational fluency. In addition, Mgidi (2022) emphasises that even if learners know the procedural steps, a single miscalculation can still produce an incorrect solution. Therefore, computational errors can be understood as the incorrect application of arithmetic operations and operational signs when solving mathematical problems.

2.9.2 Mathematical anxiety as a source of learners' challenges in algebra

Gabriel et al. (2020) highlight that numerous students encounter anxiety when confronted with challenging mathematical concepts, which can significantly impede their learning and involvement in algebra. The argument posits that a high level of abstraction in algebra may contribute to learners' high levels of anxiety. According to Mediana and Hinacay (2025), "Algebra anxiety is characterised by emotional distress and apprehension when engaging with algebra concepts and evaluations, which poses a formidable hurdle to effective teaching and learning and eventually to the academic achievement of learners" (p. 766). This study suggests that learners who are anxious about their algebraic reasoning abilities might have challenges in engaging in problem-solving activities and also lose interest in learning, which will result in low achievement. The contention is that learners with high mathematics anxiety often become passive and less engaged in working on mathematical activities (Kariuki et al., 2021). Previous studies have shown a correlation between anxiety and learners' achievement (Aldrup et al., 2020; Mediana & Hinacay, 2025; Namkung et al., 2019). Empirically, Mediana and Hinacay (2025) revealed a moderate strength of the negative correlations between algebra anxiety and performance, wherein it was concluded that the higher the anxiety level, the lower the performance will be. However, their findings also suggested that not all learners with high mathematics anxiety perform poorly in mathematics.

Consequently, teachers should play a crucial role in alleviating learners' mathematical anxiety (Aldrup et al., 2020; Jörn, 2021; Kyttälä & Björn, 2021). The emotional atmosphere of the learning environment, influenced by elements like teacher support and peer dynamics, has the potential to either intensify or reduce learners' anxiety levels. Anxiety not only hinders cognitive engagement with mathematical learning activities but also frequently leads to avoidance behaviours and reliance on superficial learning strategies instead of deep learning approaches (Quintero et al., 2021). Zhang (2018) emphasises the importance of creating learning materials that align with students' cognitive levels and effectively address both their prior knowledge and the anxieties that arise in mathematics classrooms. Furthermore, integrating strategies to activate prior knowledge during class discussions or problem-solving effectively can reduce anxiety and promote a sense of competence among students (Turşucu et al., 2020).

2.9.3 Conventional teaching approaches as a source of algebraic reasoning challenges

Learners' challenges in algebraic reasoning are attributed to traditional (conventional) teaching strategies, which emphasise rote skills rather than foster a deeper interaction with the content (Birgin & Demiroren, 2020). The concern is that such strategies depend exclusively on the rote memorisation of algebraic rules without promoting understanding of the fundamental principles, and as a result, they intensify learners' misconceptions (Fardah & Palupi, 2023). In this case, it can be argued that routine memorisation of procedures does not equip learners with the confidence to justify their thought processes. Kobandaha et al. (2019) highlight that in such learning environments, learners often struggle to grasp the concept of variables and their applications in algebra, leading to misunderstandings and misapplications of rules and concepts. Consequently, they tend to approach questions requiring algebraic thinking inadequately. Empirically, an earlier study by Ajai et al. (2013) revealed that learners taught algebra using the conventional method perform lower than their counterparts taught through problem-based learning. Similarly, Fumador and Agyei (2018) found that conventional methods are less effective in addressing learners' errors and misconceptions in algebra. The contention is that conventional methods, which are teacher-centred, do not promote learners' algebraic proficiency and conceptual understanding (Chirove & Ogbonnaya, 2021; Luneta & Legesse, 2023). My understanding of these findings is that conventional methods do not provide learners with the opportunity to interact with each other and their teachers, where they can develop questioning and reasoning skills in algebra. Thus, conventional methods limit learners' algebraic cognitive abilities to identify, generalise, or manipulate patterns or relationships using symbols.

Furthermore, challenges perpetuated by a dominant reliance on conventional methods suggest the need for innovative, learner-centred approaches. Hence, Irshid et al. (2023) challenge teachers to adopt a broader perspective, embracing the responsibility not only to distribute knowledge in classrooms but also to shift their thinking towards fostering conceptual understandings, particularly in algebraic concepts. The challenge presented by Irshid et al. (2023) is supported by another research study that shares a common perspective, emphasising the significance of implementing effective new teaching methods. These methods equip learners with the necessary skills to accurately interpret problems that require the application of algebra (Chan et al., 2023). Eriksson et al. (2020) and Groth (2017) asserted that these instructional strategies should not only equip learners with essential algebraic skills but also foster a supportive learning environment where mistakes are regarded as valuable opportunities for growth.

A teaching approach framed by the theoretical construct of "folding back", as suggested by Martin and Towers (2016), places learners at the centre of their learning. This approach allows them to engage in collaborative problem-solving while effectively utilising their prior knowledge without teacher interference. It also provides an opportunity for contextual feedback from either the teacher or their peers, thereby enhancing students' cognitive processes (Hähkiöniemi et al., 2022). Consequently, this study employed folding back to improve the algebraic reasoning of grade 10 learners from the same perspective. It is crucial to recognise that incorporating folding back into mathematics instruction not only aids in fostering deeper conceptual understanding among students but also offers educators essential feedback that can guide teaching strategies tailored to various student backgrounds and learning styles (Patmaniar et al., 2021).

2.10 PROBLEM-BASED LEARNING AND ALGEBRAIC REASONING

Numerous studies in the literature have articulated various perspectives on the concept of problem-based learning (PBL) as a learner-centred instructional method in mathematics education (Hokkanen, 2017). Meilasati et al. (2020) define PBL as an instructional approach that focuses on engaging students in the resolution of realistic and complex problems, thereby promoting active learning, critical thinking, and self-directed inquiry. This definition aligns with the conceptualisations of other scholars regarding PBL as a pedagogical strategy that utilises complex, real-world problems to enhance conceptual understanding, reasoning, and communication skills in mathematics (Hendriana et al., 2018). Masitoh and Fitriyani (2018) similarly perceive the principles underlying PBL as multifaceted, encompassing authentic contexts, collaborative inquiry, self-directed learning, and reflective practice. These principles align with the perspective of Liljedahl et al. (2016), who argue that mathematics should be perceived not just as an abstract procedure but as a human endeavour rooted in problem-solving and practical application in real life.

Santoso et al. (2019) emphasise that motivating learners to link mathematical concepts with their everyday experiences through problem-solving encounters enhances intrinsic motivation and promotes a more profound understanding. Their work focused on the genuine essence of PBL, illustrating that real, contextual problems encourage ongoing inquiry and frame mathematics as a practical tool, rather than a detached academic discipline — a viewpoint also highlighted by Zulixanti et al. (2024). In this context, PBL enables learners not only to apply techniques previously taught but also to develop innovative strategies for addressing unfamiliar challenges (Santoso et al., 2019; Zulixanti et al., 2024).

The design of PBL curricula generally encompasses intricate scenarios that lack a single correct answer, prompting learners to incorporate various viewpoints and utilise multiple disciplines (Anuaraheni, 2018). This educational method is grounded in constructivist principles, emphasising that learning takes place through active engagement, collaboration, and reflection (Meilasari et al., 2020). Research indicates that students engaged in project-based learning (PBL) surpass their counterparts in conventional educational environments, demonstrating improved critical thinking, creativity, and advanced reasoning abilities (Efendi & Wardani, 2021). Meta-analytical studies indicate notable enhancements in critical thinking skills within elementary and secondary educational environments, emphasising the importance of PBL in cultivating abilities vital for lifelong learning (Putra & Mascui, 2019).

Numerous studies conducted over the years have demonstrated that problem-based learning environments are exceptionally effective for both teaching and learning (Gündüz et al., 2016; Orji & Ogbuanya, 2018; Nadeak & Naibaho, 2020). Houghton (2023) indicates that this effect is a result of its emphasis on learner-centredness and the opportunity it offers for learners to collaborate effectively in addressing assigned problems. Mustofa and Hidayah (2020) investigated the impact of PBL on learners' lateral thinking skills within the framework of environmental change. Their findings revealed that students engaged in PBL environments exhibited more significant improvements compared to those instructed through traditional methods. In a similar study, Amin et al. (2020) discovered that the application of PBL significantly enhanced students' critical thinking abilities and environmental attitudes in contrast to traditional teaching approaches. Boye and Agyei (2023) further extend these findings to mathematics education in Ghana, demonstrating that PBL encouraged collaboration, teamwork, and conceptual understanding among preservice teachers, which ultimately enhanced their mathematical reasoning skills. These studies collectively affirm that PBL environments are exceptionally effective in fostering thinking skills, which have considerable potential for enhancing algebraic reasoning in learners.

The implications for algebraic reasoning hold significant importance in the South African context, where ongoing challenges in mathematics education encompass low pass rates, procedural teaching methods, and a restricted conceptual grasp of algebraic concepts (DBE, 2024). Syawahid (2019) emphasises the significance of addressing algebraic reasoning challenges at an early stage of education to prevent escalating difficulties in later grades and higher education. This subject holds significant importance in Grade 10, the initial grade of the Further Education and Training (FET) phase, where students move from arithmetic to a more abstract form of algebraic reasoning. It is essential to address misconceptions, such as an insufficient understanding of fractions, weak connections between arithmetic and algebra, and limited integration of procedural and conceptual knowledge, to enhance learners' performance in algebraic tasks.

Research conducted by Mhlolo (2018) and Julie (2016) emphasises the significance of authentic contexts, especially those derived from learners' immediate surroundings, for fostering engagement and enhancing comprehension of algebraic structures. For example, presenting issues related to budgeting in local households, calculating taxi fares, or exploring patterns in traditional beadwork offers students culturally relevant ways to engage with algebraic thinking. This approach reflects the emphasis on authenticity highlighted by Santoso et al. (2019), while situating it in the daily lives of learners. Research has demonstrated that connecting mathematics to lived experiences can reduce disengagement and math anxiety (Sibanda & Makgakga, 2021).

Studies conducted in Gauteng and Limpopo (Setati & Barwell, 2008; Ramatlapeng, 2022) demonstrate that, in the context of collaborative inquiry, when learners engage in group work to tackle complex problems, they express their reasoning more frequently and negotiate the meanings of algebraic symbols with greater effectiveness compared to traditional teacher-led discussions. This finding is consistent with Efendi and Wardani's (2021) international study, which also highlights the distinct linguistic challenges present in multilingual classrooms, where students may engage in code-switching between their native languages and English to comprehend various problems. When thoughtfully organised, problem-based learning can use linguistic diversity as an asset instead of an obstacle to algebraic reasoning.

The concept of self-directed learning, a fundamental aspect of PBL, has also been the subject of research. Prinsloo and Vorster (2020) argue that in schools with limited resources, fostering self-directed inquiry enables learners to develop adaptive problem-solving strategies, which help mitigate the challenges posed by insufficient textbook availability and overcrowded classrooms. This view is especially relevant for algebra, where misunderstandings regarding variables, equality, and operations necessitate that learners revisit and reconstruct their understanding both independently and collaboratively. Problem-based tasks inherently promote this iterative reconstruction, enabling learners to revisit prior knowledge and advance their understanding (Martin & Towers, 2016).

Reflective practice, the final principle of PBL, has garnered attention in research, as seen in the work of Naidoo (2019), who investigated the reflections of Grade 10 learners following their involvement in problem-based tasks centred on linear and quadratic functions. Students indicated that contemplating their errors and examining various solution approaches allowed them to perceive algebra as adaptable, rather than inflexible, thereby boosting both their confidence and reasoning skills. This finding resonates with Groth's (2017) call to view errors as valuable learning opportunities, which is especially pertinent in South African classrooms where numerous learners come with deeply rooted misconceptions.

Evidence gathered from intervention studies demonstrates the effectiveness of PBL in enhancing algebraic reasoning. A quasi-experimental study conducted by Mavhunga and Rollnick (2019) applied PBL with Grade 9 learners in a township school, revealing notable improvements in the learners' capacity to generalise patterns and justify algebraic procedures when compared to a control group that received traditional instruction. In a similar vein, a study conducted by Khuzwayo (2021) in rural KwaZulu-Natal revealed that PBL significantly improved learners' perseverance in problem-solving, especially when the problems were framed within familiar socio-economic contexts, such as small-scale farming and informal trading. The findings suggest that PBL has the potential to connect abstract algebraic concepts with concrete experiences, a connection that is often absent in conventional teaching methods.

Dolmans et al. (2016) caution that insufficient scaffolding in PBL may overwhelm learners who lack foundational skills, resulting in frustration rather than fostering deep learning. This concern is evident in the research by Mthethwa and Mji (2022), who observed that students with substantial deficiencies in arithmetic encountered difficulties in effectively engaging with open-ended problems. Therefore, successful PBL design in algebra requires striking a balance between challenge and support, providing entry points for every learner while also encouraging higher-order thinking.

Furthermore, the integration of PBL in South African classrooms necessitates addressing systemic challenges, including large class sizes, high-stakes assessments, and insufficient teacher training in constructivist pedagogy (Venkat & Spaull, 2015). Teachers frequently express that they feel compelled to "cover the syllabus" instead of exploring open-ended problems, which can hinder opportunities for in-depth reasoning (Govender, 2020). Initiatives focused on professional development, such as the Mpumalanga Secondary Science Initiative (MSSI) and comparable programmes, have demonstrated potential in providing teachers with strategies to implement PBL effectively while maintaining curriculum coverage (Maree, 2019).

2.11 CHAPTER SUMMARY

In this chapter, the literature was reviewed under the following headings: an overview of algebra in South Africa’s mathematics curriculum, algebraic reasoning, learners’ challenges in algebra, problem-based learning and algebraic reasoning, and the role of folding back in mathematics education. In addition, the theoretical framework was discussed. Selected studies that employed folding back also provided insights into the research methodology, which is described in the next chapter.

CHAPTER 3 RESEARCH METHODOLOGY

3.1 INTRODUCTION

The previous chapter presented a review of the literature that establishes the relevance of this study. In this chapter, I provide a detailed account of the methodology that guided the data collection process. The methodological account is organised as follows. First, the research paradigm is explained. Second, the research approach and design, as well as their suitability, are discussed. This is followed by a description of the research site and the sampling strategy employed. The data collection methods and instruments are then outlined. Furthermore, an explanation of how narrative analysis was applied to analyse the data is presented. In addition, issues of rigour and trustworthiness are addressed through the quality criteria. An account of the ethical considerations upheld in this autoethnographic study is also provided. Finally, the chapter concludes with a summary.

3.2 RESEARCH PARADIGM

The term *paradigm* is often defined as a philosophical worldview for seeing, explaining, and understanding a phenomenon in order to make sense of it (Cohen et al., 2018; Lincoln et al., 2011; Mukherji & Albon, 2015). It constitutes “a set of beliefs about the way in which particular problems exist and a set of agreements on how such problems can be investigated” (Fraser & Robinson, 2004, p. 59). Thus, the role of the paradigm in this study is to provide philosophical insight into understanding learners’ challenges in algebraic reasoning, as well as the extent to which teaching framed by the notion of folding back can play a pivotal role in alleviating such challenges. Furthermore…

A research paradigm consists of four main philosophical assumptions, namely ontology, epistemology, methodology, and axiology (Cohen et al., 2000, 2018; Creswell, 2014; Lincoln et al., 2011). Ontology generally refers to the form and nature of reality (Cohen et al., 2018). Epistemology refers to the nature of the relationship between the knower and what can be known (Guba & Lincoln, 1994). Methodology emphasises “the logic and flow of the systematic processes followed in conducting a research project, so as to gain knowledge about a research problem” (Kivunja & Kuyini, 2017, p. 26). Lastly, axiology focuses on the nature of ethics or ethical behaviour (Kivunja & Kuyini, 2017). Through these philosophical assumptions, researchers systematically "lay the foundations for how we, as individuals, understand the world we live in, the determinations we make about issues relating to truth, and the matters we consider to be of value to us individually and to society at large" (Edelheim, 2014, pp. 30–31).

Education researchers commonly employ a range of paradigms, including positivism and constructivism or interpretivism (Kivunja & Kuyini, 2017; Kumatongo & Muzata, 2021). In this autoethnographic study, I align with the interpretivist paradigm, as justified below.

3.2.1 Interpretivist ontological assumptions

The interpretivist paradigm places significant emphasis on the construction of meaning and the interpretation of information to understand reality (Nickerson, 2022). In contrast to positivism, which maintains that the researcher should remain detached from the phenomenon under investigation, interpretivism argues that reality can only be fully understood through active participation in the process of interpretation (Alharahsheh & Pius, 2020). Consequently, interpretivists emphasise grounding research in the first-hand experiences of individuals rather than relying solely on external observation (Pope & Mays, 2020). As a Grade 10 mathematics teacher seeking to promote learners' algebraic reasoning by framing instructional decisions around the folding back framework, I am well positioned to gain first-hand insights into my learners' experiences and meaning-making processes within the classroom context.

3.2.2 Interpretivist epistemological assumptions

Interpretivist epistemological assumptions postulate that the "researcher makes meaning of their data through their own thinking and cognitive processing of data informed by their interactions with participants" (Kivunja & Kuyini, 2017, p. 33). Accordingly, reality is viewed as a social construct. As a result, classroom interactions among learners and myself, in my role as their teacher, became central to interpreting and understanding how teaching framed by the notion of folding back promotes learners' algebraic reasoning. I contend that through these interactions, characterised by learners' mathematical actions such as writing, reasoning, questioning, listening, and correcting one another, multiple perspectives are acknowledged and a socially constructed reality is formed.

3.2.3 Interpretivist methodological assumptions

Interpretivist methodological assumptions are often grounded in approaches that seek to gain an in-depth understanding of the phenomenon under investigation (Kivunja & Kuyini, 2017). Accordingly, in this autoethnographic study, I employed learning activities, video recordings, semi-structured interviews, and documentary evidence, as described later in Section 3.7, to develop a rich understanding of how folding back promotes Grade 10 learners' algebraic reasoning within their natural classroom setting. These data collection methods were considered fit for purpose, as described by Cohen et al. (2018), in addressing the research questions that guided the study.

3.3 RESEARCH APPROACH

Education researchers generally choose among three research approaches, namely qualitative, quantitative, or mixed-methods research (Cohen et al., 2018; Creswell & Creswell, 2018; Leavy, 2022). The choice of a research approach is influenced by the philosophical worldview adopted by the researcher. In other words, coherence is required between the research paradigm and the research approach. Accordingly, as this study is framed within the interpretivist paradigm, I adopted a qualitative research approach aimed at "exploring and understanding the meaning individuals or groups ascribe to a social or human problem" (Creswell, 2014, p. 32). In this regard, the emphasis is placed on developing an in-depth understanding of the folding back phenomenon from the participants' perspectives (Creswell & Creswell, 2018). As noted by Creswell and Poth (2016), qualitative research is well suited to exploring complex social phenomena and individual experiences, as it generates rich and detailed data that are often overlooked by quantitative methods. It is argued that quantitative approaches may struggle to capture lived experiences and behaviours with sufficient depth and accuracy (Singh et al., 2021; Tracy, 2024). Thus, interactions between the researcher-teacher and learners within their natural classroom setting were considered valuable in enabling learners to express their mathematical ideas and reasoning, thereby generating meaningful qualitative data.

Furthermore, qualitative research is grounded in the principle of studying phenomena within their natural settings, where participants experience the phenomenon under investigation (Creswell, 2014). This aligns with the study's research questions, which are centred on the researcher-teacher's enactment of folding back in promoting Grade 10 learners' algebraic reasoning within their mathematics classroom. In this regard, I concur with Mabotja (2017), who notes that "in its conceptualisation, the study was to take place as part of the day-to-day occurrences of a classroom" (p. 52). In addition, qualitative research typically positions the researcher as a key instrument who is directly involved in the design of the study, as well as in data collection and analysis (Cohen et al., 2018). This is particularly relevant to the present study, as I am directly involved in the daily activities of the Grade 10 mathematics classroom in my role as a teacher. This involvement is achieved through my position as an insider, which is consistent with the autoethnographic research design and is justified in the following section.

3.4 RESEARCH DESIGN

Cohen et al. (2018) define a research design as "a plan or strategy that is drawn up for organising the research and making it practicable, so that research questions can be answered based on evidence" (p. 173). The choice of a research design is therefore influenced by the research questions and the objectives or purpose of the study. Accordingly, Cohen et al. (2018) contend that fitness for purpose underpins the selection of a research design. Common qualitative research designs include historical studies, grounded theory, phenomenological studies, action research, case studies, ethnography, and autoethnography (Creswell & Poth, 2016; Tomaszewski et al., 2020). In this regard, autoethnography was identified as the most appropriate research design for this study. According to Adams et al. (2017), autoethnography "uses personal experience ('auto') to describe and interpret ('graphy') cultural texts, experiences, beliefs, and practices ('ethno')" (p. 1). The intention of this autoethnographic study was to gain insights into the role of folding back in promoting Grade 10 learners' algebraic reasoning through my personal experiences as a mathematics teacher (Tilley-Lubbs, 2016). Thus, this research design provided an opportunity to generate insight into personal narratives related to the promotion of algebraic reasoning through folding back within a problem-based learning environment, as learners reflected on their experiences. Autoethnography enables this by foregrounding the researcher's experiences with participants, allowing personal truths and experiences to be conveyed in ways that differ from traditional research methods (Tilley-Lubbs, 2016).

Cohen et al. (2018) note that autoethnography is often written in the first person and may include emotional language, in contrast to conventional academic writing, which typically adopts a third-person, passive voice and a neutral, objective tone (p. 298). It is within this description that the justification for the frequent use of the first-person voice in this study is located. Accordingly, I narrate stories of my lived experiences of using folding back to frame the promotion of Grade 10 learners' algebraic reasoning within their mathematics classroom as a cultural context (Gorichanaz, 2021). It is important to acknowledge that this research design goes beyond storytelling, as it adheres to academic conventions, particularly through the provision of justifiable interpretations (Anderson, 2020). Thus, the data in autoethnography are not solely reliant on the researcher's perspective, but are strengthened through additional sources of evidence, such as field notes, reflective journaling, video recordings, interviews, and learners' written work, to substantiate interpretations (Le Roux, 2017).

Creswell and Poth (2016) indicate that researchers who employ autoethnography possess extensive knowledge of the context of the group they are writing about, having experienced it first-hand. As a result, I "can talk about these issues in ways different from others who have limited experiences with these topics" (Adams et al., 2017, p. 3). However, my position as an insider does not imply that I provide more truthful or accurate accounts than outsiders (Adams et al., 2017). Rather, it suggests that I am able to narrate these experiences in novel ways when compared to how others might do so (Adams et al., 2017). This positioning enabled a deeper engagement with the analysis of Grade 10 learners' algebraic reasoning through folding back. As the teacher in this context, I was well positioned to gain insight into learners' algebraic reasoning as they interacted with peers during the resolution of learning activities. My daily involvement in teaching provided insider knowledge of classroom dynamics, learner interactions, and the complexities involved in promoting algebraic reasoning, thereby offering a distinctive perspective.

The experiences of Grade 10 learners within a folding back environment are described in a manner intended to engage the reader in the narrator's subjective world. These narratives illustrate how learners reflected on their experiences and how such reflections influenced their algebraic reasoning skills, as viewed from their own perspectives rather than from an objective standpoint. Furthermore, the reflective nature of autoethnography offers considerable potential for the researcher to critically examine their role in enhancing learners' algebraic reasoning. Through reflection on personal values and practices, the researcher is able to identify areas for improvement and develop strategies to enhance teaching practice (Liu et al., 2021). This process aligns with the reflective nature of folding back, allowing the researcher to interrogate and refine teaching practices in order to better support learners in the Grade 10 classroom. Nonetheless, it is important to acknowledge that autoethnography has inherent limitations, which are discussed in the confirmability section of this study. These limitations are recognised as part of ensuring the overall quality and rigour of the research.

3.5 RESEARCH SITE

This autoethnographic study was conducted at a high school located in the vicinity of a mining town in Thabazimbi, Limpopo Province. The school has a population of more than 400 learners who speak Setswana and approximately 150 learners who speak isiXhosa. Learners are drawn from nearby townships as well as from the immediate surrounding area. The school offers subjects in the mathematics, science, and technical streams.

The study was conducted in a Grade 10 mathematics classroom comprising 22 learners. It is important to note that English is the learners' First Additional Language. Consequently, learners were permitted to communicate with one another using their home languages, a practice that is also reflected in the vignettes presented in Chapter 4. This approach was adopted to ensure that language did not act as a barrier to learners' interaction and participation in the study.

3.6 SAMPLING PROCEDURE

Sharma (2017) describes sampling as a procedure used by researchers to select a subset from a predetermined population, which then serves as the primary data source. The choice of sampling strategy is influenced by the research design adopted for a study (Creswell & Poth, 2016). In this autoethnographic study, I employed heterogeneous purposive sampling, also referred to as judgement sampling (Etikan et al., 2016). This sampling approach aligns with the exploratory nature of autoethnography, which seeks an in-depth understanding of participants' experiences (Prabandari et al., 2024).

Accordingly, the study initially involved all 22 Grade 10 learners who were exposed to exploratory teaching framed by the notion of folding back during the fieldwork phase. The inclusion of the entire class was intended to minimise potential selection bias, as highlighted by Cohen et al. (2018). However, for reporting purposes, a purposive sample of only 12 learners was selected as data sources for the learning activities. The selection of these learners was based on the nature of their interactions with one another during the problem-solving process with different learners selected depending on how clearly their interactions demonstrated folding back. These interactions involved questioning one another, justifying their reasoning, responding to teacher intervention, or revisiting earlier layers of understanding to solve a mathematical task at hand.

From this group of 12 learners, four were further selected to participate in semi-structured interviews as they were able to provide insightful and diverse reflections of their folding back learning experiences. This sample size enabled the researcher to ask follow-up questions, gain a deeper understanding of participants' responses, and explore emerging themes in depth, which is essential for achieving data saturation in qualitative research (Ahmed, 2025).

3.7 DATA COLLECTION

The following data collection methods were adopted to provide an autoethnographic account of promoting Grade 10 learners' algebraic reasoning through teaching framed by the notion of folding back.

3.7.1 Learning activities

The study emphasises the importance of engaging learners in activities that involve algebraic reasoning and other forms of mathematical reasoning. Given the objective of the study, it is essential to recognise that learners' verbal expressions are as significant as their written work when evaluating their use of mathematical reasoning in mathematical concepts (Chuene et al., 2023; Gill, 2020). The use of learning activities as a data collection method in this study aligns with Pirie's (1996) assertion that "folding back can only be observed by close attention to the activities and talk of students as they work at a mathematical problem" (p. 4). Accordingly, learners were assigned learning activities to complete as part of the data collection process during teaching episodes. The researcher-teacher designed learning activities focusing on financial mathematics, drawing on the CAPS document, the Mind Action Series textbook, and the 2025 Grade 10 annual teaching plan (Annexure I).

3.7.2 Video recordings

The mathematical interactions between learners and the teacher-researcher during the completion of learning activities were captured through video recordings, in line with the integration of digital and social media tools in ethnographic studies (Coombes & Jones, 2020). On a daily basis, the researcher recorded separate videos of small groups of learners engaged in problem-solving activities that involved algebraic reasoning. These video recordings enabled the teacher-researcher to capture learners' meaningful and insightful interactions. The video recordings of learners working through the financial mathematics tasks were repeatedly viewed in order to identify episodes in which learners exhibited evidence of folding back (questioning, justifying their thinking or responding to the teacher's intervention).

In this autoethnographic study, the use of video recordings as a data collection tool is consistent with Pirie's (1996) assertion that "to elicit moments of folding back, I needed to have a continuous, detailed record of the working of specific students, together with a record of any teacher interventions that impinged on their working" (p. 5). In addition, the teacher-researcher encouraged learners to maintain video diaries in which they reflected on their overall learning experiences. The learning environment fostered an atmosphere in which learners could freely express themselves and engage with the teacher-researcher, thereby providing rich data that supported reflection and the adaptation of their thinking.

3.7.3 Semi-structured interviews

When conducting a study, teacher-researchers often employ interviews to gain a comprehensive understanding of a phenomenon (Husband, 2020). In this study, it was important to examine how learners fold back and how this process influences the development of their algebraic reasoning skills. Accordingly, semi-structured interviews (Annexure H) were used to gather in-depth individual insights into learners' challenges and achievements when engaging in algebraic reasoning activities within a problem-based learning environment characterised by folding back. As indicated earlier, four Grade 10 learners were invited to participate in the semi-structured interviews. Each learner took part in a face-to-face interview lasting approximately 35 minutes. The interview recordings were transcribed verbatim so as to capture the learners' response in full. In addition, the interviews were audio recorded, in line with the recommendation by Rutakumwa et al. (2019) that audio recordings enhance the validity and trustworthiness of interview data compared to interviews conducted without such technology. The interview questions and prompts were open-ended and designed to elicit rich data relevant to the research questions, while also allowing participants to guide particular aspects of the interviews (Merriam & Tisdell, 2016).

3.7.4 Document analysis

According to Flick (2018) and Merriam and Tisdell (2016), document analysis in qualitative research refers to the examination of a range of text-based materials. As this autoethnographic study sought to generate detailed descriptions of learners' written work, document analysis was an essential data collection method. In this study, documentation of learners' responses to the learning activities was analysed as part of the data set. Photographs of learners' written work are included to support the narrative analysis, as presented in Chapter 4.

3.8 DATA ANALYSIS

This study employed Polkinghorne's (1995) narrative analysis to explore how learners' algebraic reasoning is enhanced through folding back. According to Polkinghorne (1995), narrative analysis involves using collected data to construct a compelling story. However, it is important to note that these compelling "stories were not descriptions of these conversations but interpretations and extrapolations of latent meanings so that the conversations could be coherently understood", as emphasised by Mabotja et al. (2018, p. 3). The interpretations used to produce these narratives were informed by the theoretical framework of folding back, as articulated by Martin (2008).

In analysing the video data, I firstly watched the recordings multiple times in searching for instances in which the video recordings consisted of information-rich interactions drawn from learners' algebraic reasoning activities in which folding back was observed. I therefore made sense of the data through repeatedly viewing the recordings. This selection was purposeful rather than random and was guided by three key features of folding back, as highlighted within the theoretical framework. These included the source of intervention, where attention was paid to instances of learners' folding back; the forms of folding back, which considered whether learners engaged at an inner layer using existing understanding, expanded at an inner layer, diverged from the topic and worked at that level, or experienced a discontinuity; and the outcomes of folding back, which assessed whether the process resulted in effective or ineffective learning. Thereafter, I produced verbatim transcripts of the videotapes as written text was of paramount importance in this study. The transcription of videotapes to text provided even a broader opportunity to repeatedly read through the interactions and make sense of the data. In addition, interview data and documentation of learners' written work were used to triangulate and strengthen the narrative analysis.

Consistent with Polkinghorne's (1995) narrative analysis, I repeatedly read and interpreted each interview transcript in relation to the theoretical framework of folding back and the research questions. Similarly, learners' written work that exhibited rich evidence of folding back during the financial mathematics learning activities was selected and repeatedly read for sense making. In making sense of the written data, attention was given to written responses that reflected peer interaction, teacher intervention, and shift in learners' understanding as these were narratively analysed. As such, to strengthen the trustworthiness of the narrative analysis, the interpretation of learner's written work was compared with the semi-structured interview transcripts and video recordings.

3.9 QUALITY CRITERIA/ TRUSTWORTHINESS

The following measures were adopted to ensure the quality criteria of this study:

3.9.1 Credibility

Credibility is defined as "the degree to which the findings accurately reflect the reality that the participants experienced" (Ahmed, 2024, p. 1). Several measures were undertaken to ensure the credibility of this autoethnographic study. First, prolonged engagement was achieved through four weeks of exploratory teaching, which enabled the development of trust and rapport with the Grade 10 learners over time, as recommended by Ahmed (2024) and Lincoln and Guba (1985). This was further strengthened by my role as their mathematics teacher, which afforded me a nuanced understanding of learners' experiences and behaviours over an extended period. Second, vignettes in the form of extracts of interactions and dialogues, verbatim scripts of written tasks, and direct quotations from Grade 10 learners were included to support the analysis and interpretation of the data. In this regard, multiple data sources were used to triangulate the findings, thereby enhancing the study's credibility (Creswell & Creswell, 2018).

3.9.2 Dependability

According to Singh et al. (2021), dependability refers to the consistency with which the data collected and the findings derived from that data can be traced. In this autoethnographic study, dependability was enhanced through the provision of detailed methodological documentation by thoroughly recording each step of the research process, as evidenced throughout this chapter. Such detailed documentation is regarded as an essential measure for achieving dependability (Ahmed, 2024, p. 2). In addition, an audit trail was maintained, which entailed the systematic documentation of research decisions, data collection procedures, data analysis processes, and interpretive decisions. This approach was adopted to ensure the dependability of the research findings, as recommended by Ahmed (2024) and Cohen et al. (2018).

3.9.3 Confirmability

Amankwaa (2016) defines confirmability as the researcher's ability to demonstrate that the data genuinely represent participants' viewpoints and are not unduly influenced by the researcher's biases. I acknowledge that autoethnographic studies have been critiqued for potentially reflecting a singular perspective, given that the researcher occupies a central role in the research process (Borges, 2022; Liu et al., 2021; Vasconcelos, 2022). Although I occupied a central position as an insider, I sought to narrate the Grade 10 learners' lived realities of learning through folding back, in line with the recommendations of Adams et al. (2017). Accordingly, I reflected on my own experiences alongside those of the Grade 10 learners in order to construct a meaningful narrative of our shared experiences within a folding back classroom. Throughout this reflection, I was mindful that my interpretation of the data collected in this folding back classroom could be influenced by my experiences and beliefs as a mathematics teacher valuing a learner centered leaning environment.

Confirmability was further strengthened through peer debriefing, in which my supervisor, as an expert, provided constructive feedback that assisted in validating interpretations and minimising personal bias (Ahmed, 2024). In addition, participants were afforded the opportunity to review the representations of their experiences to ensure that their viewpoints were accurately captured (Ahmed, 2024; Cohen et al., 2018). This process of member checking contributed to reducing researcher bias and ensured that the data were analysed in a manner that reflected participants' realities rather than the researcher's perspectives, as advocated by Adams et al. (2017). Moreover, the triangulation of data sources provided a broader perspective and reduced reliance on a single narrative, thereby enhancing the confirmability of the study (Fyffe et al., 2019).

3.9.4 Transferability

Transferability refers to the extent to which the findings of a study can be applied to other groups or contexts (Ahmed, 2024; Hadi & José, 2016). As indicated earlier, this study does not seek to generalise its findings. Consequently, transferability is not pursued in the conventional sense, as generalisation is not the intention of autoethnographic research. Borges (2022) further cautions that expectations of replication and generalisation in autoethnography are misplaced, as the findings emerge from the researcher's personal experiences and interpretations, which inherently limit their applicability to broader populations (Pratt, 2020). Nevertheless, to support contextual understanding, I provided rich and detailed descriptions of the research setting, the Grade 10 learner participants, and the methods employed. This enables readers "to evaluate the similarities between their context and the study, allowing them to judge the relevance and applicability of the findings to their own settings or situations" (Ahmed, 2024, p. 2).

3.10 ETHICAL CONSIDERATIONS

The following ethics measures guided the study:

3.10.1 Ethics application and permission

Prior to conducting the study, an ethics application was submitted to the UNISA CEDU Ethics Committee and approval was granted (Annexure A). In addition, permission was sought and obtained from the Limpopo Province Department of Education (Annexure B) and from the school where the research was conducted (Annexure C and Annexure D). As the Grade 10 learners who participated in the study were classified as minors, informed consent was also obtained from their parents or legal guardians (Annexure E).

3.10.2 Confidentiality

Pseudonyms were used to protect the confidentiality of both the research site and the participants (Edwards, 2020). Furthermore, to minimise the risk of raw data leakage and to ensure participants' safety during the video-recording process, the researcher obscured the faces of all 12 Grade 10 learners involved, as well as any details that could potentially identify them (Er et al., 2022). Participants were also assured that all information collected during the study would be treated with the utmost confidentiality and would not be shared with anyone outside the research.

3.10.3 Informed Consent

Before participating in the study, informed consent forms were issued to participants (Annexure E), outlining the purpose of the study and the procedures for data collection. According to Nusbaum (2017), informed consent ensures that individuals are fully aware of their rights and voluntarily choose to participate, with the assurance that those rights will be protected. In this study, informed consent forms were provided to both the Grade 10 learner participants and their parents or legal guardians for review and signature. It is important to note that participation in this study was entirely voluntary, as emphasised by Nusbaum (2017). Accordingly, the 12 sampled Grade 10 learners were free to withdraw from the study at any stage and for any reason, and the researcher undertook to respect and honour such requests.

3.11 CHAPTER SUMMARY

In this chapter, I presented the rationale for adopting a qualitative research paradigm and provided justification for selecting autoethnography as the research design. The selection of participants for the study was also carefully addressed. In addition, the chapter discussed issues related to data collection, including the methods of data collection, data analysis, ethical considerations, and quality criteria. The rigour of the study within an autoethnographic methodology, including trustworthiness in qualitative research, was examined to ensure adherence to quality criteria. The next chapter focuses on the analysis of the data that were collected.

CHAPTER 4 RESULTS AND DISCUSSION

4.1 INTRODUCTION

The previous chapter outlined the autoethnographic methodological approach employed in this study. In this chapter, I present the narrative analysis of how folding back promotes Grade 10 learners' algebraic reasoning. As previously indicated, this autoethnographic study focuses on learners' application of algebraic reasoning skills within the CAPS Grade 10 financial mathematics content area. The selection of financial mathematics is not necessarily due to learners' difficulties with this topic; rather, it offers a rich learning environment for the application of algebraic reasoning. First, an overview of the fieldwork conducted in the study is provided. Second, each phase of the fieldwork and its related classroom episodes are presented. In this regard, classroom episodes comprise learning activities, vignettes, and the narrative analysis of those vignettes. The narrative analysis is guided by the folding back framework (Martin, 2008) and is based on vignettes, that is, extracts of interactions among learners and the researcher-teacher. In line with Pirie and Martin's (2000) assertion, this study recognises that brief extracts of interaction are sufficient to highlight shifts in learners' thinking during mathematical learning processes. Furthermore, relevant literature is integrated into the narrative analysis to support or challenge the interpretations made. The chapter concludes with a summary.

4.2 DATA GENERATION PROCESS: OVERVIEW OF THE FIELDWORK

In this study, the teaching and learning episodes are referred to as fieldwork. Accordingly, four fieldwork phases were conducted. Fieldwork 1 focused on the introduction of the simple and compound interest formulae, which served as prior knowledge, as learners had previously been exposed to these concepts in Grade 9. Although the Grade 9 curriculum does not explore financial mathematics in depth, it establishes a foundation by explaining what each variable in the formula represents for substitution purposes. At this level, learners mainly focus on using the formulae to determine the value of A, the final amount of an investment over a specified period. In Grade 10, however, the curriculum allows for algebraic manipulation of the formulae, requiring learners to solve for P (principal amount), i (interest rate), and n (investment period). Fieldwork 1 comprised two classroom episodes. Classroom Episode 1 focused on manipulating the simple and compound interest formulae to determine missing values, particularly n and P. Classroom Episode 2 centred on determining the value of i in both formulae.

Fieldwork 2 also comprised two classroom episodes. The first episode focused on solving problems related to population growth, while the second addressed hire purchase agreements. Fieldwork 3 consisted of a single classroom episode that primarily focused on determining monthly repayments for loans involving either simple or compound interest. Fieldwork 4 comprised two classroom episodes, with one focusing on understanding the implications of exchange rates and the other on solving problems using timelines. Learners' responses to the learning activities are presented in the form of vignettes, capturing their communication and interactions, as reported in this study.

4.3 FIELDWORK 1: APPLYING ALGEBRAIC REASONING TO SOLVE SIMPLE AND COMPOUND INTERESTS

4.3.1 Classroom Episode 1: Application of algebraic reasoning to determine the time period (n) and the value of principal amount (p).

In this classroom episode, learners were given a financial mathematics task that required them to analyse a given statement, select an appropriate formula, and solve for the missing value. The primary focus was on determining the value of n and, in some cases, P. The learning activities used in this study were drawn from the *Mind Action Series* textbook (2021), as well as additional supplementary sources. A total of two learning activities (Activities 1 and 2) were purposefully selected to provide a comprehensive account of their effectiveness and outcomes.

The learning activity comprised two main components. The first component included subparts (a–d), while the second focused on learners' verbal and written expressions as they worked through the tasks. This latter component was central to the data collection process. Accordingly, the sampled vignettes capture key moments of learner interaction during the classroom episodes, illustrating the development of their algebraic reasoning.

The first activity presented to learners was phrased as indicated in Learning Activity 1.

1.2. How long would it take an investment of R500 to increase by R350 if the interest rate is 4,3% p.a simple interest?

Figure 4.1: Learning Activity 1

An expected response from the learners is presented below:

$$A = P(1 + in)$$
$$850 = 500(1 + 0{,}043n)$$
$$\frac{850}{500} = 1 + 0{,}043n$$
$$\frac{17}{10} - 1 = 0{,}043n$$
$$\frac{0{,}7}{0{,}043} = \frac{0{,}043n}{0{,}043}$$
$$16{,}28\ years = n$$

The following vignette presents a sampled conversation from Group A, involving Jane, Tebello, and Rudo, as they worked on Question 1.2 of Learning Activity 1. This is followed by the narrative analysis.

Vignette 1

1.1.	Jane:	*(looks at Tebello's attempt and screams)* Hai! Tebello, why are you transposing 500?
1.2.	Tebello:	The question asks us to solve for *n*, so we must remove all the terms that are not *n* from the substitution we have made.
1.3.	Rudo:	Okay, let me ask you this. You remember when we solve for x, say we have $4x = 3$, how would you solve for x?
1.4.	Tebello:	*(stares with confusion)*
1.5.	Jane:	Would you say $x = 3 - 4$?
1.6.	Tebello:	No, you must divide both sides with 4 so I can remain with x.
1.7.	Rudo:	Why must we divide?
1.8.	Tebello:	Because we want to solve for x and $4x$ is like 4 multiply by x. we cannot transpose to remove 4 from x.
1.9.	Rudo:	Okay so let's go back to isolating terms from "n" here (points at Tebello's working). Wena you transposed 500 when the bracket implies multiplication. Meaning you have 500 multiply by $(1 + 0{,}043n)$.
1.10.	Tebello:	Oh, I was supposed to divide. Eish, so this thing is like solving for x?
1.11.	Jane and Rudo:	Yes!
1.12.	Tebello:	(*laughs*) Oh, I see, so meaning here (points at $2{,}25n$) I was not supposed to add the two because they are not like terms.
1.13.	Jane:	Yes.
1.14.	Tebello:	Oh, I see now, eish meaning this whole thing is incorrect.

Vignette 1 is influenced by Tebello's attempt to solve the given activity and reflects his efforts to address the task. In his attempt, as illustrated in Figure 2 below, he uses the appropriate formula, accurately substitutes the variables, and proceeds to solve for the required variable. Although his solution is incorrect, it indicates that he was operating at a formalising layer. In this regard, Tebello seemed to have an idea of what was required; he appeared to lack the skill of correctly isolating variables using inverse operations within the context of financial mathematics. Thus, albeit incorrect, his mathematical actions suggest that he recognised the knowledge he had developed and was forming mathematical definitions based on observed properties and generalisations of concepts, which, according to Yao and Manouchehri (2020b), is evidence of a formalising layer of understanding.

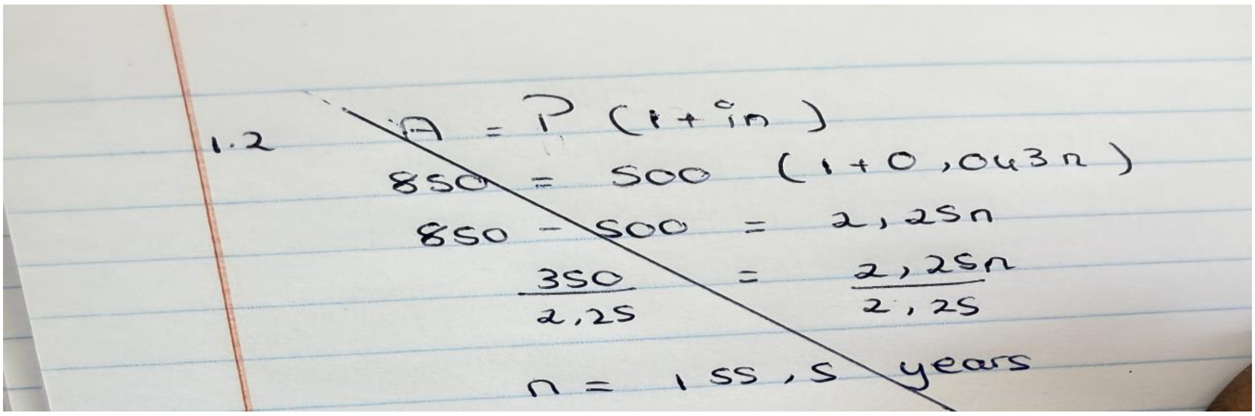

Figure 4.2: Tebello's initial attempt

Consequently, Jane questioned why he had transposed R500 (line 1.2), which indicated her strong disagreement with his approach. In other words, Jane understood that the amount of R500 should not have been transposed when solving for *n*. In response to Jane's question, Tebello's utterance, *"so we must remove all the terms that are not 'n' from the substitution we have made"*, further confirms that he was reasoning at a formalising layer, as previously indicated. Upon realising that Tebello did not grasp the essence of Jane's question, Rudo's illustration using algebraic equations was intended to serve as a source of explicit and intentional intervention. The aim was for Tebello to reflect on concepts he might revisit or reconsider from his prior knowledge (Martin, 2008; Yao and Manouchehri, 2022a), which would prompt him to fold back to the property noticing layer. In this instance, Rudo's utterance, *"you remember when we solve for x"*, was intended to help Tebello access the concept used when simplifying algebraic equations. Her follow-up question, *"…how would you solve for x?"*, sought to address Jane's concern regarding when and why transposition is applied.

However, Rudo's initial attempt did not yield the desired outcome, as Tebello appeared confused in line 1.4. This suggests that he struggled to recognise the connection between algebraic concepts and financial mathematics. Importantly, this confusion did not imply that he lacked the skill of isolating variables in algebraic contexts. This is evident in his response to Jane's question, *"would you say $x = 3 - 4$?"* (line 1.5), to which he replied, *"no, we must divide both sides by 4, so I can remain with x"*, as reflected in line 1.6. Furthermore, he was able to justify why this was mathematically correct by stating that *"we cannot transpose to remove 4 from x"* (line 1.8). This type of utterance further demonstrates exploratory talk, which is an important enabler of folding back, as reported by Chuene et al. 2023.

Despite demonstrating proficiency in algebra, Tebello initially failed to relate this understanding to his act of transposing R500 until Rudo prompted him to revisit the earlier step (line 1.9). Rudo then used the same ideas Tebello had articulated in lines 1.6 and 1.8 to guide him to fold back to the formalising layer. By linking the statement *"…because we want to solve for x and 4x is like 4 multiplied by x, we cannot transpose to remove x"* to her remark, *"…wena you transposed 500 when the bracket implies multiplication"* (line 1.9), Tebello reached a key moment of realisation. His utterance, *"Oh…"* (line 1.10), signifies this realisation, as he began to understand why his peers prompted him to revisit the algebraic concept of simplification at the property noticing layer, leading him to recognise that *"…I was supposed to divide"*. This moment illustrates his engagement with mathematical met-befores, as described by Martin and Towers (2016).

According to Martin (2008), this moment constitutes effective folding back. In this instance, Tebello's remarks, *"…Eish, so this thing is like solving for x?"* (line 1.10) and *"…so meaning here… I was not supposed to add the two because they are not like terms"* (line 1.12), indicate that he temporarily moved away from the immediate topic to make sense of the underlying mathematical concept. The *"Oh"* moment appears to have strengthened his algebraic reasoning, as evidenced by his ability to self-correct in line 1.12 and justify his reasoning appropriately. The form of folding back observed here can be classified as collecting at an inner layer, where the learner retrieved prior algebraic knowledge to address the demands of the current financial mathematics problem.

Similarly, interview data revealed that learners rely heavily on algebraic reasoning when solving financial mathematics problems. For instance, one learner reflected that *“algebra is in almost every maths topic because as soon as I was stuck, what I could think about was algebra simplify like in solve for x. And algebraic the… formulas… like solving for x, we are only solving for the unknowns in financial maths”* (Phuti). In the same vein, another learner noted that the learning activity *“requires a lot of algebra reasoning because without it you cannot solve”* (Teko, semi-structured interview). Likewise, another learner stated that *“financial mathematics for me just became easy because of the strong foundation I already carry with me when it comes to solving algebraic equations. From the onset, I just saw algebra throughout”* (Rudo, semi-structured interview).

In this case, the thickening effect of folding back, as described by Pirie and Kieren, is evident, as Tebello acknowledged the incorrectness of his initial attempt, reconstructed his solution, and successfully applied algebraic skills to solve the financial mathematics task, as shown in Figure 4.3. Learners’ mathematical actions in this episode align with the findings of Al-Mutairi and Marzouq (2025), who reported that learners use abstract symbols to define variables and apply algebraic properties to arrive at correct financial mathematics solutions.

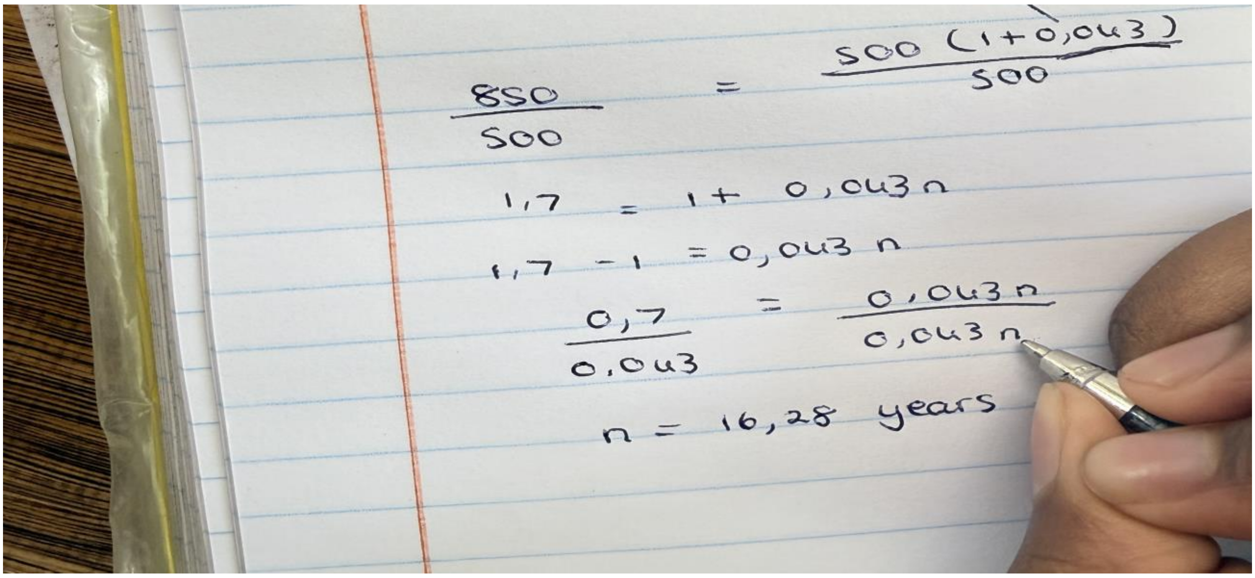

Figure 4.3: Tebello’s second attempt

Following the above discussion (Figure 4.3), Tebello’s second attempt was significantly influenced by his peers’ questions (lines 1.1, 1.3, 1.5, and 1.7). It is evident that, in reattempting the question, he applied appropriate mathematical manipulation skills. Initially, Tebello appeared to struggle with correctly isolating variables using inverse operations in the context of financial mathematics. However, his subsequent response (Figure 4.3), after engaging with his peers, indicates that he required only a prompt to activate and effectively apply the relevant algebraic skills. This action illustrates one of the benefits of a folding back classroom, as noted by Patmaniar et al. (2021), where learners revisit layers of understanding in order to address knowledge gaps necessary for success in mathematical tasks.

Furthermore, meaningful peer collaboration contributed to this pivotal moment, as Tebello strengthened his algebraic knowledge. This aligns with the principles of a problem-based learning environment, where collaboration plays a central role, as highlighted by Segerby and Chronaki (2018). The episode also demonstrates an enhancement in Tebello's algebraic reasoning, as he was able to reflect on his initial attempt and apply appropriate algebraic reasoning to reach the correct solution. The time spent in collaborative discussions allowed learners to take intellectual risks and make mistakes, thereby enriching their reasoning skills through experience. The peer-driven intervention enabled Tebello to recognise the shortcomings in his initial approach, leading to a successful second attempt.

Pirie and Kieren's (1994) framework, together with problem-based learning principles, positions learners at the centre of the learning process, with the teacher acting as a facilitator. Following this classroom episode, the teaching and learning environment appeared effective, as Tebello was able to reflect on and correct his errors.

The second activity of the classroom episode, undertaken by Group B, focused on analysing a given statement in order to solve for a missing variable. This learning activity is presented as Learning Activity 2 and was structured as follows:

1.6 A store is running a contracted deal for a mobile phone as depicted below:

Determine:

a) the total amount a customer would pay at the end of the 36 months period.

b) the cash price of the cellphone if the monthly instalments quoted on the advertisement are on simple interest agreement which offer an interest rate of 8,2% on the cash price of the cellphone.

Figure 4.4: Learning activity 2

An expected response from the learners is presented below.

a) $R1029 \times 36 = R37\ 044$

b)

$$A = P(1 + in)$$

$$37\ 044 = P\big(1 + 0{,}082(3)\big)$$

$$\frac{37\ 044}{\big(1 + 0{,}082(3)\big)} = P$$

$$R29\ 730{,}34 = P$$

The following section presents an analysis of Learning Activity 2 responses by Group B, which comprised Bonolo, Phuti, and Yolanda. In their attempts to solve the mathematical task, the learners did not appear to experience difficulty in selecting the appropriate formula. However, they first needed to analyse the question to identify the given values and the variable to be determined. Once the given information and the unknowns were identified, they proceeded to implement the solution. Notably, Yolanda observed that *"these amounts do not make sense"* and invited her peers to revisit their solution. This observation demonstrates the learners' ability to evaluate and verify their solutions, as suggested by Al-Mutairi and Marzouq (2025). Yolanda's concern triggered further peer interaction, which unfolded as follows.

Vignette 2

2.1. Yolanda: Hey, guys these amounts do not make sense. How do we know that P is R1029? Isn't the amount given the money one is required per month?
2.2. Phuthi: Yes, it is our starting amount. Here they said R1029 per month and here is P. So I just took the simple route and just substituted P.
2.3. Yolanda: But per month does not mean P.
2.4. Phuti: Eish, yes but I now realise that this might be incorrect.
2.5. Yolanda: What makes you say that?
2.6. Phuthi: Because when I am trying to explain to you now, I can realise that this amount of R1282,134 cannot be the cash price of the cellphone when the 36 monthly instalments will total to R37044.
2.7. Yolanda: Exactly!
2.8. Phuthi: Also, here (pointing at R37044) the amount is really high and here (pointing R1282,134) the amount is very low. The two just do not correspond.
2.9. Yolanda: And if this amount is the total price that you will pay after 36 months then it makes this money the value of A.
2.10. Phuthi: Oh, meaning we do not have P which is the original cash price of the cellphone.

Following the conversation initiated by Yolanda, she appeared to possess the procedural algebraic skills required to simplify the expression and solve for the missing variable, which she correctly identified as A. However, her algebraic reasoning seemed to be underdeveloped, as she initially treated an amount designated as a monthly payment over 36 months as the principal amount. She then immediately questioned this approach herself, which indicates that she was operating at an image making layer. Her attempt to answer the question is presented below.

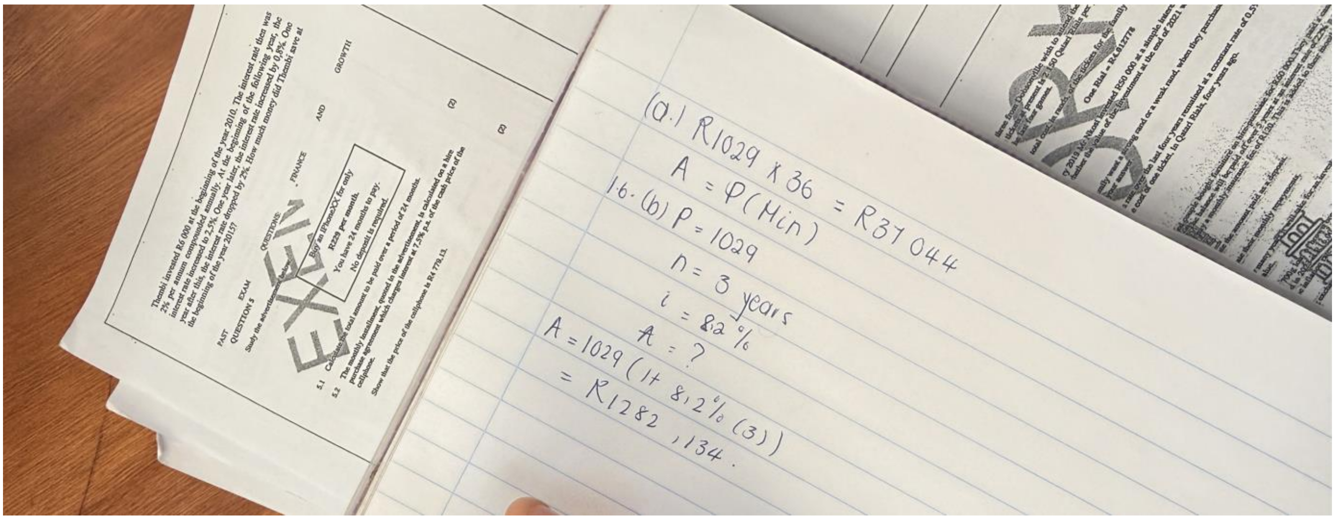

Figure 4.5: Yolanda's attempt

Yolanda's question, *"…isn't the amount given the money one is required to pay per month?"*, represents an attempt to make sense of the value she obtained as the cash price of the cellphone. This question further suggests that she was not fully confident in how she had conceptualised the problem. In contrast, Phuti appeared confident that R1

029 represented the principal amount, as reflected in his response, *“Yes, it is our starting amount”*, to Yolanda’s expressed doubts. His insistence on substituting *P* as R1 029 indicates the presence of symbolic reasoning, although this reasoning had not yet been validated against the underlying conceptual relationships, as evidenced by his statement, *“Here they said R1 029 per month”*. The learners’ mathematical actions illustrate their attempts to apply algebraic thinking to interpret the financial mathematics problem through repeated reading, as described by Al-Mutairi and Marzouq (2025). Phuti’s response further reveals that he was drawing on knowledge at the image having layer and relying primarily on procedural algebraic skills rather than conceptual understanding.

The interaction between Yolanda and Phuti suggests that they were operating at different layers of mathematical understanding. It is therefore reasonable to argue that they were not necessarily disagreeing, but rather approaching the task from different cognitive positions. Phuti appeared confident in his approach, as indicated by his remark, *“So, I just took the simple route and substituted P”*, which aligns with the image having layer. In contrast, Yolanda’s question, *“How do we know that P is R1 029?”*, reflects engagement at the image making layer, characterised by uncertainty and sense-making.

Yolanda’s statement in line 1.5 served as a catalyst for folding back, as she recognised that *“per month”* does not equate to the principal amount. Her utterance, *“But per month does not mean P”*, signals an emerging awareness of a conceptual inconsistency and indicates a fold back to the property noticing layer, where she distinguished between monthly instalments and the principal amount. This action aligns with Al-Mutairi and Marzouq’s (2025) description of algebraic thinking through repeated reading and extraction of known and unknown information. Moreover, Yolanda’s questioning functioned as an unintentional peer intervention that prompted Phuti to fold back from the image having layer to the image making layer. Although Yolanda did not explicitly direct Phuti towards a specific mathematical concept, her questioning interrupted his initial line of reasoning and prompted reflection. This is evident in Phuti’s utterance, *“Eish, yes, but I now realise that this might be incorrect”*, which suggests that he began reconstructing his understanding through reasoning at the image making layer, where he engaged in activities aimed at helping her to develop particular representations of the mathematical concept to get an idea (Martin & Towers, 2016).

As Phuti reflected further, he identified an inconsistency in the values used for substitution, which led to additional reconstruction of his mental image. His remark, *"I can realise that this amount of R1 282.134 cannot be the cash price of the cellphone when the 36 monthly instalments will total to R37 044"*, indicates rational reasoning and a transition into a deeper layer of understanding. At this point, it can be argued that Phuti folded back to the property noticing layer as a result of refined reasoning.

Although Yolanda initially expressed doubt regarding her solution, her response in line 1.9 reflects growing confidence as she affirmed Phuti's observation in line 1.8. This increased confidence appears to stem from her sustained engagement with Phuti's reasoning, which allowed her to internally refine her understanding. Her confident articulation then prompted Phuti to compare the magnitudes involved, noting that *"…the amount is really high and here… the amount is very low. The two just do not correspond"*. Through this comparison, Phuti began to demonstrate conceptual understanding that had not been evident earlier (line 1.4).

At the same time, Yolanda's refined reasoning enabled her to accurately link the total of the monthly payments to the final amount payable for the cellphone. Her statement, *"…and if this amount is the total price that you will pay after 36 months, then this money is the value of A"*, resolved her initial confusion. This utterance demonstrates her ability to draw a valid conclusion using algebraic reasoning about the quantities involved and to generalise the role of *A* within the financial mathematics formula. Consequently, her reasoning indicates that she had progressed to a formalising layer, where she recognised the knowledge she had developed based on observed properties and generalisations of concepts (Amin & Sulaiman, 2021; Chuene et al., 2023).

Ultimately, Phuti recognised the symbolic implications of Yolanda's reasoning, which prompted him to fold back into the formalising layer. His utterance, *"Oh, meaning we do not have P, which is the original cash price of the cellphone"*, reflects a refined understanding of the role of *P* as the principal amount. This progression demonstrates growth in his algebraic reasoning, as he moved beyond a purely procedural orientation towards a more structured and conceptually grounded understanding of the relationship between the cash price, the total amount paid, and the algebraic structure of the financial mathematics formula.

Meanwhile, Bonolo appeared to have listened quietly to the interaction between Phuti and Yolanda until a point at which she recognised that they were now operating at a similar layer of understanding, as suggested by their utterances in lines 2.11 and 2.12, while she was not yet aligned with them. Vignette 3 below captures the subsequent interaction between the teacher and Bonolo, which followed up on her concerns.

Vignette 3

3.1.	Bonolo:	Nna I hear you guys talking but I am lost.
3.2.	Teacher:	If you were buying something and it is R50 but you do not have the money to buy it. You buy it on credit and at the end you find out that the money you paid in total is R70. Between the two amounts, which one is the initial price(original) of the item you bought?
3.3.	Bonolo:	R50.
3.4.	Teacher:	So, P would be R50?
3.5.	Bonolo:	Yes, and A would be R70, the final amount paid for the item bought.
3.6.	Teacher:	Good, how much money would you pay for the cellphone after 36 months?
3.7.	Bonolo:	R37044
3.8.	Teacher:	Do you know the initial cost of the cellphone? How much is the cellphone?
3.9.	Bonolo:	No, that is what we are looking for. *(pause)* Ohooo, R37044 is A and we do not have P.
3.10.	Teacher:	Precisely.

Bonolo's utterance, *"Nna I hear you guys talking, but I am lost"*, emerged from the interaction between Phuti and Yolanda. Although she appeared to be listening attentively, she struggled to make sense of their discussion and lacked a structured understanding of the mathematical ideas being exchanged, as reflected in her responses in lines 3.3, 3.5, and 3.7. This lack of structure was not necessarily due to limited algebraic reasoning, but rather to difficulty in connecting the abstract mathematical symbols used by her peers to the underlying mathematical ideas. As a result, she was unable to meaningfully engage with the concepts under discussion.

Recognising this, the teacher intervened by introducing an example drawn from an everyday financial context, with the intention of helping Bonolo follow her peers' reasoning about principal and accumulated amounts. This intervention aligns with Chuene et al.'s (2023) assertion that when teachers attend closely to learners' mathematical actions, they are better positioned to intervene appropriately. In this instance, the teacher used a concrete context that Bonolo could easily relate to. The teacher's question, *"…between the two amounts, which one is the initial price of the item you bought?"*, served as an explicit and intentional intervention aimed at refining Bonolo's mathematical met-befores. Bonolo's response in line 3.3 appeared intuitive, which prompted the teacher's follow-up question, *"So, P would be R50?"* Through this exchange, the teacher guided Bonolo towards recognising that mathematical symbols in financial mathematics represent specific quantities rather than isolated values.

This intervention proved effective, as evidenced in line 3.5, where Bonolo was able to associate R70 with the value of *A*. Her utterance, *"…A would be R70, the final amount paid for the item bought"*, demonstrates her emerging ability to connect everyday financial examples to the symbols *A* and *P*, thereby forming a mental image of their relationship. At this point, Bonolo began to internalise the concept and notice the properties of the quantities involved. Consequently, she can be regarded as operating at the property noticing layer, despite initially not being situated within any layer due to her expressed sense of being lost (line 3.1), which reflected cognitive disengagement rather than misunderstanding.

Upon recognising Bonolo's developing understanding at the property noticing layer, the teacher acknowledged her effort through praise, as reflected in the utterance *"Good"*. This affirmation served to reassure Bonolo that she was on the right track and helped reinforce her emerging confidence. The teacher's use of praise also played an important motivational role, encouraging Bonolo to adopt and apply the same line of reasoning within the context of the learning activity. This highlights the importance of positive reinforcement in supporting the development of algebraic reasoning skills and sustaining learner engagement in challenging tasks. The subsequent questions, *"How much money would you pay for the cellphone after 36 months?"* and *"Do you know the initial cost of the cellphone? How much is the cellphone?"*, were intended to prompt Bonolo to transfer her newly developed understanding to the actual mathematical context of the problem.

The effectiveness of the teacher's intervention is evident in Bonolo's response in line 1.10, where she correctly identified R37 044. This response reflects a justifiable mathematical understanding rather than random recall. It can be argued that the teacher's role in recognising and affirming learners' progress is crucial for building confidence within a problem-based learning environment. Bonolo's utterance, *"Ohooo, R37 044 is A and we do not have P"*, indicates that she was now able to clearly articulate what was known and what was missing in the equation, thereby demonstrating reasoning at a formalising layer. This episode highlights learners’ algebraic ability to identify known and unknown quantities (Al-Mutairi & Marzouq, 2025), as well as their awareness of the knowledge constructed through image formation and property noticing (Amin & Sulaiman, 2021; Mabotja et al., 2018). Ultimately, folding back in this classroom episode was effective, as learners were able to apply appropriate algebraic reasoning skills to successfully solve the financial mathematics task.

4.3.2 Classroom Episode 2: Application of algebraic skills to determine the rate of interest (value of i).

Classroom Episode 2 focused on identifying the interest rate in both the simple and compound interest formulae. Learners were not explicitly taught how to manipulate the variables to obtain the required value; instead, they were encouraged to draw on the algebraic knowledge acquired in previous grades. This approach is characteristic of problem-based learning environments, where learners actively engage with the content, collaborate with peers to construct understanding, and the teacher assumes the role of a facilitator. Accordingly, this classroom episode centres on learners’ attempts in Learning Activities 3 to 5, which were sampled from the afternoon sessions.

Learning Activity 3 was undertaken by Group C and focused on determining the value of the interest rate in a simple interest formula.

Activity 3

a) Joseph invests R18 000 and it grows to R25 000 over a period of 2 years. Calculate the interest p.a to one decimal if the interest earned was simple interest.

b) Paul invests R35 000 and it accumulates to R55 000 over a period of five years. What

Figure 4.6: Learning Activity 3

An expected response from the learners is presented below.

$$A = P(1 + in)$$

$$25\ 000 = 18\ 000\big(1 + i(2)\big)$$

$$\frac{25\ 000}{18\ 000} = 1 + 2i$$

$$\frac{25}{18} - 1 = 2i$$

$$0{,}3888888 \ldots = 2i$$

$$\frac{0{,}3888888}{2} = i$$

$$0{,}19444 \ldots \times 100 = i$$

$$19{,}4\% = i$$

Vignette 4 below presents the responses of the learners, John, Khethiwe, and Thania, to the learning activity, as well as the sources of folding back observed during their interaction.

Vignette 4

4.1.	John:	We are not done; we should divide the interest rate with 12, then we would have answered the question.
4.2.	Khethiwe:	Kanti, why are you dividing your percentage with 12?
4.3.	John:	(*hides his book and giggles*)
4.4.	Khethiwe:	Ma'am look (*points at John's book*)
4.5.	Teacher:	Why did you decide to divide with the 12 here?
4.6.	John:	So basically, what I did I got the uhm…(*points at the "i"*), the interest rate which is that one. What I thought was that this interest rate is for the entire year, so I wanted to get just for one month, that's why I divided by 12 here (points at his attempt) and I got that one (pointing at 1,62%). Then I thought *gore* that is the interest rate for one month.
4.7.	Teacher:	Oh! okay, but we want the interest for the entire period meaning for the whole 2 years, so were you supposed to divide with the 12?
4.8.	John:	No, I was not supposed to divide by 12, I just had to conclude here (*pointing at 19,445%*) and get my final answer because I can now see that this answer is for the entire 2 years and that the question did not look for the interest per month.
4.9.	Khethiwe:	He saw p.a ma'am that is why.
4.10.	Teacher:	John, what do you think p.a stands for?
4.11.	John:	I thought it meant per month but now I see that it is per annual, meaning every year.
4.12.	Teacher:	We refer to it as per annum.
4.13.	John:	Okay ma'am so i was supposed to end here (*pointing at 19%).*

The conversation is sparked by John's awareness that the group had not completed the question. As indicated in line 4.1, it can be argued that he had no doubts about the algebraic procedure used in their attempt to solve the learning activity. He recognised the formula used and the substitution of variables and consequently stated that they *"should divide the interest rate by 12"* (line 4.1). In line with Pirie and Kieren (1994), this suggests that the learner was engaging in the process of examining a concept or its relevant properties by noticing distinctions, relationships, and combinations between concepts. In this regard, John can be regarded as operating at the property noticing layer.

Nevertheless, his understanding was incomplete, as his subsequent calculations were incorrect.

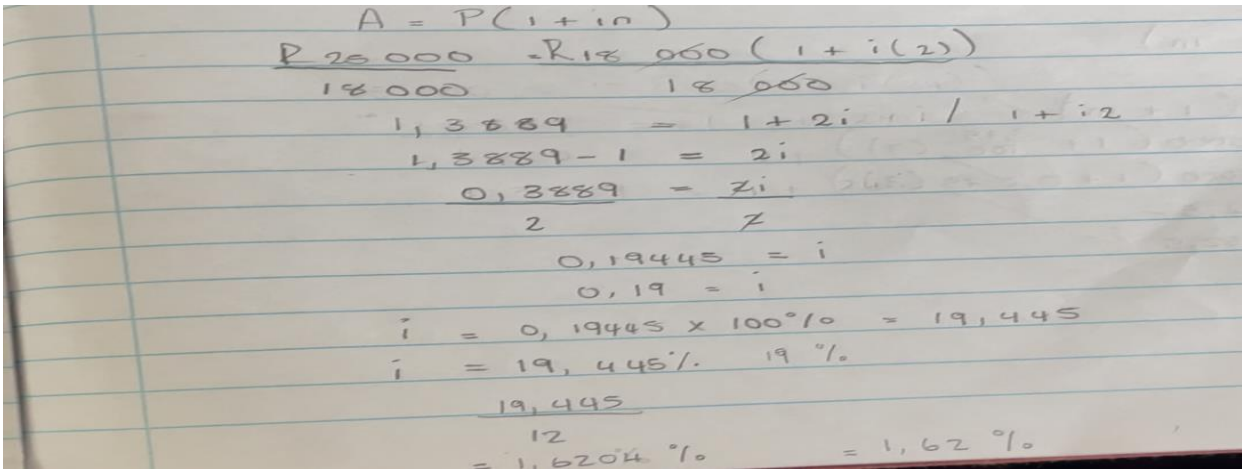

Figure 4.7: John's attempt

As a result, Khethiwe's question, *"Kanti, why are you dividing your percentage by 12?"*, suggests that she strongly disagreed with John's mathematical actions. She appeared to recognise a possible inconsistency between John's decision to divide by 12 and the actual context of the mathematical question. John's action in line 4.3, where he hides his book and giggles, suggests that he may have been retreating from his initial approach upon noticing that Khethiwe held a different view, rather than defending or justifying his reasoning. Realising that her question may not have achieved its intended purpose, Khethiwe invited the teacher to join the discussion. The teacher observed John's reluctance to respond and emphasised the importance of the question by rephrasing it and directing it back to John, as evident in line 4.4. In intervening, the teacher did not dismiss Khethiwe's challenge to John's reasoning, but acknowledged both her disagreement and her uncertainty in articulating it clearly.

The teacher's intervention underscores the importance of collaborative learning and encourages learners to engage critically with one another's reasoning. In this instance, the teacher fostered an environment in which questioning and justification were valued, thereby promoting deeper algebraic reasoning. It became apparent that John was not entirely unaware of the relationship between interest rate and time, but rather that he had divided the interest rate by 12 without a sound conceptual basis. This created an opportunity for John to justify his thinking. His explanation, *“what I thought was that this interest rate is for the entire year, so I wanted to get just for one month, that’s why I divided by 12 here… and I got 1.62%. Then I thought that is the interest rate for one month”*, demonstrates his ability to articulate relationships between mathematical ideas and to attempt to validate them. His phrasing, *“what I thought was…”*, suggests that he was reviewing rather than defending his reasoning, indicating engagement at the property noticing layer rather than an immediate act of folding back.

Having followed John’s line of reasoning, the teacher’s question, *“We want the interest for the entire period, meaning for the whole two years, so were you supposed to divide by 12?”*, served as an explicit and intentional intervention. By foregrounding the full duration of the investment before posing the question, the teacher redirected John’s focus away from monthly interest. As a result, John shifted from the property noticing layer to the image having layer. This shift is evident in his utterance, *“No, I was not supposed to divide by 12; I just had to conclude here… and get my final answer”*, which indicates that he began to recognise the relevant properties of the mathematical image he had constructed. He was now distinguishing between annual and monthly interest and moving away from purely procedural reasoning towards conceptual analysis.

John’s subsequent statement, *“I can now see that this answer is for the entire two years and that the question did not ask for interest per month”*, provides evidence that he folded back to the property noticing layer and successfully resolved the task. His mathematical actions demonstrate not only the ability to evaluate a solution through verification, but also the use of algebraic thinking to manipulate symbols and ensure correctness (Al-Mutairi & Marzouq, 2025).

Recognising that John’s confusion had been addressed, Khethiwe reflected on the possible source of his misunderstanding. Her comment in line 4.9 suggests that she was drawing on John’s earlier explanations in lines 4.6 and 4.8. In doing so, she offered insight into why the error occurred, reinforcing the idea that the interest rate was annual rather than monthly. Building on Khethiwe’s observation, the teacher reframed it as a question in line 4.10, which served as a further source of folding back for John. By line 4.11, John appeared to be operating at a formalising layer, as evidenced by his utterance, *“I thought* *it meant per month, but now I see that it is per annual, meaning every year"*. This reflects a clear refinement of his earlier reasoning and an explicit correction of his misunderstanding. Although his terminology was informal, the teacher addressed this in line 4.12 by linking his expression to the correct mathematical term, *per annum*.

The next learning activity, Learning Activity 4, focused on determining the compound interest rate. Group D, together with the other learners in the classroom, engaged with this activity.

1.

1.1. Calculate the total amount to be paid for the cycling bicycle after 36 months.
1.2. Calculate the cash price of the bicycle if the interest is charged at 7,0% p.a simple interest.
1.3. After 3 years, the same model's cash price has increased to R25 644,12 due to inflation. Calculate the annual inflation rate over this period assuming the increase in price is compounded annually.

Figure 4.8: Learning Activity 4

An expected response from the learners is presented below:

1.1. $R850 \times 36 = R30\ 600$

1.2.

$$A = P(1 + in)$$
$$R30\ 600 = P\big(1 + 0{,}07(3)\big)$$
$$\frac{30\ 600}{\big(1 + 0{,}07(3)\big)} = P$$
$$R25\ 289{,}26 = P$$

1.3.

$$A = P(1 + i)^n$$
$$25\ 644{,}12 = 25\ 289{,}26(1 + i)^3$$

$$\frac{25\ 644{,}12}{25\ 289{,}26} = (1+i)^3$$

$$\sqrt[3]{1{,}01405} = \sqrt[3]{(1+i)^3}$$

$$1{,}00465 = 1 + i$$

$$1{,}00465 - 1 = i$$

$$0{,}00465 \times 100 = i$$

$$0{,}465\% = i$$

The conversation captured in Vignette 5 between Teko and the teacher emerged from a learning activity comprising several sub-questions. Before engaging with the percentage sub-question, which is the primary focus of the conversation, Teko was expected to have completed Questions 1.1 and 1.2, as these were interlinked with Question 1.3, the percentage-related task. The interactions then unfolded as follows:

Vignette 5

5.1.	Teko:	Yes I knew it, it's gonna hurt. It's gonna be difficult but we gonna keep on moving. Why is the percentage 192,1%?
5.2.	Teacher:	How about you tell me Teko?
5.3.	Teko:	What I know is that an interest cannot be more than 100% but this one is going towards 200% and that is not normal.
5.4.	Teacher:	So, what are you implying?
5.5.	Teko:	Looking at my simplification leading to the answer, I can see that my steps are correct. I eliminated multiplication with division, exponent of 3 with cube root and addition with transposing. I am also realising that my substitution matches up with my previous answers (1.1. and 1.2) which is making me to now question my answers.
5.6.	Teacher:	Ow, so which answer of yours 1.1 or 1.2 or both you do think it is not correct?
5.7.	Teko:	1.2. ma'am, what I know is that a percentage cannot be more than 100% that is a fact that is proven to be true.
5.8.	Teacher:	What makes you think it is not correct? It is because it is giving you a percentage above 100? Or?
5.9.	Teko:	I am thinking that my answer in 1.2 is incorrect because it is far off from 1.1 for it to be a cost price of a phone but at first I did not realise that until now. Initially thought I did everything correct. But now I can see that this amount I found in 1.2 will not give me the correct percentage, hence I am going back to where I found the R1028,5 to find out if there is a problem there and now I am seeing that there is. My cash price is incorrect. That is why I am getting an incorrect percentage.
5.10.	Teacher:	So if the next question did not require the interest or you got a value less than 100% you were not going to question your previous answers?
5.11.	Teko:	Yes ma'am, I would not dare to even question if I am on the right path.
5.12.	Teacher:	Analysing your answers is always good to check if they make sense. We do not have to only do so when there is a mistake.

In Vignette 5 above, Teko raises an important question regarding a percentage value that exceeds 100%. However, an examination of his written attempt to solve the task indicates that he demonstrates procedural fluency in manipulating the variables and in identifying that answers from the preceding sub-questions are required, as illustrated in Figure 9.

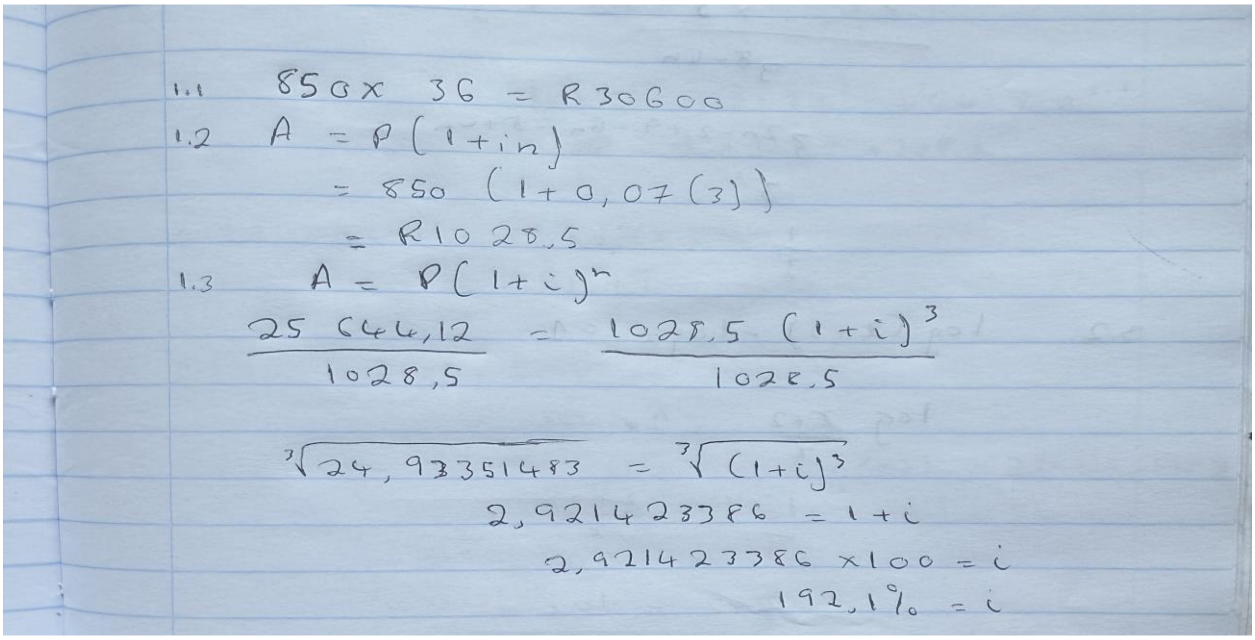

Figure 4.9: Teko’s attempt for 1.1 - 1.3

Although Teko demonstrated algebraic fluency, the solution he obtained did not make sense to him. This is evident in his question, *“Why is the percentage 192.1%?”*, which indicates his ability to reflect critically on the outcome of his calculations. According to Pirie and Kieren (1994), learners reflect on prior knowledge in an attempt to make sense of new situations, thereby extending their thinking. Martin (2008) identifies such self-reflection as a key component in the development of mathematical understanding, arguing that learners’ ability to question their own mathematical thinking enables them to construct broader and more coherent understandings through the refinement of existing knowledge. Following his independent attempt at the task, Teko appeared to be operating at a formalising layer, as he demonstrated the ability to solve the problem and arrive at a solution independently. Importantly, he was not questioning the algebraic procedures themselves, but rather the plausibility of the resulting percentage value. One possible explanation for his concern is a comparison between real-life expectations and the outcome of his mathematical manipulation. This is evident in his utterance, *“What I know* *is that an interest cannot be more than 100%"*, which further confirms that he was reasoning at a formalising layer by reflecting on the properties of the mathematical image he had constructed. At this point, Teko was able to recognise the relationship between financial mathematics quantities and their associated meanings.

The teacher's prompt, *"How about you tell me, Teko?"*, provided a space for Teko to articulate his thinking, reinforcing the notion that learners should not rely solely on peers or teacher explanations. The intention of this prompt was to encourage self-reflection, not only on potential errors, but also on the validity of the solution itself. In this context, Teko began to reflect on the contradiction between his answer and the mathematical properties of percentages. The teacher's subsequent questioning enabled him to further unpack the reasoning behind his statement in line 5.3, thereby creating an opportunity for deeper explanation. This interaction aligns with principles of problem-based learning, where teachers create structured opportunities for learners to verbalise misconceptions, test conjectures, and iteratively refine their reasoning (Häkkinen et al., 2022). As noted by Chuene et al. (2023), such environments promote dialogic interactions in which learners' reflective talk drives deeper engagement with algebraic concepts.

Teko's self-reflection is further evident in his statement, *"Looking at my simplification leading to the answer, I can see that my steps are correct"*, which suggests strong procedural fluency. His explanation that he *"eliminated multiplication with division, exponent of 3 with cube root, and addition with transposing"* demonstrates correct application of algebraic operations, despite his continued doubt about the outcome. This prompted him to reconsider his earlier answers, as reflected in his remark, *"I am also realising that my substitution matches up with my previous answers (1.1 and 1.2), which is making me now question my answers"*. This indicates that he was attempting to reconcile his confident procedural steps, his previously unquestioned results, and the unexpected percentage value of 192.1%.

Recognising Teko's concern, the teacher's question, *"Which answer, 1.1 or 1.2, or both, do you think is not correct?"*, prompted him to fold back from the formalising layer to the property noticing layer. In his response, *"Initially, I thought I did everything correctly. But now I can see that this amount I found in 1.2 will not give me the correct percentage"*, Teko actively connected his procedural confidence with the emerging doubt about his results. This moment is significant, as it reflects a deeper level of reflection on the source of his error. His subsequent statement, *“I am going back to where I found the R1 028.5 to find out if there is a problem there, and now I am seeing that there is”*, suggests a deliberate attempt to trace the origin of the inconsistency. This is further supported by his realisation that *“the cash price is incorrect, that is why I am getting an incorrect percentage”*, as indicated in line 5.9. At this stage, Teko was deepening his understanding at the property noticing layer by intentionally linking algebraic steps to the reasonableness of the solution.

The teacher’s follow-up question, *“…if you had obtained a value less than 100%, would you still have questioned your previous answers?”*, built on Teko’s explanation and encouraged further reflection on the importance of verification. Teko acknowledged that his earlier answer in 1.2 was likely incorrect because it deviated substantially from the value obtained in 1.1, even though this inconsistency had not initially appeared problematic. Ultimately, Teko was able to return to the formalising layer and, with his refined understanding, determine the correct percentage, which was less than 100%.

A similar pattern of solution verification emerged in the interview data. One learner reflected that *“while I was doing the work myself, I was able to go through my mistakes and I also realised that after writing something it is always best to go back and check what I have written”* (Phuti). The learner further explained that verification helps in tracing the source of errors, stating, *“If I see a mistake in something like 1.1.3, it is always best to go back and check 1.1.1, because the mistake might have started there”*. These mathematical actions demonstrate learners’ ability to verify solutions, which is a critical component of algebraic reasoning and solution evaluation, as highlighted by Al-Mutairi and Marzouq (2025).

In the subsequent Learning Activity 5, learners were engaged in determining the interest rate using the compound interest formula.

2. What is the interest rate if Nosipho invested R2 000 at compound interest for a period of three years, and received R2 800?

Figure 4.10: Learning Activity 5

An expected response from the learners is presented below:

$$A = P(1+i)^n$$

$$2800 = 2000(1+i)^3$$

$$\frac{2800}{2000} = (1+i)^3$$

$$\sqrt[3]{1{,}4} = \sqrt[3]{(1+i)^3}$$

$$1{,}118688942 = 1+i$$

$$1{,}118688942 - 1 = i$$

$$0{,}118688942 \times 100 = i$$

$$11{,}87\% = i$$

In this learning activity, the teacher observed that the learners, John, Khethiwe, and Thania, were able to determine the interest rate using the appropriate formula without encountering difficulties. Their calculations indicate that they possessed the necessary algebraic skills to solve the problem. Although Thania arrived at the correct answer, she used the square root to eliminate the cube root, which suggests a procedural or computational error rather than a conceptual misunderstanding. Consequently, the teacher not only commended the group for successfully completing the task but also asked Thania to explain the reasoning behind her approach, as illustrated in Vignette 6:

Vignette 6

6.1.	Teacher:	I can see that you guys have successfully managed to solve for the interest rate but Thania, take me through your working please.
6.2.	Thania:	Yes, ma'am we identified that the value of A is 2800, P is 2000, n is 3. Then substituted into the compound interest formula to find the interest rate.

6.3.	Teacher:	Oh okay, let us track back your working a bit (*pointing*)
6.4.	Thania:	Yes, ma'am we did that to eliminate the exponent of 3.
6.5.	Teacher:	Check on the right-hand side, you have the square root and yet the aim was to eliminate the exponent of 3.
6.6.	Thania:	Yes that was the aim, that is why I put the square root of 3 both sides.
6.7.	Teacher:	You mean cube root?
6.8.	Thania:	Cube because we are removing exponent of 3?
6.9.	Teacher:	What is a square root used for?
6.10.	Thania:	To eliminate the exponent of 2 and ohhh, cube to remove 3 and fourth root to remove 4. I get it.

In this vignette, the teacher identified the algebraic errors made by Thania in her calculations. However, the teacher's utterance, *"Thania, take me through your working…"*, indicates an intention to encourage the learner to identify her own mistakes. In doing so, learners are afforded opportunities to build confidence in articulating their thought processes. In this regard, the teacher's mathematical action creates valuable opportunities for learners to fold back and refine underdeveloped mathematical understanding. Thus, learners' algebraic errors are treated as productive opportunities for deepening conceptual understanding.

Thania then confidently explained that *"…A is 2800, P is 2000, n is 3. Then I substituted into the compound interest formula to find the interest rate"*, which demonstrates her understanding of the meaning of each variable and her ability to carry out the required algebraic manipulation. According to Pirie and Kieren (1994), Thania's explanation exhibits characteristics associated with the formalising layer of understanding.

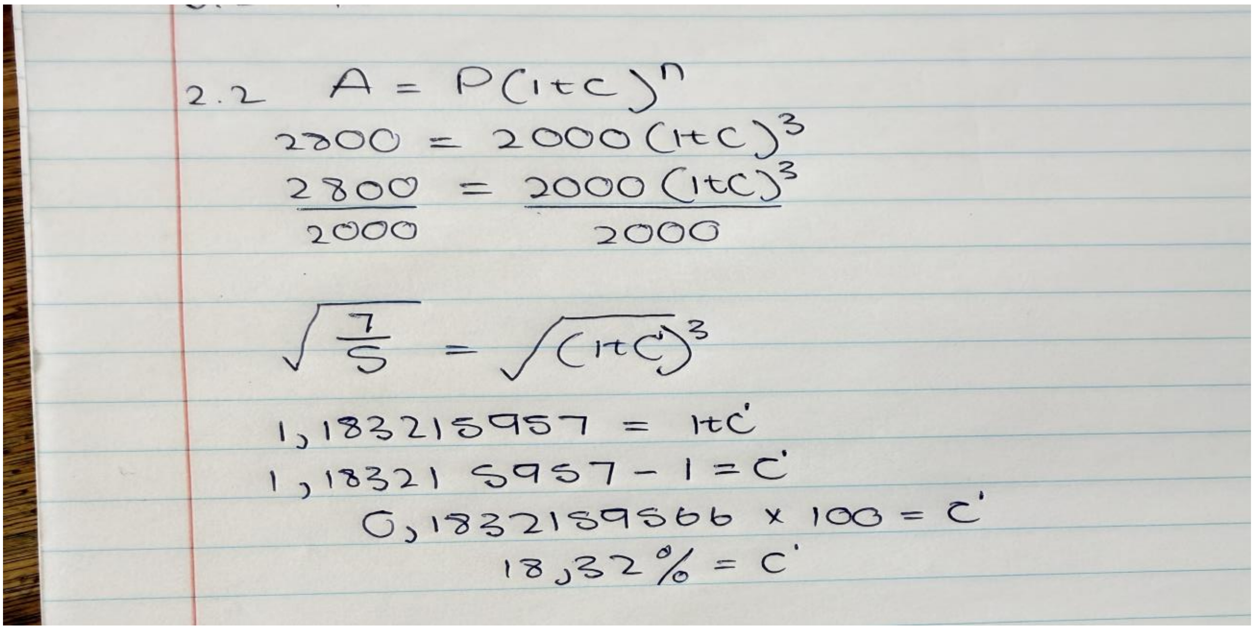

Figure 4.11: Thania's attempt

Despite Thania's ability to work with generalised properties at the formalising layer, she was unable to identify the errors in her algebraic manipulation (lines 6.3 and 6.4). Consequently, the teacher's actions, as reflected in lines 6.3 and 6.5, were aimed at supporting Thania in recognising her mistakes. In this instance, it can be argued that the teacher intentionally directed Thania's attention to a specific mathematical concept within her calculations, which constitutes an explicit and intentional intervention (Martin, 2008). Rather than correcting her directly, the teacher invited Thania to reflect on her own steps, consistent with practices encouraged in a folding-back learning environment (Patmaniar et al., 2021). Specifically, the teacher guided Thania to reconsider her understanding of square roots and cube roots.

As a result, Thania folded back from the formalising layer to the image having layer to rework her understanding. Her question, *"Is it a cube because we are removing the exponent of 3?"*, reflects uncertainty and an underdeveloped conceptual understanding. Instead of providing a direct explanation, the teacher redirected Thania's attention through a probing question, as seen in line 6.9, encouraging her to draw on her existing knowledge of square roots. This prompted Thania to use prior understanding to make sense of the task, which Chuene et al. (2023) identify as characteristic of engagement at the image having layer.

Thania's reconstructed reasoning in line 6.4, particularly regarding the algebraic laws applicable across different contexts, further indicates a return to the formalising layer. Her assertion that *"…cube to remove 3 and fourth root to remove 4"* demonstrates a developing structural understanding of algebraic manipulation and the inverse relationship between exponentiation and root operations. Her concluding remark that she now *"…gets it"* signifies a moment of conceptual clarity. This progression aligns with the findings of Dhlamini et al. (2019), who highlight that learners' ability to connect mathematical concepts supports the development of robust mathematical reasoning skills.

4.4 FIELDWORK 2: USING ALGEBRAIC REASONING TO SOLVE FINANCIAL MATHEMATICS IN REAL LIFE CONTEXT

4.4.1 Classroom Episode 1: Application of algebraic reasoning to calculate an estimated population after a given period

In Fieldwork 2, the classroom episodes linked financial mathematics concepts to real-life situations, specifically compound interest with population growth and simple interest with hire purchase. To provide context, hire purchase involves acquiring an item on credit and paying for it over time while making use of it, whereas population growth refers to the rate at which a population increases over a given period. Graphs were used to introduce these concepts to learners. In this context, an exponential graph represents growth that depends on what already exists, meaning that the increase is not constant over time. In contrast, a straight-line graph represents fixed growth, where interest is applied once to the original cost price and does not accumulate on previous growth, resulting in a constant increase.

Learners were therefore required to relate the given mathematical tasks to appropriate financial mathematics formulae based on the real-life contexts presented and the type of graph they believed best represented each situation. This process ultimately guided their choice of formula. Such an approach is characteristic of a problem-based learning environment, where the teacher is not the sole source of knowledge and learners are placed in situations that require them to take active responsibility for their own learning. Learning Activities 6 to 9 were sampled, and learners' conversations were analysed narratively.

The following analysis focuses on the interactions among learners Queen, Temoso, and Tebatso during their attempt to solve Learning Activity 6, which centred on population growth.

3.

3.1 The population of a city in South Africa increased by 5,25% for the year 2017

At the beginning of 2017 the population of the city was 2560000.

Assuming that the population will continue to increase at a constant rate of 5,25% each year, estimate the population of the city at the beginning of 2020.

Give your answer correct to the nearest whole number

Figure 4.12: Learning Activity 6

An expected response from the learners is presented below:

$$A = P(1+i)^n$$
$$A = 2\,560\,000(1+0{,}0525)^3$$
$$A = 2\,984\,738$$
$$\therefore \textit{The population at the beginning of } 2020 \textit{ would be } 2\,984\,738$$

The interactions among the learners, Queen, Tebatso, and Temoso, as they worked on the learning activity, are captured as follows:

Vignette 7

7.1.	Queen:	The population we are looking for it's an estimate for 2020, right?
7.2.	Temoso:	Yes, so now we are working with people. We cannot control the growth so it is exponential.
7.3.	Tebatso:	Compound interest.
7.4.	Queen:	Now that we have the formula, let us read the statement to take out the given information for substitution.
7.5.	Tebatso:	A=2560000, the interest is 0,0525 for 1,2,3 from 2017 to 2020 is three years, right?
7.6.	Temoso and Queen:	Yes
7.7.	Temoso:	We start counting from 2018-2020.

(*Continue with an intention of solving for P*)

It is evident from the learners' interactions that Queen, Tebatso, and Temoso were collaboratively engaged in making sense of the mathematical task presented. This is reflected in Queen's invitation to her peers to clarify what the question required, thereby creating space for collective problem-solving. Despite this collaboration, it is important to note that the learners were not operating at the same layer of understanding.

On the one hand, Temoso's utterance, *"we are working with people, we cannot control the growth, so it is exponential"*, suggests that he was contributing from the image having layer of understanding. In this instance, he appeared to draw on a mental image of exponential growth, informed by his knowledge of graphs and their relationship to financial mathematics formulae. This aligns with the assertions of Pirie and Kieren (1994) and Martin (2008), who argue that learners can rely on mental constructs about a topic without necessarily engaging in the activities that originally gave rise to those constructs.

On the other hand, Tebatso's suggestion to use the compound interest formula (line 7.3) indicates an awareness of the properties associated with population growth, exponential functions, and the application of financial mathematics concepts in non-financial contexts. In this case, Tebatso appeared to be developing a representation of the learning activity, which can be associated with the image making layer. This action is consistent with Yao's (2020) description of the image making layer, where learners engage in specific mental actions to develop an understanding of the concept under exploration.

After considering her peers' contributions, Queen's utterance, *"Now that we have the formula, let us read the statement to take out the given information for substitution"*, demonstrates her awareness of the problem-solving process and the importance of carefully interpreting the question. Queen can also be classified as operating at the image having layer, as she appeared to accept the selected formula without yet manipulating the variables. When attempting to identify and use the given information, Temoso substituted A with 2 560 000 and used the population growth rate for three years, and his peers agreed with this substitution (line 7.6) without verifying the contextual meaning of each variable. Both Queen and Tebatso then applied the compound interest formula to solve for the unknown variable, focusing primarily on procedural application rather than on a relational understanding of the quantities involved, as illustrated in Figure 4.13.

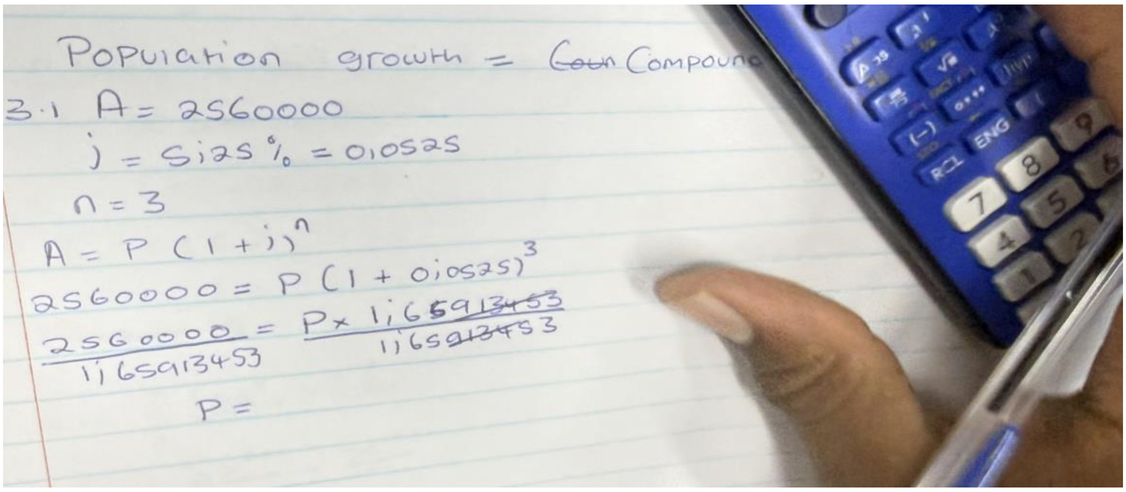

Figure 4.13: Queen's initial attempt

As shown in Figure 4.13, the learners were unable to correctly interpret the given information, particularly with regard to the value of *P*. Consequently, Queen was able to identify the errors in their calculations, as illustrated in the following vignette:

Vignette 8

8.1	Queen:	Mara wait, P is the initial amount meaning initial population in this instance. The statement is stating that the population in 2017 was 2560000 and now we are looking for the population in 2020. Therefore, the population we already know is for 2017 and we are looking for the one in 2020. Meaning we are expecting an increase in the number of people making the number 2560000 given is for the initial population (2017).
8.2	Temoso:	So, this is incorrect. (Queen *proceeds to cancelling out the initial attempt*)
8.3	Teacher:	What made you decide to cancel out your initial attempt?
8.4	Queen:	It is because I am realizing here that we read the question completely wrong and here (points at the statement) we are given the value for P and not A.
8.5	Teacher:	How do you know that?
8.6	Queen:	It does say... ma'am, population in 2017 is 2560000 and we must find the one in 2020. So, we start 2017. 2017, beginning and 2020 end meaning final.

Queen's utterances in line 8.1 suggest that the incorrect calculations stemmed from a misinterpretation of the variables. This is further confirmed in her response to the teacher's question, *"…I am realising we are given the value for P and not A"*. Consequently, she revisited her earlier emphasis on reading and interpreting the statement carefully (see line 7.9), this time with clearer understanding, as reflected in lines 8.1 and 8.6. Queen's mathematical actions demonstrate algebraic reasoning characterised by repeated reading of the problem and deliberate attempts to extract and organise relevant information, as noted by Al-Mutairi and Marzouq (2025).

The identification of variables in this instance was not random, but grounded in an understanding of the meaning of each variable, as evident in her utterance, *"Mara wait…"*. This moment prompted Queen to fold back from the image making layer to the property noticing layer, where she began to manipulate and combine elements of her existing images to construct contextually relevant properties. In line with Martin (2008), the learner engaged in self-conscious reflection by questioning her understanding and considering what could be generalised from the situation. In this case, the source of folding back can be understood as the learner herself.

As she refined her reasoning, Queen articulated that one would *"…expect an increase in the number of people, making the number 2 560 000 the initial population in 2017"*. This indicates that she extended her understanding by revising her earlier constructs of the concept. Queen reasoned that the population after three years (A) should not be less than the initial population (P) in 2017, drawing on the real-life meaning of population growth. Her choice of words suggests that she reconstructed her initial understanding of the problem by aligning the mathematical symbols with their contextual meanings. As a result, she was able to return to an outer layer of understanding and apply this extended reasoning to correctly solve the problem, as illustrated in Figure 4.14 below.

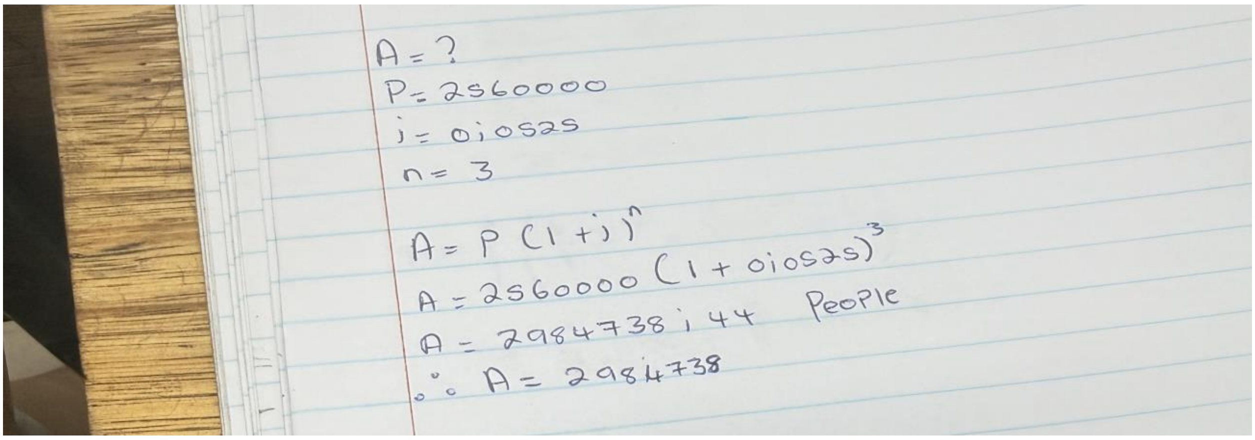

Figure 4.14: Queen's second attempt

4.4.2 Classroom Episode 2: Application of algebraic reasoning to solve hire purchase agreement problems.

Classroom Episode 2 focused on learners' use of the simple interest formula to address questions related to hire purchase agreements. The activities were curated from the *Mind Action Series* textbook (2021) and other relevant sources. Three learning activities (Activities 7 to 9) were sampled for reporting. Learners' interactions as they worked through the tasks were central to the study and are therefore presented below in the form of vignettes.

The analysis that follows is based on the interactions among Karabo, David, and Tshepo during their attempt to solve Learning Activity 7.

Mrs. Reyem wants to buy herself a new laptop. After completing her monthly budget she realizes that she does not have the money to buy a new laptop right now.

Mrs. Reyem decides to buy a laptop on a hire purchase agreement from a computer store that advertises the following special:

> Cash Price: **R4 999**
>
> OR
>
> Hire purchase: 12% Deposit and 24 equal monthly payments to the value of **R6 598,68**

3.1. Calculate how much Mrs. Reyem would pay for the deposit.

3.2. Calculate the total amount that Mrs. Reyem will pay for her new laptop on the hire purchase agreement.

Figure 4.15: Learning Activity 7

An expected response from the learners is presented below:

3.1. $R4\,999 \times 12\% = R599{,}88$

3.2. $R599{,}88 + R6\,598{,}68 = R7\,198{,}56$

The interactions among the learners, Karabo, David, and Tshepo, as they worked on the learning activity are captured in Vignette 9 below:

Vignette 9

9.1.	Karabo:	Mrs Reyem's agreement is on hire purchase, which implies simple interest. Therefore, to calculate the total amount Mrs Reyem will pay for her new laptop. We would have to use the simple interest formula.
9.2.	Tshepo:	What do we then substitute on the value of P? R6598, 68 or R599,88?
9.3.	Karabo:	What about interest?
9.4.	David:	We use 12% and I will use R599,88 as P because it is the deposit (the money she paid, the only one she had) for the period of 2 years.
9.5.	Karabo:	I will use R6598, 68. Because it is the loan value at the end of the 24 monthly instalments. (*Proceeds to doing the calculations*)

Vignette 9 is influenced by Karabo's intention to actively engage her group members in the mathematical task at hand. In her reasoning, she relates Mrs Reyem's hire purchase context to the concept of simple interest. This connection appears to stem from prior knowledge acquired during the teaching and learning sessions preceding the task they were attempting as a group. This action indicates that she is able to draw on previously established knowledge linking hire purchase to simple interest, providing evidence that she was operating at a formalising layer of understanding.

In response, Tshepo's question, *"What do we then substitute for the value of P? R6 598.68 or R599.88?"*, suggests that he agrees with Karabo's assertion that simple interest should be used to solve hire purchase problems. His question further reflects uncertainty regarding which value should be used as the principal amount. Although Tshepo appeared relatively quiet throughout the interaction, his question played a central role in shaping the subsequent group discussion, as evidenced in the following two vignettes.

In response to Tshepo's question, his peers appeared to follow different lines of reasoning concerning the principal amount to be used. On the one hand, David selected R599.88 on the basis that *"it is the deposit, the money she paid, the only one she had, for the period of two years"*. In this instance, David appeared to equate the deposit with the principal amount, suggesting a conceptual misunderstanding, as illustrated in Figure 4.16 below. Although misapplied, David's reasoning reflects engagement at the primitive layer.

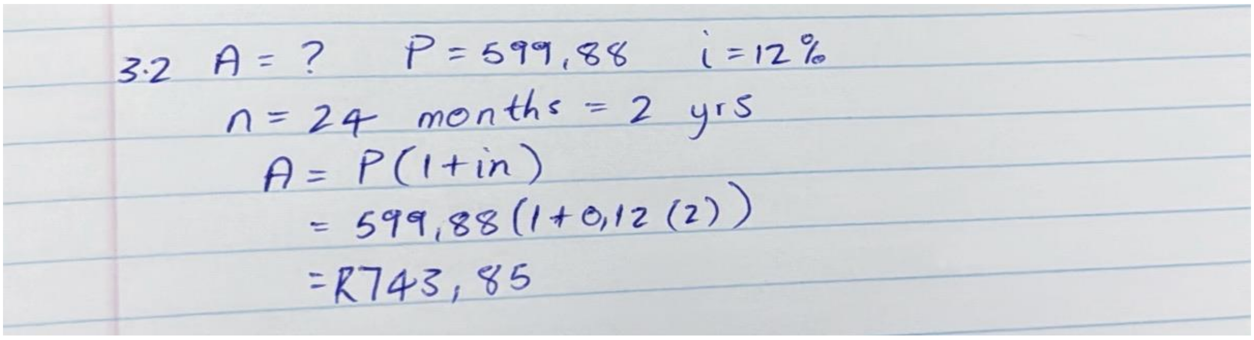

Figure 4.16: David's attempt

On the other hand, Karabo focuses on R6 598.68, stating that *"it is the loan value at the end of the 24 monthly instalments"*. In this regard, Karabo regards the loan value as equivalent to the principal amount, as illustrated in Figure 4.17 below:

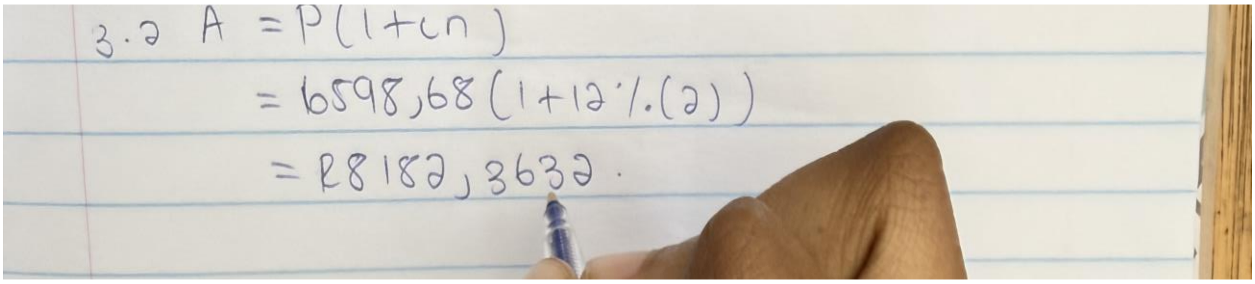

Figure 4.17: Karabo's attempt

Learners' mathematical actions reveal different reasoning paths, indicating that they were operating at different layers of understanding in relation to the learning activity. Noticing this, the teacher intervened by engaging with each learner's contribution separately, allowing both to articulate their reasoning without interference from the other. This vignette is characterised by the provision of equal opportunities for learners to express their thinking, despite holding differing views on the same aspect of the task. The interaction then continued, as captured in Vignette 10 below:

Vignette 10

10.1.	Teacher:	Okay, let me start with David, why did you use R599,88?
10.2.	David:	It is the deposit she paid, that is why.
10.3.	Teacher:	So, do you think Mrs Reyem will pay back an amount of R743,85 at the end of her term while the laptop costs R4999? And on hire purchase she is bound to make 24 monthly instalments to the value of R6598, 68.
10.4.	David:	Eish, it really does not make sense.
10.5.	Teacher:	What is the hire purchase agreement saying?
10.6.	David:	12% deposit and 24 equal monthly payments to the value of R6598,68.
10.7.	Teacher:	How much do you think Mrs Reyem will pay back to the bank as part of her hire purchase agreement from reading that statement?
10.8.	David:	12% deposit and the value of the 24 monthly Instalments is how much Mrs Reyem will pay back.
10.9.	Teacher:	How much would that be?
10.11.	David:	R6598,68+ R599,88 which is R7198,56. Ohh, I missed it. I just saw "hire purchase" and jumped to simple interest formula.
10.12.	Teacher:	Oh, I see. It was just all in the statement given.
10.13.	David:	Eish this thing of not reading statements with understanding and analysing answers found will work against me but I now see it.

The teacher's question, as reflected in line 10.1, created an opportunity for David to explain his reasoning for identifying R599.88 as the principal amount. In his response, David reaffirmed his initial focus on the deposit, which indicates engagement at the primitive layer of understanding. In this instance, he drew on prior algebraic knowledge related to percentages and applied it to the financial mathematics context. Although he was able to use this prior knowledge to calculate a percentage, he was unable to apply it effectively to address the demands of the current problem. Rather than correcting him directly, the teacher employed probing questions to guide his thinking. This is evident in the question, *“So, do you think Mrs Reyem will pay back an amount of R743.85 at the end of her term while the laptop costs R4 999?”* (line 10.3). This prompt encouraged David to think critically and to establish connections between the numerical results and the real-life context of the problem. Consequently, David recognised that his initial conclusion, which suggested paying less than the cost of the laptop at the end of the hire purchase agreement, *“really does not make sense”* (line 10.4). This interaction aligns with the findings of Al-Mutairi and Marzouq (2025), who highlight learners’ ability to use algebraic and functional thinking to identify meaningful relationships between variables in financial mathematics contexts.

The teacher’s probing techniques (lines 10.5, 10.7, and 10.9) can be regarded as explicit and intentional interventions that prompted David to fold back from the primitive knowing layer to the image having layer, focusing more carefully on the structure of the hire purchase arrangement. As a result, David was able to recognise the general properties of hire purchase and the relationships between the relevant quantities. This is evident in his utterance, *“R6 598.68 plus R599.88, which is R7 198.56. Ohh, I missed it. I just saw hire purchase and jumped to the simple interest formula.”* Through this realisation, David was able to connect the deposit and the total of the monthly instalments, as illustrated in Figure 4.18, while also acknowledging the shortcomings in his initial approach (line 10.13).

In this shift, David moved from an arithmetic focus towards a more conceptual understanding that linked the deposit to the total repayments. The teacher’s probing therefore created space for reflective thinking, allowing David to reassess and refine his initially constructed knowledge. This episode illustrates how effective folding back can provide learners with opportunities to reflect on their strategies, identify errors, and deepen their mathematical reasoning.

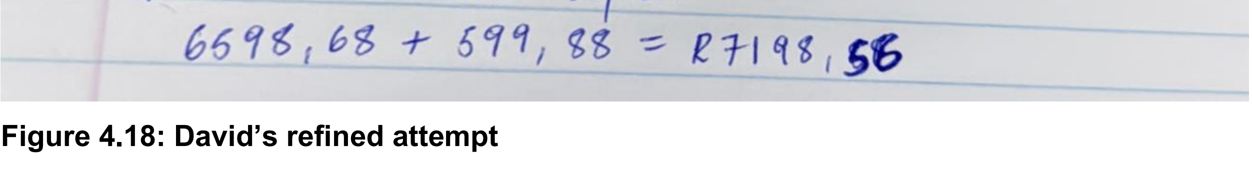

Figure 4.18: David’s refined attempt

Although the total amount payable by Mrs Reyem for her new laptop under the hire purchase agreement was successfully determined, the teacher proceeded to revisit Karabo's contribution, as illustrated in the following vignette.

Vignette 11

11.1.	Teacher:	Let's see, Karabo, why R6598,68 as the value of P?
11.2.	Karabo:	It is because R6598,68 is the money that Mrs Reyem owes.
11.3.	Teacher:	If that is so, then why did you not write it down as the money Mrs Reyem will pay back as part of her hire purchase agreement?
11.4.	Karabo:	The rate is given at 12%, hence the simple interest formula.
11.5.	Teacher:	What is a deposit?
11.6.	Karabo:	Money one puts down as a down payment.
11.7.	Teacher:	So, what does the 12% deposit tell you?
11.8.	Karabo:	Mrs Reyem put a down payment but in this case we do not know the amount but know that it is 12% of the laptop price.
11.9.	Teacher:	Now we are getting somewhere. What is the hire purchase agreement stating?
11.10.	Karabo:	12% deposit and the value of the 24 monthly installments.
11.11.	Teacher:	So how would you calculate that?
11.12.	Karabo:	12% down payment which we calculated in 1.1. then add it to the R6598,68.
11.13.	Teacher:	How much would that be?
11.14.	Karabo:	(*takes out calculator*) R7198,56.

The teacher's actions were intended to provide Karabo with an opportunity to articulate her thought process. In doing so, the teacher demonstrated a commitment to valuing each learner's contribution and reasoning within a problem-based learning environment. Consequently, the discussion centred on the principal amount (P) and the amount of money, R6 598.68, owed by Mrs Reyem, as reflected in Karabo's reasoning. Similar to David, Karabo initially conflated the deposit percentage with the overall interest rate, as indicated in line 11.4. This suggests that both learners were associating the percentage with the interest rate without sufficient consideration of the context in which it was applied. Although the use of simple interest was appropriate, they first needed to determine the correct interest rate.

While Karabo understood that the deposit represented a down payment, she was unable to link this to the deposit amount calculated in Question 3.1 of Learning Activity 7. This is evident in her utterance, *"we do not know the amount but know that it is 12%"*. This statement does not necessarily indicate that Karabo was unable to calculate the 12% deposit, but rather that she lacked understanding of how this information could be applied to solve the problem. In this context, it can be argued that Karabo was attempting to make sense of the information by visualising and articulating it, relying heavily on emerging representations to reason through the task. This suggests that she was operating at the image making layer of understanding. Her actions further indicate that she was working within an inner layer using existing knowledge.

The teacher's remark, *"Now we are getting somewhere"*, acknowledged Karabo's effort to make sense of the information and served as encouragement. This was followed by a probing question that explicitly directed Karabo's attention to the structure of the hire purchase agreement as described in the learning activity. This intervention prompted Karabo to fold back from the image making layer to the property noticing layer, where she began to examine the properties of the constructed image more critically. At this stage, she was able to link the deposit and the value of the 24 monthly instalments to the appropriate calculation procedure, rather than relying solely on the formula without contextual understanding.

Through this process, Karabo recognised the relationship between the 12% deposit and the total monthly payment of R6 598.68, which enabled her to determine the total amount payable under the hire purchase agreement. This progression aligns with Yao's (2020) assertion that learners operating at the property noticing layer are able to identify attributes and features of an image. In this instance, Karabo was able to fold back to the formalising layer of understanding with the support of an external prompt and apply her extended understanding to solve the problem, as evidenced in line 11.12 and Figure 4.19 below. This episode demonstrates learners' ability to implement solutions by applying algebraic properties within a financial mathematics context, as proposed by Al-Mutairi and Marzouq (2025).

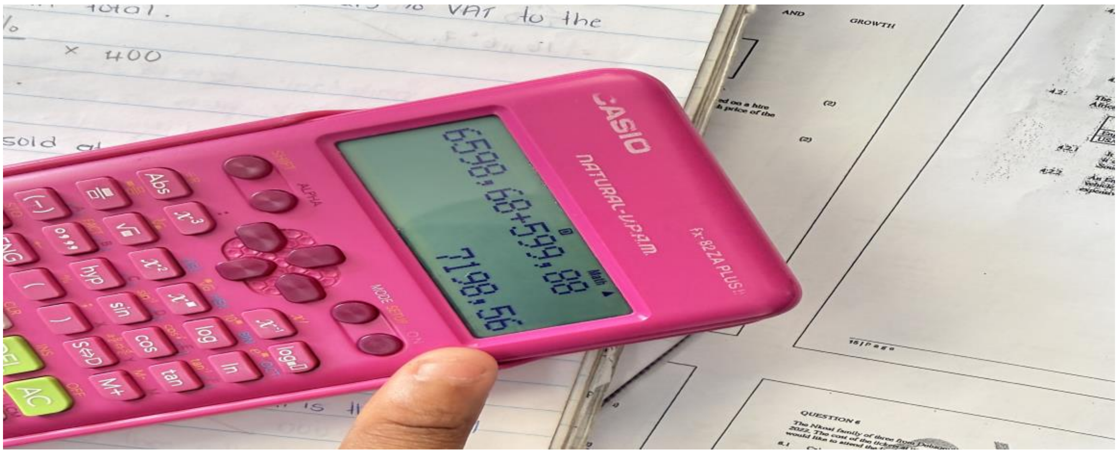

Figure 4.19: Karabo's calculator computation

In the following learning activity, 8 learners were working within a hire purchase context.

d) Tyler purchases a sound system for R7,999 and pays an 11% deposit. The store charges Tyler 20% interest per year, with the loan payable over 4 years. Calculate the total amount that he will have to pay back?

Figure 4.20: Learning activity 8

An expected response from the learners is presented below:

$$R7\,999 \times \frac{11}{100} = R879{,}89$$

$$R7\,999 - R879{,}89 = R7\,119{,}11$$

$$A = P(1 + in)$$

$$A = R7\,119{,}11\big(1 + 0{,}2(4)\big)$$

$$= R12\,814{,}398$$

The interactions among the learners, Karabo, David, and Tshepo, as they worked on the learning activity, are captured in Vignette 12 below:

Vignette 12

12.1.	Tshepo:	The sound system is R7999 on hire purchase.
12.2.	David:	The period is 4 years and the bank charges Tyler 20%.
12.3.	Karabo:	So, we use the simple interest formula. (write down the *calculations*)

Tshepo's act of highlighting the sound system price and the hire purchase agreement, *"the sound system is R7 999 on hire purchase"*, appears to have prompted David to contribute further by noting, *"four years and the bank charges Tyler 20%"*. This collaborative exchange supported the group in identifying the key aspects of the problem that needed to be addressed. Consequently, Karabo's suggestion to use *"…the simple interest formula"* was justified.

As reflected in the vignette above, the learners identified the principal amount, the repayment period, and the interest charged, and collectively agreed to use the simple interest formula, as proposed by Karabo. Their actions demonstrate an ability to identify known and unknown information, which Al-Mutairi and Marzouq (2025) describe as a crucial algebraic thinking process in understanding and solving mathematical problems. It can therefore be argued that the learners were operating at the formalising layer of understanding, as they were able to agree on an appropriate formula and recognise the connection between hire purchase and simple interest concepts. However, they were unable to identify the correct value of the principal amount, which ultimately led to an incorrect solution, as illustrated in Figure 4.21 below.

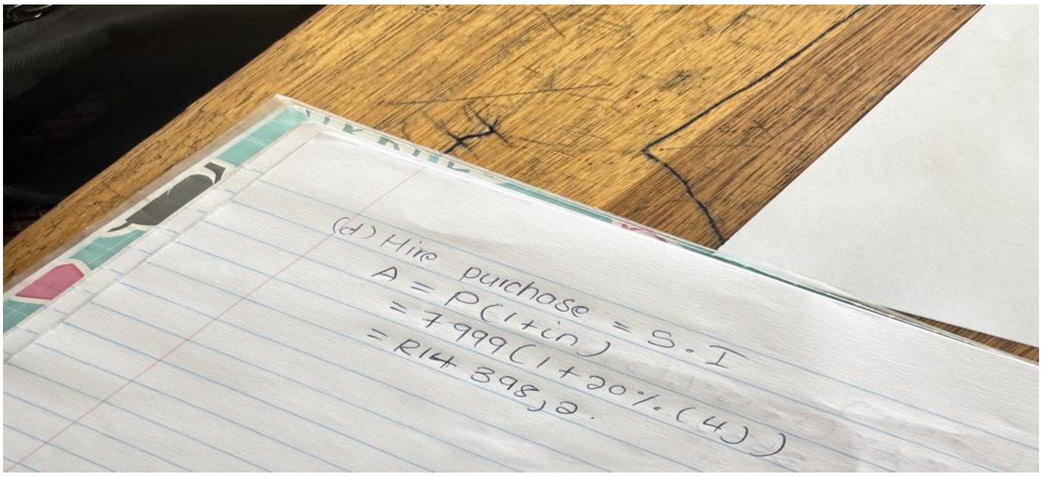

Figure 4.21: Karabo's initial attempt

Although the learners demonstrated certain aspects of algebraic fluency, they showed a lack of conceptual understanding in interpreting the deposit paid, as they did not consider its effect on the amount of money that needed to be borrowed to supplement what was already available. The interaction continued, as captured in Vignette 13 below:

Vignette 13

13.1. Karabo: But wait guys, Tyler paid a deposit already of 11%. Then now we want to see how much is the remaining amount that she owes. When the bank charges the interest

		is on the money that Tyler is owing not the price of the sound system. It is not like Tyler went to the store empty handed. Tyler had something "11% deposit" a down payment like that Mrs Reyem question.
13.2.	David:	Are you suggesting that we subtract the down payment from the price of the sound system?
13.3.	Karabo:	I am not suggesting, that is what we should do because the bank cannot charge you interest on the money you did not borrow from them. Tyler had 11% and not the full amount. The reason Tyler goes to the bank for the loan is because they were running short.
13.4.	Tshepo:	Ohooo, I see it now. Tyler takes out a loan for the shortfall. As such, we are trying to see how much Tyler will pay back to the bank.
13.5.	Karabo:	Yes, meaning we should use R7999 – R879,89 which equals R7119,11. We should use R7119,11 as P not R7999.
13.6.	Teacher:	Great contribution Karabo, I am impressed with the progress.

In this vignette, Karabo's utterance, *"But wait, guys"*, demonstrates her ability to reflect on the group's solution and recognise that an important aspect of the problem may have been overlooked. Her mathematical action was not necessarily triggered by her peers' contributions, but rather by her own awareness of the solution in relation to her understanding of the problem. This aligns with algebraic evaluation of solutions in financial mathematics, where learners manipulate symbols to verify the reasonableness of a solution (Al-Mutairi & Marzouq, 2025). Accordingly, her self-awareness prompted a self-initiated act of folding back from the outer formalising layer to the inner property noticing layer of understanding. At this layer, she began to identify distinctions and connections between key constructs. For instance, her utterance, *"Tyler paid a deposit already of 11%"*, indicates that she was considering the deposit as a partial payment rather than part of the loan. She further recognised that *"…the bank charges the interest on the money that Tyler is owing, not the price of the sound system"* (line 13.1).

In addition, Karabo's remark that *"it is not like Tyler went to the store empty-handed"* reflects her awareness of the relationship between the cost price, the deposit, and the loan amount, further confirming that she was operating at the property noticing layer. This description reflects the learner's ability to make sense of the problem through repeated reading, as noted by Al-Mutairi and Marzouq (2025). In this way, Karabo conceptualised the deposit as a non-interest-bearing payment made towards the sound system. Her form of folding back can be classified as collecting at an inner layer, as she retrieved prior knowledge about deposits and applied it to the demands of the current mathematical problem. Her mathematical actions are consistent with Yao and Manouchehri's (2022a) description of property noticing, where the learner is able to recognise distinctions, combinations, or connections between multiple mental images (p. 245).

As evident in line 13.3, Karabo treated the deposit as a critical element of the learning activity that needed to be accounted for when solving the problem. She strongly suggested that the group should not use R7 999 as the principal amount (P) for the hire purchase, but rather calculate the difference between the purchase price (R7 999) and the 11% deposit. In doing so, Karabo recognised that the group had overlooked the relationship between the deposit and the borrowed amount, and how this relationship informed the calculation process. This shift indicates that she moved from procedural awareness towards conceptual justification by developing a generalised understanding of the problem.

Karabo's mathematical actions prompted Tshepo to fold back from the formalising layer to the image having layer, where he began to form a clearer mental image of the concept. This is evident in Tshepo's utterance, *"Ohooo, I see it now"* (line 13.4), which suggests that he was able to recognise the properties of the image he had constructed using the new information introduced by his peer. His utterances further suggest the learner's ability to utilise mental constructs to grasp mathematical ideas rather than relying solely on physical activities (Jannah, 2023; Putri, 2024). In this instance, peer explicit intervention facilitated effective folding back. This episode illustrates the learner's ability to apply mental representations to solve the task, as described by Arnal-Bailera and Manero (2024). At the image having layer, Tshepo engaged in collecting as a form of folding back, extending his understanding through the construction of new meanings. As a result, he was able to recognise that the loan cost represented *"…the shortfall"* and that they needed to determine *"how much Tyler will pay back to the bank"*, as shown in Vignette 13. In agreement, Karabo further clarified that they *"should use R7 119.11 as P"* when determining the total amount Tyler would repay, as illustrated below.

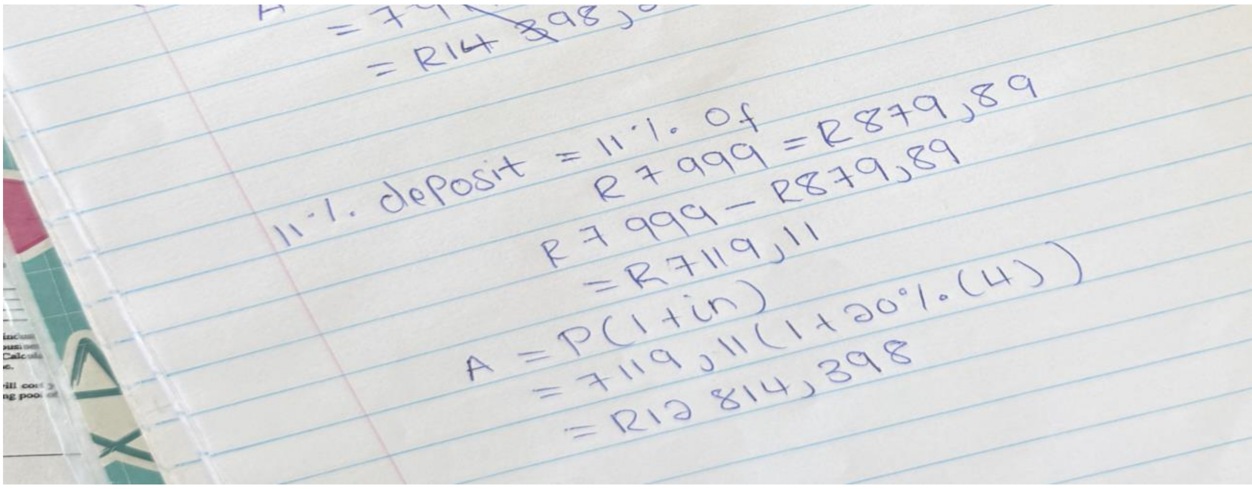

Figure 4.22: Karabo's second attempt

Therefore, folding back prompted by peer intervention is deemed effective in this vignette, as learners were able to return to earlier layers of understanding and apply their extended comprehension of the implications of the deposit on the hire purchase agreement to determine the amount Tyler should pay. Learners' algebraic actions are consistent with Al-Mutairi and Marzouq's (2025) assertion of *"expanding the solution by generalising the general pattern of the simple interest formula"* (p. 9).

In the subsequent Learning Activity 9, learners were required to apply their knowledge of hire purchase agreements.

4.

BRAND NEW HISENSE washing machine **ONLY** for R39845 cash price.

OR

Payable over a hire purchase agreement:

10% deposit and the value of the 36 monthly equal Instalments to the value of R44 467,02

4.1. Write down cost price of the displayed washing machine.

4.2. Given Mrs Mothipa takes the washing machine on the hire purchase agreement:

a) Calculate the total Mrs Mothipa would have to pay.

b) How much interest would she pay in total?

Figure 4.23: Learning activity 9

An expected response from the learners is presented below:

4.1. $R39\,845$

4.2.

$$(a)\ R39\,845\ \times \frac{10}{100} = R3\,984{,}50$$

$$R44\,467{,}02 +\ R3\,984{,}50 = R48\,451{,}52$$

$$(b)\ R44\,467{,}02 - R39\,845 = R8\,606{,}52$$

The interactions among the learners, Queen, Tebatso, and Temoso, are captured in Vignette 14 below:

Vignettes 14

14.1.	Temoso:	How can we approach this?
14.2.	Tebatso:	I say we approach this by taking the washing machine price and subtracting the deposit.
14.3.	Temoso:	Yes, I agree (*continues to writing*). R39845 – R3984,5 = R35 860,5.
14.4.	Teacher:	Read the hire purchase agreement.
14.5.	Temoso:	10% deposit and the value of the 36 monthly instalments to the value of R44 467,02.
14.6.	Teacher:	What do you think that means?
14.7.	Temoso:	Haiii ma'am the English. I do not understand.
14.8.	Teacher:	Interpret it in your home language perhaps it will make sense.
14.9.	Temoso:	For hire purchase agreement, Mrs Mothipa *o tla patala* 10% deposit le di instalment *ka kgwedi tse 36 tse di tlo irang* R44 467,02.
14.10.	Teacher:	So, in total, how much will Mrs Mothipa pay?
14.11.	Temoso:	Washing machine is R39845 and deposit ke R3984,5 therefore, we will add the two amounts together instead of subtraction.

In the opening line of this vignette, Temoso's question, *"How can we approach this?"*, serves as an invitation for peer contributions towards solving the learning activity. In response, Tebatso's suggestion that they should *"…approach this by taking the washing machine price and subtracting the deposit"* implies that she is reasoning from the image having layer of understanding. She appears to hold a mental image that the total cost of the washing machine under a hire purchase agreement should account for the deposit paid, although at this stage she applies this idea incorrectly. She links the washing machine price and the deposit through subtraction, implying an equation that determines the difference between the two values.

This error can be classified as a transformation error, as described by Pomalato et al. (2020), which occurs when learners do not fully understand how to translate a contextual problem into appropriate mathematical representation. In such cases, learners may apply mathematical procedures without sufficient consideration of their relevance, or use rules and concepts that do not align with the problem at hand, as reflected in Tebatso's reasoning. Temoso, on the other hand, appears to follow Tebatso's suggestion without critically engaging with it conceptually. This is evident in his agreement in line 14.3, followed by a mathematical computation that mirrors Tebatso's reasoning, as illustrated in Figure 4.24 below.

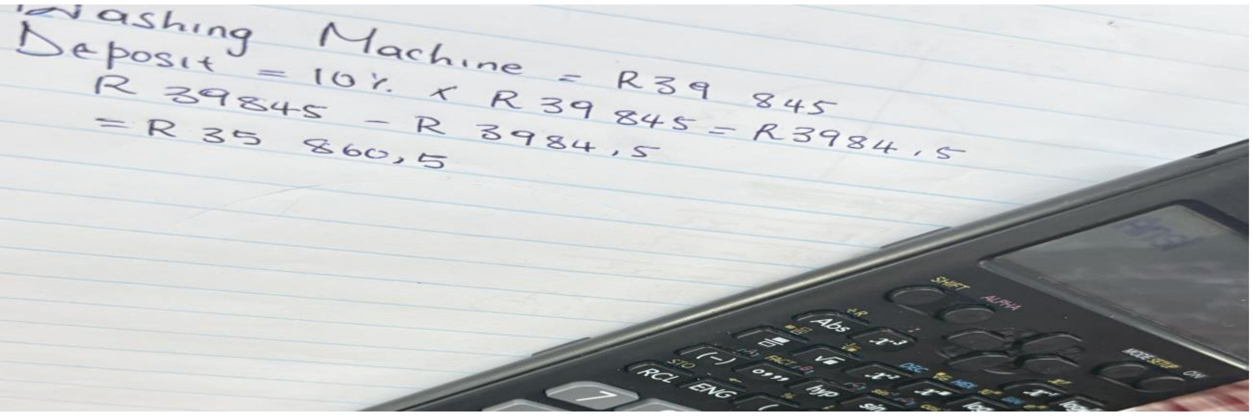

Figure 4.24: Temoso's calculation

Noticing Tebatso's influence on her peers (line 14.3), the teacher's statement in line 14.4 was intended to prompt Temoso to actively reconstruct his understanding of the hire purchase agreement. However, the teacher's prompt was initially ineffective, as Temoso was able to read and articulate the statement from the agreement (line 14.5) but could not respond to the conceptual question about its meaning. When asked, *"What do you think that means?"*, his reasoning shifted towards a language-related challenge, as evidenced by his utterance, *"Haiii ma'am, the English. I do not understand."* His willingness to articulate this difficulty suggests an awareness that being asked to reread the statement was purposeful and that he may have overlooked important information.

To bridge the gap between language and comprehension, the teacher encouraged Temoso to translate the statement into his mother tongue (line 14.8). This integration of the learner's home language proved effective in supporting his engagement, as reflected in line 14.11. As a result, Temoso began to move away from his initial line of reasoning and to establish his own understanding. He was able to articulate the mathematical operation required, suggesting that they should *"…add the two amounts together instead of subtraction."* This shift from subtraction to addition indicates that he was reconstructing meaning using mental images, situating him at the image having layer of understanding, where she is now able to carry a mental plan for these activities with them and use it accordingly (Martin & Towers, 2016). At this stage, Temoso considered the deposit and the total cost of the machine as complementary quantities based on his emerging understanding. This demonstrates his developing ability to internalise mathematical relationships meaningfully. The interactions then continued as follows.

Vignettes 15

15.1.	Teacher:	So, are you saying you will pay an interest of your own deposit (R39845 + R3984,5 = R43 829,5) yet you are expected to pay 24 monthly instalments to the value of R44 467,02 with the deposit? Are the numbers balancing?
14.13.	Temoso:	Was I supposed to add R44 467,02 and the deposit? *pauses* That is what the statement is saying.
14.14.	Tebatso:	Ma'am, I think we should have added R44 467,02 and R3984,5.
14.15.	Teacher:	Why?
14.16.	Temoso:	That is what the statement is saying.
14.17.	Tebatso:	It is the stated hire purchase agreement. It clearly outlines the money to be paid back.
14.18.	Teacher:	Absolutely.

The teacher further challenged Temoso's reasoning by asking whether he *"…will pay interest on your own deposit?"* (line 15.1). This question functioned as a focused and intentional intervention aimed at helping Temoso recognise the inconsistency between his suggestion and the structure of the hire purchase agreement. In addition, the teacher's follow-up question, *"Are the numbers balancing?"* (line 15.1), encouraged the learner to evaluate the reasonableness of his solution by comparing the calculated amounts. Specifically, adding R39 845 and R3 984.50 results in R43 829.50, which does not align with the stated *"…36 monthly instalments to the value of R44 467.02."*

Temoso's response, *"Was I supposed to add R44 467.02 and the deposit?"*, can be interpreted as an act of self-reflection, where he paused to reconsider his approach in light of the problem's requirements. At this point, teacher intervention is clearly evident. Building on the interaction between Temoso and the teacher, Tebatso suggested that they *"should have added R44 467.02 and R3 984.50"*, creating space for further clarification and correction. Tebatso's contribution appears to have been influenced by Temoso's evolving reasoning, in contrast to the earlier stages of the vignette where the influence flowed in the opposite direction. This shift suggests that Tebatso engaged in self-reflection while observing the teacher–Temoso interaction, recognising that her earlier suggestion (line 14.2) did not fully align with the hire purchase context.

Tebatso's utterance in line 14.14 indicates an emerging understanding that the total payment comprises both the deposit and the instalments, reflecting a self-initiated fold back to the property noticing layer. At this stage, both learners demonstrated a clearer understanding that a hire purchase agreement includes the deposit and the instalments as components of the total repayment. This progression illustrates how teacher intervention can support learners in reviewing and refining their mental images. Consequently, the teacher's actions align with the findings of Yao and Manouchehri (2020b), who note that teacher intervention facilitates image review, enabling learners to reflect on their mathematical activity and thereby deepen their understanding.

4.5 FIELDWORK 3: USING ALGEBRAIC THINKING TO DETERMINE THE MONTHLY REPAYMENTS OF A LOAN

4.5.1 Classroom Episode 1: Application of algebraic reasoning in exploring relationship between the time period, interest, and monthly payments

The following classroom episode focused on calculating the total loan cost and distributing it into manageable monthly payments. Learners were required to explore the relationships between the interest rate, the repayment period, and the monthly instalments. Prior to these fieldwork activities, learners had been exposed to tasks that required them to determine the values of A, P, n, and i. As such, their prior experiences provided a solid foundation for understanding the concept of monthly loan repayments. Within this problem-based learning environment, the teacher assumed the role of a facilitator, encouraging learners to participate actively and take responsibility for their own learning. This type of classroom setup not only positions learners as constructors of mathematical knowledge, but also promotes conceptual and relational understanding of mathematical concepts, which is recommended in learning environments that support folding back.

This classroom episode specifically aimed to distribute the total amount payable for a loan into equal monthly instalments. Learners were required to examine the relationships between the interest rate, the repayment period, and the monthly payments. The learning activities were inspired by the *Mind Action Series* Grade 10 textbook (2021) but were adapted to include real-life contexts that learners could relate to. In addition, Learning Activities 10 to 12 were sampled for this section of the data analysis. The conversations learners engaged in while attempting these learning activities are presented below in the form of vignettes, as they constitute the central focus of the analysis.

Learning Activity 10 required learners to apply their knowledge of hire purchase agreements, among other concepts, to determine the total amount payable and the monthly instalment, as shown below.

2. Gugu buys a double bed which costs R12 000 on hire purchase. She is charged a simple interest of 12%p.a. over six years.

2.1 Calculate the total amount she will pay for the double bed.

2.2 How much interest will she pay over this period?

2.3 Calculate her monthly instalment.

Figure 4.25: Learning Activity 10

An expected response from the learners is presented below:

6.1.1.

$$A = P(1 + in)$$
$$A = 12\ 000\big(1 + 0{,}12(6)\big)$$
$$= R20\ 640$$

6.1.2.

$$R20\ 640 - R12\ 000 = R8\ 640$$

6.1.3.

$$\frac{R20\ 640}{72} = R286,67$$

The learners were able to determine the total amount payable and the interest accrued over the duration of the hire purchase agreement. However, they were unable to immediately calculate the monthly instalment, as illustrated in Vignette 15 below.

Vignette 16

16.1.	Tebogo:	Here we are required to find the monthly repayments for the loan.
16.2.	Nelly:	Yes, but we do not know how many months is 6 years.
16.3.	Portia:	But we can get that by multiplying 6 with 12.
16.4.	Tebogo:	Agreed. (*All three proceed to doing so*)
16.5.	Nelly:	Portia, how much money is R0,00348837209?
16.6.	Portia:	It is the monthly repayments for the loan.
16.7.	Tebogo:	We are trying to find out how much Gugu will pay per month. How much money will Gugu pay back each month 72 times.
16.8.	Nelly:	Meaning at the end the money should add up to the total money Gugu owes which is R20 640.
16.9.	Portia:	Oohh, I was supposed to divide R20 640 with 72 not vice versa.
16.10.	Tebogo:	Yes, we are working with money here. It must be tangible. Therefore R286,6666667 per month for 72 months makes sense than R0,00348837209 per month. Your answers should make sense to you. Understand your calculations.

This vignette illustrates learners' attempts to make sense of the problem. In this instance, Nelly's acknowledgement that "they do not know how many months is six years" indicates underdeveloped prior knowledge regarding the conversion of years into months. Although she recognises that the calculation should be based on months, her mathematical actions suggest that she is attempting to interpret the situation through speculation rather than established mathematical facts. This reflects engagement at the primitive knowing layer.

In contrast, Portia's assertion, supported by Tebogo, demonstrates an understanding that the number of months can be determined by multiplying the number of years by twelve. This observation provides evidence that they were able to identify a valid mathematical rule, which can be attributed to the property noticing layer of understanding. Their ability to recognise the multiplicative relationship and apply it to the problem suggests that they were reasoning at this layer. These actions are consistent with Yao and Manouchehri's (2020a) description of property noticing, which emphasises learners' capacity to recognise connections between multiple mental images.

Despite Portia's ability to determine the correct number of months at the property noticing layer, she was unable to apply this information accurately to calculate the monthly payment, as illustrated in Figure 4.26 below.

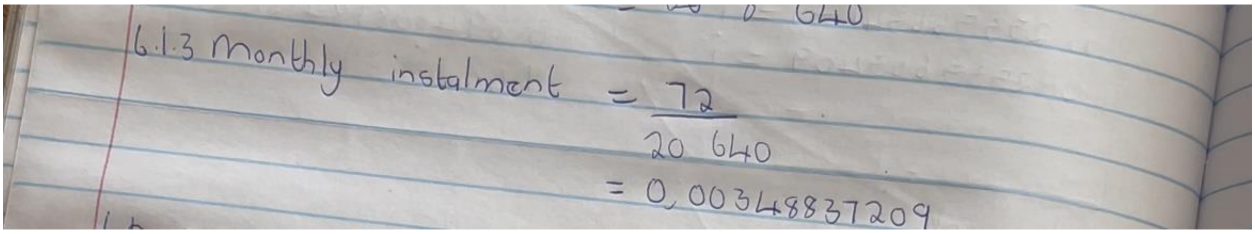

Figure 4.26: Portia's attempt

Portia's attempt prompted Nelly to question her by asking, *"How much money is R0.00348837209?"* It is important to note that Nelly's question does not necessarily reflect a lack of understanding of the numerical value obtained by Portia, but rather concerns its practicality. One possible explanation is that the amount calculated by Portia did not make sense in a real-life context of monthly repayments. Nelly appeared to be forming a mental image of money based on everyday experience and was therefore questioning the validity of such a small amount. Indeed, it would not be realistic to pay R0.00348837209 as a monthly instalment.

Portia's response, *"it is the monthly repayments for the loan"*, suggests that she did not fully grasp the concern raised by her peer. This prompted further explanation from the group, with peers noting that the total of the monthly repayments should amount to R20 640, which represents the total amount owed by Gugu, as reflected in lines 15.7 and 15.8. In this instance, unintentional peer intervention occurred, as the explanations offered by her peers did not directly point out the specific algebraic error, but nevertheless prompted Portia to reconsider her reasoning. This interaction led Portia to fold back from the property noticing layer to the formalising layer of understanding. At this stage, she demonstrated the ability to evaluate her solution and identify her error, as evidenced by her realisation, *"Oohh, I was supposed to divide R20 640 by 72, not vice versa."* Her use of the phrase *"not vice versa"* indicates awareness of having interchanged the numerator and denominator in her initial calculation, which resulted in an incorrect and contextually meaningless answer.

This moment reflects Portia's growing awareness that calculating monthly repayments requires dividing the total amount owed by the number of months in the loan term. Such reasoning aligns with Chuene et al.'s (2023) assertion that learners operating at the formalising layer are able to explain their solutions using appropriate mathematical language and recognise connections between concepts. In addition, Portia's mathematical actions resonate with Al-Mutairi and Marzouq's (2025) notion of evaluating solutions through verification.

Recognising Portia's self-correction, Tebogo further clarified the contextual meaning of the problem with the intention of helping Portia avoid similar errors in future calculations. This is reflected in her comment, *"your answers should make sense to you; understand your calculations"* (Tebogo, semi-structured interview). The learners' ability to attach meaning to their solutions was also corroborated during the interviews, where one learner stated:

> *You'll also be able to judge your own attempts like for instead, you cannot have an investment where eliminating the value of P is greater than the value of A so if I solve for P and find money that is greater than the starting amount then I know I did not do the right thing entirely* (Phuti, semi-structured interview).

As such, learners' mathematical actions exemplify the benefits of a problem-based classroom, where learners actively interrogate and check one another's reasoning. This fosters the understanding that mathematics is not solely about obtaining correct answers, but also about making sense of solutions and their contextual meaning (Novikasari, 2020). In the subsequent Learning Activity 11, learners Bonolo, Phuti, and Yolanda engaged with the concept of equal monthly payments for an item purchased on credit.

An iPhone 14 Pro Max is offered on a monthly payment plan as shown below:

a) If a customer was to take up this offer, calculate the amount they will pay in total for the phone over the 36 months period.

b) If the cash price of the iPhone 14 Pro Max is R31 560, calculate the total amount of interest paid over the 36-month period.

c) Determine the percentage increase from the cash price to the total amount paid on the monthly plan.

Figure 4.27: Learning Activity 11

An expected response from the learners is presented below:

$$\text{R1 029} \times 36 = \text{R37 044}$$

The following vignette is shaped by the conversations that unfolded among the learners as they attempted the learning activity:

Vignette 17

(*Phuti's cancelled attempt*)

17.1.	Teacher:	Why did you cancel your attempt here (points at Phuthi's cancelled attempt). You had divided 1029 with 36 months for the total amount to be paid for the iphone 14 pro max.
17.2.	Phuti:	I made a mistake ma'am
17.3.	Teacher:	How did you see it was a mistake?
17.4.	Phuti:	The reason I saw it was a mistake it was because I got 28 comma this number (*shows me a calculator with 28,58333333*) and I was like, how can you buy an iPhone with such a small amount. Mara, only per month is this amount of R1029? So, I was like, oh okay no no, something is wrong.
17.5.	Teacher:	Oh okay, so that is what made you think that it is incorrect?
17.6.	Phuti:	Yes (*Turns to Bonolo and Yolanda*)
17.7.	Bonolo:	Same ma'am
17.8.	Yolanda:	Me too ma'am (while hiding her attempt), that is why we were saying to you initially that we are not yet done. Mara we don't know what to do to get to the answer even when we understand what the question is asking. Please help us.
17.9.	Teacher:	Okay now let us form a better understanding, this R1029, is the money for how many months?
17.10.	Phuti and Bonolo:	Per month.
17.11.	Teacher:	Per month neh? Do we refer to a single month as 1 or 2?

17.12. Bonolo: One.
17.13. Teacher: It is one, good, so now the question is; calculate the total amount to be paid over, how many months?
17.14. Yolanda: 36 months.
17.18. Phuti: Oh, so you'll say R1029 multiply by 36 months.
17.19. Bonolo: Oh yes, R1029 for 36 months.
17.20. Teacher: Precisely.

In this vignette, the teacher observed that Phuti cancelled his initial attempt. Attention was therefore drawn to his reasoning for doing so. His original attempt showed that he divided 1029 by 36 months to obtain the total amount to be paid for the iPhone 14 Pro Max. Having recognised that this approach was incorrect, he chose to attempt the problem differently. In this case, Phuti's mathematical action can be classified as self-invoked folding back. His decision was not influenced by teacher, peer, or material intervention, but rather by his own realisation. This aligns with Martin's (2008) description of self-invoked folding back, where he notes that *"some learners do possess a natural ability to fold back in response to becoming self-aware of the limitations of their present understandings"* (p. 83). This indicates a shift in Phuti's thinking. The initial route he followed appears to have been influenced by reasoning at the formalising layer, where he recognised that the question required the total amount to be paid for the cellphone after 36 months but incorrectly associated this requirement with an inappropriate mathematical operation (Figure 4.28). His action reflects what Yao and Manouchehri (2020b, p. 20) describe as *"the abandonment of the origins of one's mental action"* at the formalising layer of understanding.

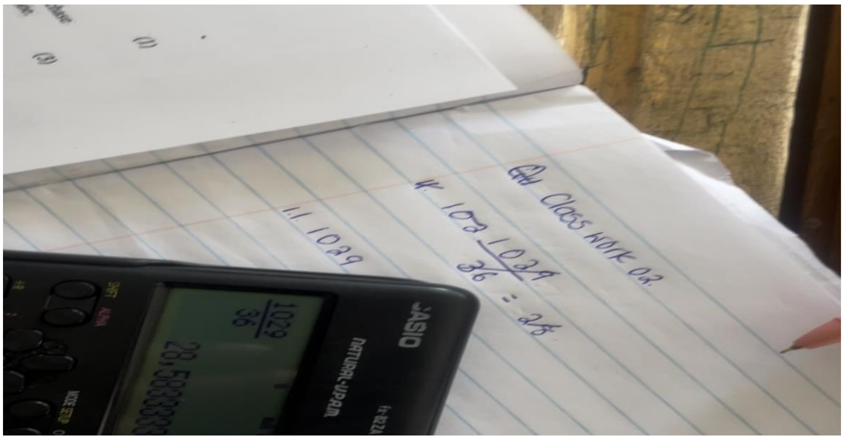

Figure 4.28: Phuti's cancelled attempt

Moreover, the abandonment of Phuti's initial idea was also influenced by his ability to evaluate the obtained value against what he knows to be realistic in everyday life. This was evident in his self-question, *"…how can you buy an iPhone with such a small amount…"*, which indicates his recognition that the R28 obtained could not reasonably represent the cost of an iPhone 14 Pro Max. His reasoning demonstrates an ability to make sense of mathematical computations by relating them to familiar real-life contexts. Additionally, Phuti was not the only learner who experienced difficulty in determining the amount to be paid, as similar challenges were expressed by his peers (lines 17.7 and 17.8). Yolanda, for instance, acknowledged that they *"don't know what to do… even when we understand what the question is asking"*, suggesting that understanding the question does not automatically translate into knowing how to implement appropriate solution strategies. Although learners demonstrated awareness of what the question required, they lacked the necessary computational strategies. In this instance, Yolanda can be located at the image having layer, as she displayed a general understanding of the task without procedural clarity. Consequently, the learners sought assistance from the teacher (line 17.8), reflecting a learning environment characterised by folding back, where learners recognise their limitations and actively seek support. This was corroborated in interview data, where one learner noted, *"I am able to seek help if I do not understand a certain question because with the topic we were doing, I actually realised that I at times struggled to grab what exactly they were asking us to solve"* (Learner 2). In responding to the learners' request for assistance, the teacher did not provide a direct answer but instead employed questioning techniques aimed at helping learners form a clearer understanding. The teacher's questions (lines 17.9, 17.11, and 17.13) functioned as explicit intentional interventions, directing learners' attention to the meaning of paying R1029 per month over a period of 36 months. This guidance supported learners in reconstructing their understanding of the structure of the problem and prompted folding back to the property noticing layer. At this layer, learners began to recognise the relationship between a single monthly payment and the total repayment period, enabling them to reason meaningfully within the context of the task (lines 17.18 and 17.19). Phuti's explanation clarified the appropriate mathematical computation that had initially been overlooked, illustrating how effective folding back, prompted by teacher intervention, facilitated the reconstruction of learners' thinking and led to successful problem-solving. This finding aligns with Mabotja's (2017) study, which highlights the role of intentional teacher intervention in supporting learners' reconstruction of mathematical ideas and the development of nuanced understanding.

1. Reabetswe decides to buy a home theatre system which costs R9 500 on a hire purchase agreement.
   i. If Reabetswe pays a 25 % deposit, calculate the balance after Reabetswe has paid the deposit.
   ii. The store charges Reabetswe 20 % interest per year, with the loan payable over 4 years. Calculate the total amount that he will have to pay back.
   iii. Reabetswe is also required to pay a monthly insurance of R30 and an administration fee of RI2. How much will Reabetswe's monthly repayments be?

Figure 4.29: Learning Activity 12

An expected response from the learners is presented below:

12.1.

$$R9\,500 \times 0{,}25 = R2375$$

$$R9\,500 - R2375 = R7125$$

12.2.

$$A = P(1 + in)$$

$$A = R7125\big(1 + 0{,}20(4)\big)$$

$$A = R12\,825$$

12.3.

$$\frac{R12\,825}{48} = R267{,}1875$$

$$R30 + R12 = R42$$

$$R267{,}1875 + R42 = R309{,}19$$

The learners' interactions among Tshepiso, Teko, and Lukhanyo, as they worked through the learning activity, are captured in the following vignette.

Vignette 18

18.1.	Lukhanyo:	There is actually an administration fee of R30 and insurance of R12. So I am suggesting that we add those values together before.
18.2.	Teko:	The formula for monthly instalments is equals to total amount owed over months.
18.3.	Tshepiso:	But here we do not have the number of months.
18.4.	Lukhanyo:	Let us put it as x. (*proceeds to making an equation*)
18.5.	Teko:	The value of x is 305,1 this is a long time, there is no way. Also, we did not answer the question.

18.6.	Lukhanyo:	Yes, this is because we were still finding the months first so we can see how much the monthly repayments will be but ey 305,1 I have never seen before. It is a lot.
18.7.	Tshepiso:	Aii ma'am we are stuck, our answer is incorrect and as such cannot use it. For monthly repayments.
18.8.	Teacher:	Okay, I see here you were trying to solve for x, why?
18.9.	Lukhanyo:	It is because we do not have the number of years hence, we cannot calculate the monthly repayments.
18.10.	Teacher:	Did you read the statement well?
18.11.	Lukhanyo:	Yes.
18.12.	Teacher:	Please re-read it.
18.13.	Lukhanyo:	The store charges Reabetswe 20 % interest per year with the loan payable over 4 years.
18.14.	Tshepiso & Teko:	Ohhh
18.15.	Lukhanyo:	48
18.16.	Tshepiso:	48 months (*Continues to replacing* x *with 48*).

In approaching the mathematical task, the group noted that the *"number of months"* was missing and therefore needed to be determined. Lukhanyo, in line 18.1, acknowledged that although the number of months was unknown, there were additional amounts that, in his view, contributed to the monthly repayments and thus had to be considered. His suggestion to add the values, as evident in line 18.1, indicates that he was reasoning at a property noticing layer of understanding. He recognised that the total monthly payment should include *"an administration fee of R30 and insurance of R12,"* demonstrating an awareness of how different components in a financial mathematics context combine to form a total cost.

Furthermore, his proposal to represent the missing number of months as an unknown, *"Let us put it as x,"* supports the claim that he was operating at the property noticing layer, as he was able to link the symbol x to an unknown quantity. Through this suggestion, Lukhanyo addressed both Tshepiso's concern about the missing number of months and Teko's reference to a formula (line 18.2), thereby integrating multiple ideas into a coherent approach. Tshepiso, on the other hand, appeared to contribute while operating at a formalising layer of understanding. This is evidenced by his ability to relate the concept of monthly payments to a general mathematical structure, stating that "monthly instalments are equal to the total amount owed divided by the number of months." He demonstrated the capacity to express this relationship using mathematical language and symbolic representation, reflecting a structured understanding of the relationship between total cost, time, and monthly repayments, as illustrated below.

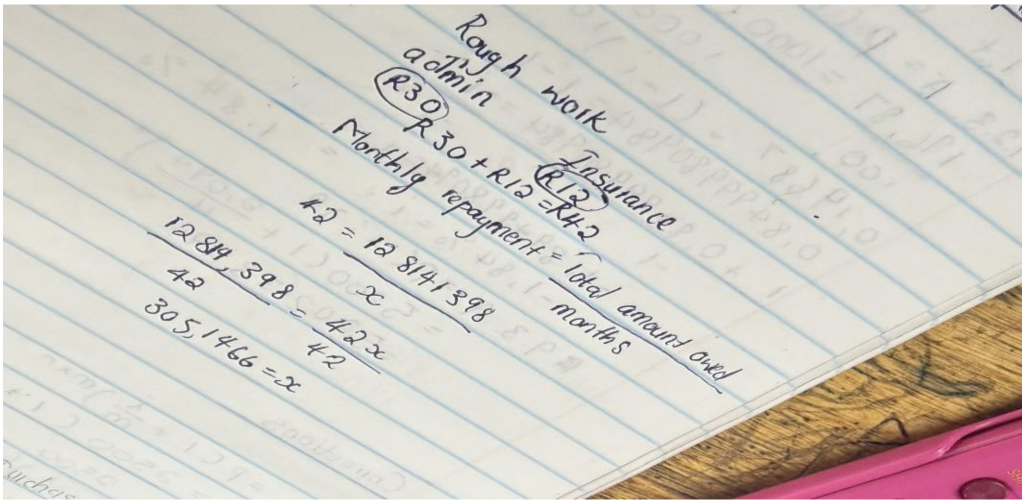

Figure 4.30: Teko's attempt

However, the answer obtained raised concerns for Teko, as he realised that *"…305,1… is a long time"* for the number of months in the context of the question they were working on. In line 18.5, Teko acknowledged that the mathematical computations performed were not only erroneous but also *"…did not answer the question."* He appeared to be fully aware that the value of x they had just solved for was merely a stepping stone towards addressing the main question of monthly payments, further supporting the claim that he was reasoning at a formalising layer. Lukhanyo seemed to agree with Teko by pointing out that the calculations they had completed were for *"…finding the months first so we can see how much the monthly repayments will be…"*. He further concurred that the number of months obtained was unrealistic, stating that *"…305,1… is a lot,"* which suggests that this result prompted him to return to the image, forming a mental representation based on his experience of what constitutes a reasonable number of months in a financial mathematics context. In recognising that the group had reached an impasse by merely highlighting that 305,1 was an implausible value without proposing a way forward (lines 18.5 and 18.6), Tshepiso invited the teacher into the discussion by stating, *"Ma'am, we are stuck,"* and further acknowledging that *"our answer is incorrect and as such cannot use it for monthly repayments."* Tshepiso's response demonstrates an important awareness that an incorrect value for the number of months would inevitably result in an incorrect calculation of monthly repayments, an insight that is crucial in a folding-back classroom where self-reflection plays a central role in developing strong algebraic thinking. Upon joining the discussion, the teacher did not immediately direct the learners to the concept they had overlooked but instead showed interest in their reasoning by asking, *"…you were trying to solve for x, why?"* This 'why' question built on what the learners had already written and elicited insight into their mathematical thinking. Lukhanyo's response, *"…we do not have the number of years, hence we cannot calculate the monthly repayments,"* revealed his awareness of the role of time in determining monthly payments, although his interpretation still relied on noticed properties rather than a fully contextualised understanding. From this response, the teacher redirected the group by asking Lukhanyo to re-read the statement, and it was this action that triggered a cognitive shift in Tshepiso's and Teko's initial understanding. The ensuing *"ohhh"* moment can be linked to their realisation that the information they believed to be missing was, in fact, embedded in the problem statement. This is evident in Lukhanyo's utterance "48" and Teko's clarification *"48 months,"* indicating that re-engagement with the text prompted Lukhanyo to fold back to the image having layer, where he reasoned using a mental construct that four years can be converted into months by multiplying by twelve, while simultaneously prompting Teko to operate at a formalising layer, where he was able to clearly and contextually establish the relationship between time and monthly repayments.

4.6 FIELDWORK 4: INTERPRETING EXCHANGE RATES AND FINANCIAL TIMELINES THROUGH REASONING ALGEBRAICALLY

4.6.1 Classroom Episode 1: Application of algebraic reasoning to analyse financial mathematics situations involving exchange rates and timelines

This fieldwork consists of one classroom episode that explores two concepts within the Grade 10 financial mathematics context. The first focus is on understanding the implications of exchange rates, particularly in relation to the South African rand. The second focus addresses how invested money grows, taking into account that it does not remain fixed and is influenced by multiple deposits and withdrawals. The learning activities were inspired by the Mind Action Series Grade 10 textbook (2021) and several YouTube videos. A total of two learning activities (13 and 14) were sampled for reporting purposes. Learners were expected to work in groups when engaging with the learning activities during the afternoon sessions, drawing on their prior experiences, as both concepts are closely linked to real-life situations. Learners' interactions and discussions as they worked collaboratively were considered central to the study and are therefore captured in the vignettes.

Learning activity 13 focused on exchange rates between different countries and was designed as illustrated in Figure 29:

1.

In June 2019, the pound to rand exchange rate was £1 = R18,18. Zola, travelled to the United Kingdom to watch some WWE wrestling matches. The total cost needed for the trip was £3 569. Convert this amount into rands.

Figure 4.31: Learning Activity 13

An expected response from the learners is presented below:

$$1 \text{ pound} = \text{R}18{,}18$$

$$3\ 569 \text{ pound} = \text{x}$$

$$\therefore \text{x} = \text{R}64\ 884{,}42$$

Vignette 19 captures a key moment involving Karabo, David, and Tshepo as they worked collaboratively through the learning activity.

Vignette 19

19.1.	Karabo:	Ehhh now we are working with pounds.
19.2.	David:	The one pound= R18,18 is a conversion rate we must use for this question.
19.3.	Karabo:	Yes, let us do it individually.
19.4.	Tshepo:	I agree.
19.5.	David:	I am getting R64 884,42 and you guys?
19.6.	Karabo:	Same.
19.7.	Tshepo:	Wait, I am getting R196,3.

In this vignette, the learners' initial actions highlight their attempt to understand the problem by identifying known and unknown information. In doing so, they recognised that the learning activity required them to work with conversions between dollars and rands, as evident in lines 18.1 and 18.2, respectively. This constitutes an act within the image having layer of understanding, where learners form a general overview by conceptually grasping exchange rates as a central aspect of the task. This interpretation aligns with Yao's (2020) description of the image having layer, which is characterised by carrying out specific mathematical actions guided by a general mental plan.

Having identified the given information, Karabo's suggestion that they "do it individually" highlights the value she places on equal participation and shared responsibility, which was supported by her peer in line 19.4. In this regard, Karabo views individual contributions as an important feature of problem-based learning within a collaborative environment. Her utterance reflects confidence in her peers' ability to apply the algebraic reasoning skills required to analyse and solve problems involving exchange rates. Although she does not prescribe a specific solution strategy, her action aligns with Al-Mutairi and Marzouq's (2025) notion of implementing a solution by first identifying the meaning of symbols and quantities involved. After working on the task individually, David reported that he obtained an answer of R64 884,42 and invited his peers to share their solutions (line 19.5). This mathematical action suggests that, although learners employed individual strategies, they recognised the importance of sharing and discussing their results with group members. In a learning environment characterised by folding back, such sharing enables learners to learn from one another by comparing strategies and reflecting on different approaches to problem solving. This trend was also evident in the interview data, where learners expressed positive views about the value of sharing ideas during collaborative work. For example, one learner stated:

> "*It is actually a good way of learning mathematics in a sense that it allows us as individual learners to also learn of our peer's ways of thinking. It also helps us to stay open minded as there are many instances in which we had to listen in on what the other person was saying*" (Teko, semi-structured interview).

Among the possible explanations for the above expressions is the view that *"talking to someone who is my age made me feel less pressurised, and as my peers explained, the concepts became easier to understand"* (Teko, semi-structured interview). In a similar vein, another learner echoed that sharing ideas is *"quite helpful in many ways; what stood out for me was being able to learn from my peers. I learnt multiple ways of manipulating mathematical equations"* (Rudo, semi-structured interview). Consistent with these views, another learner also embraced the sharing of ideas, noting that it creates opportunities to challenge one's own thinking, as reflected in the following quotation:

> *I found myself having the urge to relate what my peers were saying with what I was thinking as a way of moving forward. It is almost like they challenge your way of thinking, to assess if you are sure about what you think you know, and that requires a higher level of thinking and reasoning, especially when it comes to understanding why you are choosing to approach a problem a particular way*. (Jane, semi-structured interview).

The above descriptions affirm the findings of an earlier study by Mabotja (2017), which revealed that sharing ideas promotes active learner involvement in the learning process. In sharing ideas, as evident in Vignette 19, Karabo also obtained an answer of R64 884,42. Thus, Karabo and David appear to have applied the mathematical understanding that the pound increases proportionally to the rand, which led them to associate the relationship with multiplication. These learners demonstrate awareness that the rand value should be greater than the pound value. By contrast, Tshepo's solution of R196,3 suggests that he identified an inversely proportional relationship and associated it with division. Although Tshepo appears to recognise that the rand value changes as the pound changes, he applies this relationship incorrectly, as illustrated below.

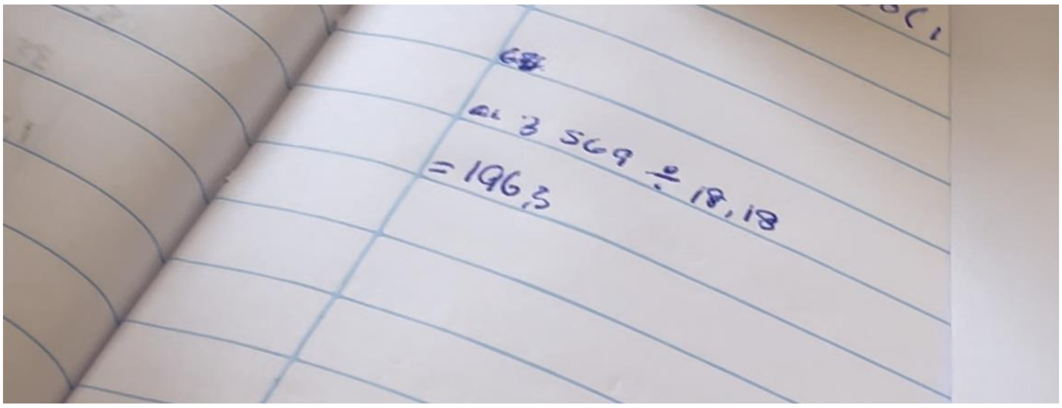

Figure 4.32: Tshepo's attempt

In noticing that he obtained a significantly smaller amount than his group members, Tshepo suggested that they *"wait…"*. This utterance implies a possible mental shift, as he appears to be consciously realising that something does not align with the expected outcome. As a result, the teacher intervened as follows.

Vignette 20

20.1.	Teacher:	I can see you divided 3569 pounds with R18,18. why is that?
20.2.	Tshepo:	I do not know why.
20.3.	Teacher:	Okay, the conversion rate is one pound= R18,18. How much do you think two pounds will be?
20.4.	Tshepo:	It would be R36,36.
20.5.	Teacher:	Can you show me how you got to the answer? Please write it down here for me.
20.6.	Tshepo:	(*Proceeds to writing*). I would say R18,18+R18,18 = R36,36.
20.7.	Teacher:	What about when I now have three pounds? What would the money be in rands?
20.8.	Tshepo:	I would say R18,18 times 3.
20.9.	Teacher:	Why are you saying R18,18 here (pointing at the two pounds conversion) and here you are multiplying with 3?
20.10.	Tshepo:	The thing is ma'am, I am lazy to keep saying plus-plus three times.
20.11.	Teacher:	Ohhh okay, so now you have 3569 pounds, how would you get to rands?

20.12.	Tshepo:	Eish, I would then say 3569 pounds times R18,18.
20.13.	Teacher:	Ohhh you will multiply with R18,18?
20.14.	Tshepo:	Yes.
20.15.	Teacher:	Why?
20.16.	Tshepo:	It is because ma'am I am trying to convert to rands. I cannot Add R18,18 3569 times.
20.17.	Teacher:	There you go.

In this vignette, the teacher does not provide the learner with the correct solution but instead uses probing questions to elicit his thinking process. Thus, explicit intentional teacher intervention, characterised by probing questions, is evident. The contention is that, by probing the learner's thinking, he may be able to evaluate his solution and recognise its inadequacies or shortcomings. In a folding back environment, this approach encourages learners to articulate their reasoning, enabling them to externalise their mental processes so that the teacher can make informed decisions about how best to support them. This description is consistent with Yao and Manouchehri's (2020b) assertion that teacher intervention through questioning techniques plays an essential role in advancing learners' mathematical understanding.

In response to the teacher's question, Tshepo acknowledged that he did "not know why" he divided 3569 pounds by R18,18. His inability to articulate his reasoning may stem from insufficient conceptual knowledge needed to make sense of the situation, thus revealing a gap between the mathematical manipulations he can perform and the algebraic reasoning skills required to justify those actions. In this regard, it can be claimed that Tshepo's primitive knowing layer of understanding is underdeveloped. As a result, the teacher continued to pose a series of guiding questions (for example, lines 20.3, 20.5, 20.7, and 20.9) to steer him towards a more appropriate solution strategy. In this instance, explicit intentional teacher intervention prompted the learner to fold back from the image having layer to the primitive knowing layer in order to reconstruct his understanding of exchange rates, particularly the role of multiplication and division.

The learner's actions demonstrate his ability to refine his understanding, as he was able to respond to questions related to the conversion rate (for example, lines 20.8, 20.10, and 20.12). For instance, his ability to relate two pounds to R36,36 using the conversion rate of one pound equalling R18,18 demonstrates the thickening effect of folding back, as he was able to return to the image making layer and answer the question correctly. He was also able to draw an analogy between addition and multiplication, recognising that repeated addition represents multiplication. This is supported by his statement in line 20.10, where he uses the structural property of repetition to justify multiplication rather than addition. It can therefore be argued that he was actively engaged in sense-making, attempting to restructure his mathematical manipulations through relevant algebraic reasoning skills. The teacher's intervention through questioning appears to have supported him in moving away from reliance on additive reasoning and towards an understanding of why multiplication is more appropriate in the context of exchange rates. This description resonates with the findings of Yao and Manouchehri (2020b), who found that targeted teacher questioning can direct learners towards specific knowledge that thickens their mathematical understanding.

The following Learning Activity 14 focused on timelines, where learners engaged with multiple payments and withdrawals made from the same investment account over a period of time.

a) Greg invests R9000 into an account. A year later, he deposits R7000. Three years after the initial investment, he withdraws R4000. Calculate the total amount that Greg will have after 5 years if the interest rate changes as follows:
- Year 1 and 2: 8% p.a compound interest.
- Year 3, 4 and 5: 9% p.a compound interest.

Figure 4.33: Learning Activity 14

An expected response from the learners is presented below:

$$A = P(1+i)^n$$

$$A = 9000(1+0{,}08)^2(1+0{,}09)^3 + 7000(1+0{,}08)^1(1+0{,}09)^3 - 4000(1+0{,}09)^2$$

$$= R18\ 632{,}72$$

The interactions among the learners, Jane, Tebello, and Rudo, proceeded as follows:

Vignette 21

21.1. Tebello: The interest of the R9000 put into the account is 8% for 2 years then it changed to 9% for 3 years.

21.2. Jane: Yes, here is a calculation for the R9000 for 2 years and when it changed for 3 years.

21.3. Tebello: Why are you separating the calculations because only one R9000 was deposited and when you are separating you are implying that two R9000 deposits were made.

21.4. Jane: Separate or not, we will get to the same conclusion.

21.5. Tebello: Okay we will see. Let us continue. (*Proceeds to working on the problem*)
21.6. Tebello: I got R18 632,71567.
21.7. Jane: Wait, I am almost done. (*moment of silence as Jane works on the problem*)
21.9. Jane: Okay done, I got R34 025,674.
21.10. Tebello: You see? I told you we would not get the same answer.

In this vignette, Tebello's initial action of reciting the contextual statement suggests that he intended to understand the given problem before attempting to solve it. His action highlights an algebraic thinking process characterised by repeated reading of the problem as a first step towards making sense of the task at hand, as underscored by Al-Mutairi and Marzouq (2025). Jane's confirmation of Tebello's reading does not necessarily imply that they understood the problem in the same way. On the one hand, Jane's utterance, *"here is a calculation for the R9000 for 2 years and when it changed for 3 years"*, although incorrect, demonstrates her attempt to break down a complex, multi-step problem into manageable components (Figure 4.35).

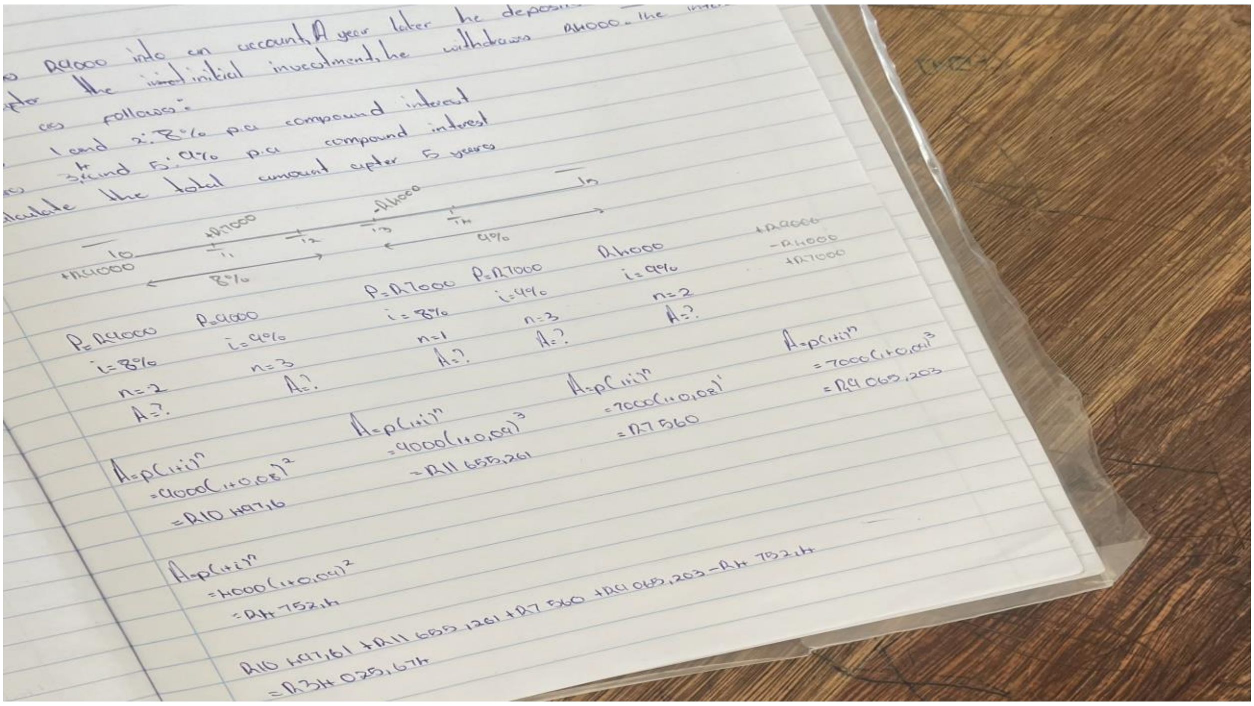

Figure 4.34: Jane's attempt

Thus, Jane understood the problem in a way that required separating the calculations into two years and three years, respectively. This description suggests that Jane demonstrates an act at the formalising layer of understanding by writing down a compound interest formula and explaining it through a breakdown of calculations. Jane's mathematical actions corroborate Chuene et al.'s (2023) assertion that the formalising layer is characterised by learners intentionally reflecting on and actively engaging with the properties they have noticed within a concept.

On the other hand, Tebello's question, *"Why are you separating the calculations?"* suggests that he does not necessarily require further explanation from his peer; instead, he considers Jane's attempt to separate the calculations as incorrect. Among the possible explanations for this reasoning is the view that *"...only one R9000 was deposited and when you are separating you are implying that two R9000 deposits were made"*. From his perspective, the real-life relationship between the money invested, its growth, and the accumulated amount implies that a single principal amount of R9000 should grow continuously at 8% and 9%, rather than being treated as separate deposits. His engagement reflects reasoning at the property noticing layer, as his focus is not solely symbolically driven but takes into account the existing relationships between quantities, as illustrated in Figure 4.34 on the next page. This description aligns with Hähkiöniemi et al.'s (2022) explanation of the property noticing layer, which emphasises learners' ability to identify similarities and differences between various images formed.

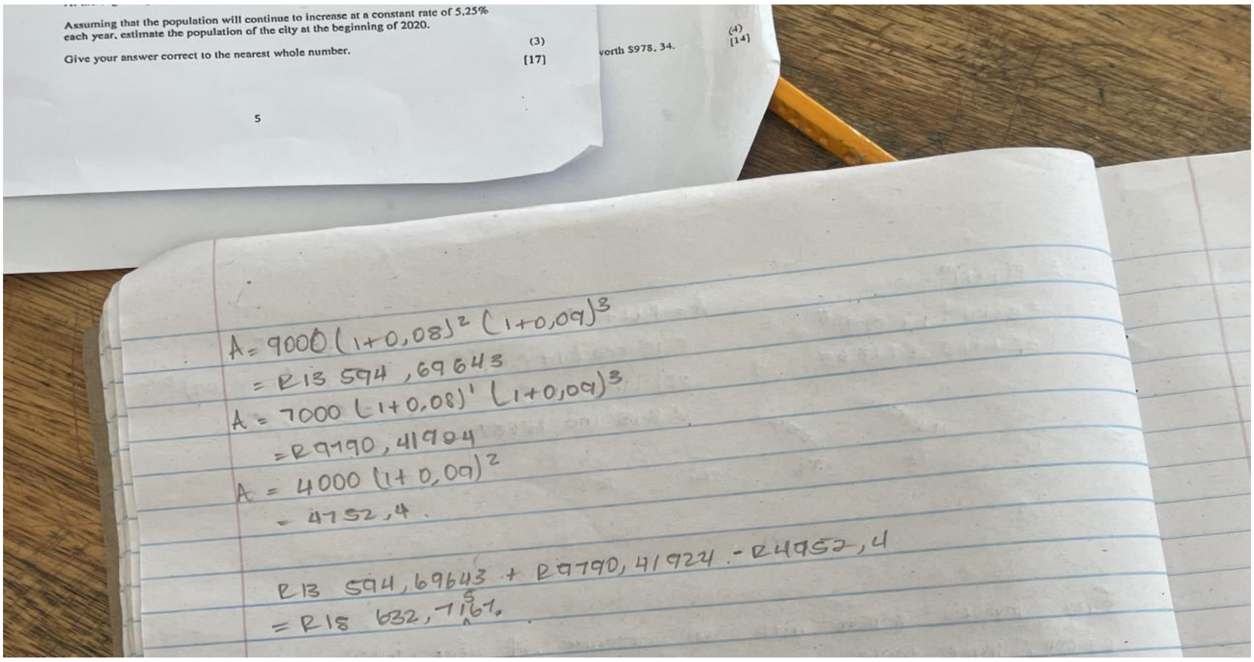

Figure 4.35: Tebello's attempt

Interestingly, in this vignette, ideas are contested and defended. Thus, both learners, despite differences in the strategies and solutions they employed, regard their respective actions as correct. They both appear confident in the algebraic manipulation skills applied up to this point, even though they interpreted the mathematical statement differently. In defending her strategy, Jane's utterance, "separate or not, we will get to the same conclusion," suggests that she is asserting her position and is likely unwilling to review her approach or consider an alternative strategy, despite not arriving at the same solution, as highlighted by Tebello (line 21.10). One might have anticipated that the learners would review their solutions; however, this did not occur.

Having noted this stalemate of ideas, both learners decided to seek the intervention of their teacher, and the interactions continued as follows.

Vignette 22

22.1	Tebello:	...Let's ask ma'am.
22.2.	Jane:	Ma'am, who is correct here?
22.3.	Teacher:	Who do you guys think is correct?
22.4.	Tebello and Jane:	Me.
22.5.	Teacher:	Okay let us break it down, R9000 was deposited into an investment account. A year later, R7000 was deposited, then three years later from the initial deposit; R5000 was withdrawn. What are the odds that the money deposited, which is R16000 can double in value?
22.6.	Jane:	Very low because the interest only changes from 8% to 9%.
22.7.	Tebello:	So the money gained from the interest should be reasonable right? Ma'am you like saying that the bank is not 'our cousin' so why give us a return of R34 025,674 instead of R18 632,71567?
22.8.	Jane:	(*starting to doubt her answer*)
22.9.	Teacher:	Jane, you want to say something?
22.10.	Jane:	Same R9000 for the 8% and 9%. Same R7000 for 8% and 9%. Mind you, all of this is happening in only 5 years. Are you suggesting I combine calculations for R9000 before and after the interest changes?
22.11.	Teacher:	Not just that, I mean think about it. R9000 was invested and for 2 years the interest is 8% and then it changes to 9% for three years, Do you think that after 2 years, the money in the account would still be the R9000 invested?
22.12.	Jane:	No, it would not. It would be R9000 plus interest the money would have gained.
22.13.	Teacher:	Exactly and for the next coming three years, which amount do you think the 9% is calculated on?
22.14.	Jane:	The new amount.
22.15.	Teacher:	Precisely.
22.16.	Jane:	Oh I see, it gives me the same amount as Tebello got for the R9000 investment. So instead of separating the amounts I can just multiply like Tebello did because it is the same thing either way it is just shorter.
22.17.	Teacher:	Of course.
22.18.	Jane:	Oh, I see my mistake. (*proceeds to rewriting*).

The teacher's actions in this vignette played a meaningful role in resolving the learners' dispute. The interview data revealed that learners regard teacher intervention through questioning as important for developing their understanding. In this regard, one of the learners commended the teacher's interventions as follows.

> *They mediated moments of arguments we had. Ehhhh to be honest ma'am sometimes felt like my group members knew how to get to the answer but dololo explanation and when they explain I could feel aii, there is something missing or I would be confused but when you asked questions and made comments it helped me understand more*. (Rudo).

The learner's interview response suggests that disagreements frequently arise among learners in the learning environment, and that the teacher plays a crucial role in intervening and supporting the development of mathematical understanding.

In intervening, the teacher, as evident in Vignette 22 and other vignettes, does not provide answers directly to the learners. In this regard, the question *"Who do you guys think is correct?"* was intended to probe the learners' thinking. However, the learners' responses (line 22.4) did not include justification for why their answers or solutions were correct. This lack of justification may indicate that the learners found the task cognitively demanding. Consequently, the teacher suggested, *"let us break it down"*, prompting an analysis of the problem into its constituent parts (line 22.5). This was followed by a question focusing on *"the odds that the money deposited, which is R16000, can double in value"* (line 22.5).

The teacher appeared to recognise that the divergent solutions stemmed from learners' failure to attach contextual meaning to their mathematical manipulations and the resulting answers (lines 21.17 and 21.23). Thus, the teacher's mathematical intervention can be classified as an explicit intentional intervention, directing learners' attention to a specific mathematical idea necessary for resolving the problem. On the one hand, Jane acknowledged that the odds are *"very low because the interest only changes from 8% to 9%"*. On the other hand, Tebello argued that *"the money gained from the interest should be reasonable.*" It is important to note that Tebello's reasoning suggests that he was operating at the property noticing layer of understanding, as discussed earlier. In this instance, he did not regard Jane's value of R34 025,674 as reasonable, as reflected in line 22.7.

As a result, Jane exhibited a moment of silence, which can be interpreted as reflective engagement with her solution. This pause may have been influenced by her realisation that the calculated value implied a doubling of the investment, which contradicted her earlier assertion of very low odds. The teacher observed this moment of silence and probed whether Jane wished *"to say something"*. In doing so, the teacher identified a shift in Jane's thinking, although this shift appeared to be internalised. This moment suggests that Jane was prompted to fold back from the formalising layer to the image having layer of understanding, where she generated a general conceptual overview of the problem. She also began to recognise the investment as a single, continuous process, informed by new insights raised by the teacher and Tebello.

Jane's actions align with Yao and Manouchehri's (2020b) description of the image having layer, in which learners carry out specific mathematical actions guided by a general mental plan. She subsequently moved to the property noticing layer, where she stated that *"instead of separating the amounts, I can just multiply like Tebello did because it is the same thing."* As a result, effective folding back was evident in this learning activity, as learners were ultimately able to arrive at a meaningful and contextually justified solution.

4.7 CHAPTER SUMMARY

In this chapter, learners' verbal and written actions during classroom interactions were presented in the form of vignettes. These vignettes consisted of data that were analysed through detailed discussion. The analysis illustrates how learners engaged in folding back to refine their underdeveloped mathematical knowledge in order to successfully complete specific mathematical tasks. As such, the data were analysed using the Pirie and Kieren (1994) theoretical framework of folding back. The following chapter discusses the findings in relation to each research question that guided the study.

CHAPTER 5 CONCLUSIONS, SUMMARY, AND RECOMMENDATIONS

5.1 INTRODUCTION

This autoethnographic study focused on promoting Grade 10 learners' algebraic reasoning through the use of folding back. The classroom environment in which the study was conducted was problem-based, positioning learners at the centre of each lesson. As learners worked through the mathematical tasks provided, their interactions with one another and with the teacher were closely observed and documented. Algebraic reasoning is widely regarded as the cornerstone of high school mathematics, and learners' ability to engage successfully with other mathematical topics depends largely on the strength of their algebraic reasoning skills. The critical role of algebraic reasoning in learners' mathematical progression highlights the need to address challenges related to algebraic thinking as early as possible. Failure to do so in Grade 10 may disadvantage learners in subsequent grades, particularly Grades 11 and 12, and beyond high school. Consequently, research emphasises the importance of clear, well-structured, and non-traditional teaching approaches that create meaningful learning opportunities and enable teachers to facilitate learners' active participation in their learning processes. It is against this backdrop that the present study was conducted to address the following research questions, which formed the core of the investigation.

- How does Folding Back promote Grade 10 learners' algebraic reasoning?
- What are the barriers that hinder the effective implementation of folding back in promoting algebraic reasoning?

This section focuses on addressing the above research questions. Firstly, a summary of the study is provided, offering an overview of the chapters. Secondly, the findings of the study are synthesised to respond to the research questions. Thirdly, the limitations of the study are outlined. Lastly, recommendations for practice and future research are presented.

5.2 SUMMARY OF THE STUDY

The chapter summary provides an overview of the chapters in this study.

5.2.1 Chapter 1

In this chapter, I provided the background to the study. The chapter also outlined the purpose of the study, the research questions, the methodology, and the significance of the study. In addition, the setting of the study was highlighted. The chapter concludes with an overview of the structure of the dissertation.

5.2.2 Chapter 2

In this chapter, the theoretical framework was discussed. Selected studies that utilised folding back provided insight into the research methodology, which is described in the subsequent chapter. In addition, the literature was reviewed under the following themes: an overview of algebra in South Africa's mathematics curriculum, algebraic reasoning, learners' challenges in algebraic reasoning, and problem-based learning in relation to algebraic reasoning.

5.2.3 Chapter 3

In this chapter, I presented the rationale for adopting a qualitative research paradigm and justified the selection of autoethnography as the research design. The participants for the study were also carefully considered. In addition, the chapter addressed issues related to data collection, including the methods used, data analysis procedures, ethical considerations, and quality criteria. The rigour of the study within an autoethnographic methodology, including trustworthiness in qualitative research, was discussed to ensure adherence to quality criteria. The next section focuses on the analysis of the data collected.

5.2.4 Chapter 4

In this chapter, learners' verbal and written actions during classroom interactions were presented in the form of vignettes. These vignettes comprised data that were analysed through detailed discussion. The analysis illustrates how learners engaged in folding back to refine their underdeveloped mathematical knowledge in order to successfully complete specific mathematical tasks. Accordingly, the data were analysed using the Pirie and Kieren (1994) theoretical framework of folding back. The following chapter discusses the findings in relation to each research question that guided the study.

5.2.5 Chapter 5

In this chapter, I provide a summary of the findings in relation to the main research question that guided this autoethnographic study. The summary is presented as follows.

5.3 SUMMARY OF THE FINDINGS

As previously mentioned, the summary of the findings is presented in relation to the research question:

5.4 How does folding back promote Grade 10 learners' algebraic reasoning?

Through this research question, I sought to share personal experiences of how teaching framed by the folding back framework promotes Grade 10 learners' algebraic reasoning. The fieldwork, particularly the classroom episode interactions in which learners were observed working collaboratively with one another and with the researcher-teacher during the resolution of mathematical learning activities, together with the semi-structured interviews, were central to understanding how folding back promotes algebraic reasoning among Grade 10 learners. The following findings emerged from the narrative analysis presented in the previous chapter in relation to this research question.

5.4.1 Learners improve their algebraic reasoning skills through evaluating their solutions in a folding back learning environment

The study found that learners improved their algebraic reasoning skills by evaluating their solutions within a folding back learning environment. Learners' ability to fold back in order to evaluate their solutions served as an important tool that assisted them in reconstructing inadequate knowledge within the inner layers of understanding. Through this process, learners were able to identify and rectify mathematical errors made in their initial attempts (for example, Vignettes 5, 8, 12, 16, and 17). Learners' expressions such as *"Mara (but) wait"* (Vignette 8, line 8.1), *"So, this is incorrect"* (Vignette 8, line 8.2), *"I am realizing here that we read the question completely wrong" (Vignette 8, line 8.4), "Oohh, I was supposed to divide"* (Vignette 16, line 16.9), and *"I saw it was a mistake"* (Vignette 17, line 17.4) demonstrate their capacity to evaluate their solutions critically. These findings suggest that learners' ability to evaluate their solutions enables them to recognise inadequacies or shortcomings in their reasoning. This evaluative process supports the development of a deeper understanding of mathematical concepts and procedures, improving accuracy while simultaneously building learners' confidence as they learn from their mistakes. As learners fold back across various layers of understanding, they do not engage with solutions passively but instead actively interrogate them beyond mere numerical correctness. This emphasis on evaluation beyond numerical values reflects a broader and more meaningful conception of algebraic reasoning, wherein learners attend to the why and how underpinning their solutions, thereby fostering stronger connections to their mathematical ideas. These findings align with Al-Mutairi and Marzouq's (2025) study, which identified solution evaluation as a critical algebraic thinking process in financial mathematics contexts. Consequently, in a Grade 10 classroom where learners regularly fold back to evaluate their solutions, algebraic reasoning skills are significantly enhanced. In addition, the study revealed that self-invoked intervention creates opportunities for learners to become self-aware of their errors. Learners were observed engaging in self-invoked folding back, where their mathematical actions were prompted not by peer or teacher intervention, but by personal reflection on the solution in relation to their understanding of the problem (for example, Vignettes 13 and 15). These findings are consistent with Martin's (2008) notion of self-invoked folding back, which is prompted by learners' "natural ability to fold back in response to becoming self-aware of the limitations of their present understandings" (p. 83).

5.4.2 Teacher intervention provides learners with opportunities to refine their algebraic reasoning

Previous studies have shown that teacher intervention, as a source of folding back, enhances learners' growth in mathematical understanding and reasoning abilities (Chuene et al., 2023; Mabotja et al., 2018; Yao & Manouchehri, 2022b). Similarly, the findings of this autoethnographic study revealed that teacher intervention provides Grade 10 learners with opportunities to reconstruct and refine their algebraic reasoning. In this study, the teacher's explicit, intentional intervention through probing questions enabled learners to refine their application of algebraic reasoning in financial mathematics contexts (for example, Vignettes 10, 11, 17, 20, and 22). The emphasis was on guiding learners towards the specific knowledge required to solve the problem, rather than simply providing answers, which contrasts with traditional teacher-centred strategies. One possible explanation is that learners should be supported to solve mathematical tasks through guided engagement rather than direct instruction. By using probing questions, the teacher assisted Grade 10 learners in refining their algebraic reasoning, thereby fostering a deeper understanding of financial mathematics. Learners were not positioned as passive recipients of solutions; instead, as they responded to the teacher’s guiding questions, they folded back and developed awareness of how the tasks could be solved. These findings underscore the importance of promoting learner independence, as the teacher’s approach encouraged critical thinking and self-reliance in problem-solving. Furthermore, the teacher’s explicit intentional interventions were informed by careful attention to learners’ initial attempts and interactions, enabling the teacher to understand their thinking. As a result, teacher interventions were not uniform but responsive to the diverse needs of learners as they engaged with the mathematical tasks.

In addition, the study revealed that teacher intervention plays a crucial role in resolving learners’ disputes during mathematics learning activities. As learners worked collaboratively, they sometimes held firmly to their ideas, regardless of whether these ideas were correct. This was evident in Vignette 22, where Jane and Tebello produced different solutions and were unable to reach consensus regarding their correctness. Upon recognising this conflict, I transformed the situation into a productive learning opportunity by posing questions that prompted both learners to fold back across different layers of understanding. This process enabled them to reorganise their thinking, identify errors, and arrive at a shared solution. By facilitating dialogue and encouraging learners to revisit and reflect on their contributions, the teacher not only resolved the disagreement but also enhanced collective understanding. This approach fostered collaboration and mutual learning among peers. The findings further suggest that the teacher’s attentiveness to individual learners’ thought processes allows for targeted and responsive support aligned with learners’ specific challenges. These results are consistent with earlier studies by Mabotja (2017) and Chuene et al. (2023), which found that when teachers closely follow learners’ reasoning paths, they are better positioned to apply appropriate intervention strategies that promote learners’ mathematical reasoning abilities.

5.4.3 Peer intervention provides collaborative opportunities for learners to learn from one another

The results of the study revealed that peer intervention provides valuable collaborative opportunities for learners to learn from one another. As learners folded back across various layers of understanding, they shared their thinking processes, thereby promoting a shared understanding of mathematical ideas. While actively engaging with financial mathematics learning activities, Grade 10 learners frequently relied on one another's explanations to progress as a group (Vignettes 12, lines 12.4–12.7; Vignette 14, lines 14.1–14.3). In this regard, peer intervention created space for learners to move beyond merely performing procedures towards making sense of the questions posed. This pattern was evident across the vignettes, as learners' explanations enabled peers to identify and correct errors in their calculations (for example, Vignette 15, lines 15.5–15.9) and to attach meaning to complex financial mathematics relationships (for example, Vignette 18, lines 18.1–18.4). Although learners employed individual strategies, sharing these strategies with group members proved essential (for example, Vignette 18), as it encouraged active participation, learner ownership, and reduced reliance on peers to solve problems on their behalf. A likely explanation for these findings is that individual contributions are a fundamental component of problem-based learning within a collaborative environment.

Findings from the semi-structured interviews further demonstrated learners' appreciation of peer intervention. One learner highlighted the benefits of learning from peers by stating, *"I learnt of multiple ways of manipulating mathematical equations"* (Rudo, semi-structured interview). Similarly, another learner noted that they *"learn of [their] peers' ways of thinking"* (Teko, semi-structured interview). Learners consistently emphasised the value of peer interaction during problem-solving, explaining that concepts became easier to understand through peer engagement. As one learner noted, *"talking to someone who is my age made me feel less pressurised and as my peers explained the concepts became easier to understand"* (Teko, semi-structured interview). This emphasis on individual contributions within a team context reinforces the notion that each learner's input is valuable, thereby enhancing the effectiveness of folding back in a problem-based learning environment. In addition, the findings revealed that both intentional and unintentional peer interventions, particularly through exploratory talk, prompted effective folding back

(Vignettes 2, 8, and 16). Peer intervention through exploratory talk plays a crucial role in enabling learners to reorganise and reconstruct their thinking. These findings align with Chuene et al.'s (2023) study, which demonstrated that exploratory talk, a key element of folding back, allows learners to build on one another's ideas to deepen their understanding.

5.4.4 Learners develop the ability to justify their thinking when they fold back across various layers of understanding

The results of the study revealed that learners develop the ability to justify their thinking when they fold back across various layers of understanding. As Grade 10 learners interacted with one another, their reasoning capabilities were enhanced through peer questioning. In this regard, peers' questions such as *"would you say…"* (Vignette 1, line 1.5), *"Why must we divide?"* (Vignette 1, line 1.5), and *"what makes you say that?"* (Vignette 2, line 2.5) created valuable opportunities for learners to justify their thinking and reflect more deeply on their reasoning processes. Such questions *"challenge your way of thinking, to assess if you are sure about what you think you know and that requires a higher level of thinking and reasoning, especially when it comes to understanding why you are choosing to approach a problem a particular way"* (Jane, semi-structured interviews). Peer questioning therefore signals a proactive approach to learning, in which learners take responsibility for both their own understanding and that of their peers. As learners engage with their own thinking and that of others, they increasingly develop the ability to justify their mathematical reasoning. Learners' utterances such as *"if this amount is the total price that you will pay after 36 months, then it makes this money the value of A"* (Vignette 2, line 2.9), as well as examples from other vignettes (e.g., Vignettes 1, 8, 9, 13, and 18), demonstrate their ability to recognise relationships between variables. Through this process, learners develop confidence in their mathematical convictions. This is further evidenced by a learner's assertion to a peer that *"I am not suggesting, that is what we should do"* (Vignette 13, line 13.3). These findings indicate that learners are not confined to mechanical manipulation to reach solutions; instead, they engage meaningfully with mathematical ideas. The results align with those of Martin (2008) and Patmaniar et al. (2021), who found that learners are better able to explain and justify mathematical information when they engage in folding back. Consequently, as learners fold back across various layers of understanding, their mathematical reasoning is strengthened.

5.4.5 Folding back creates opportunities for learners to make sense of their solutions in relation to real-life contexts

The results of the study revealed that folding back creates opportunities for learners to make sense of their solutions. In this regard, it was observed that when learners fold back across various layers of understanding, they tend to relate mathematical solutions to everyday contexts in which financial mathematics topics, such as hire purchase agreements, are applied. This was evident in several interactions between learners and the researcher-teacher as they engaged in mathematical activities (e.g., Vignettes 2, line 2.1; 5, line 5.12; 10, line 10.4; and 16, line 16.10). Among the plausible explanations for this finding is the learner's reflection that *"we are working with money here. It must be tangible. Therefore, R286,6666667 per month for 72 months makes sense than R0,00348837209 per month. Your answers should make sense to you. Understand your calculations."* (Vignette 16, line 16.10). This emphasis on plausibility was also evident in the semi-structured interviews, where one learner highlighted that *"you'll also be able to judge your own attempts, like for instance, you cannot have an investment where the value of P is greater than the value of A"* (Phuti, semi-structured interview). These findings suggest that folding back supports learners in deriving meaning from mathematical calculations, indicating that algebraic reasoning is not only about performing procedures but also about interpreting and validating results. This outcome aligns with the broader goals of algebra teaching, which emphasise learners' ability to use algebraic skills to analyse *"situations in a variety of contexts in order to make sense of them"* (DBE, 2011a, p. 10). Furthermore, the results concur with Novikasari's (2020) assertion that mathematics is not solely concerned with obtaining correct answers but also with sense-making. Therefore, in a learning environment where folding back is promoted, Grade 10 learners develop a stronger capacity for mathematical sense-making.

5.4.6 Learners fold back to collect to retrieve the previous knowledge of algebra and apply it in a financial mathematics context

The results of the study revealed that learners fold back to collect and retrieve their prior knowledge of algebra and apply it within a financial mathematics context. In this regard, it was observed that Grade 10 learners draw on their knowledge of solving algebraic equations to determine unknown variables in financial mathematics problems. Among the conceivable explanations for these findings is that algebraic rules, such as the use of inverse operations to isolate a variable on one side of an equation, support learners in solving financial mathematics equations. This was evident in Vignette 1, where learners used their knowledge of solving algebraic equations such as $4x = 3$ to explain how this approach could assist them in determining “n” (the number of time periods for a loan or investment). Learners’ interview responses further indicated that “financial mathematics becomes easy because of a strong foundation [in] solving algebraic equations” (Rudo, semi-structured interviews). Similarly, another learner stated that *“algebra is in almost every maths topic because as soon as I was stuck, what I could think about was algebra, simplify like in solve for x”* (Phuti, semi-structured interviews). The ability of Grade 10 learners to connect their foundational knowledge of algebra to real-world applications in financial mathematics demonstrates a strong conceptual understanding. These results concur with Juraev and Bozorov’s (2024) assertion that calculating investments or interest on loans necessitates the use of algebra to identify optimal strategies and forecast future outcomes. Therefore, when learners fold back, they are able to meaningfully apply their algebraic knowledge across a variety of contexts, including financial mathematics.

5.5 What are the barriers that hinder the effective implementation of folding back in developing algebraic reasoning?

This research question acknowledges that, although folding back is an effective tool for promoting Grade 10 learners’ algebraic reasoning, certain challenges or barriers may hinder its effective implementation. These challenges are discussed as follows.

5.5.1 The teacher’s limited availability and attention in following learners’ reasoning path

The results of this study revealed that teachers have limited availability and attention when attempting to follow learners’ reasoning in a folding back classroom environment. In a mathematics classroom where learners work in groups simultaneously, it is not possible for the teacher to monitor every reasoning path at once. Consequently, while the teacher focuses on one group, valuable learning moments in other groups, such as learners’ confusion or peers’ misguided prompts that should signal the need for folding back, may go unnoticed. In this study, I observed that as I facilitated the lessons, I could not be present with every group as they worked concurrently on the learning activities. This limitation is evident in Vignette 17, where a learner cancelled an initial attempt while working collaboratively within a group. In the absence of the teacher, other learners in the group adopted a readily given mathematical explanation despite not fully understanding it conceptually, ultimately leading to an incorrect solution (Vignette 17, lines 17.7 & 17.8). As such, peers may influence one another's mathematical thinking in ways that reinforce misconceptions when the teacher is not present to identify the layers of understanding at which learners are reasoning. This situation allows incorrect reasoning to remain unchallenged, thereby constraining the depth of learners' developing algebraic reasoning skills. These findings may offer insight into the reliance on small sample sizes often used in folding back studies, such as those conducted by Mabotja et al. (2018), Martin (2008), and Yao and Manouchehri (2022a).

5.5.2 Learners' reluctance to answer peer questions limit effective folding back

Sujathan and Vinayakan (2022) assert that engaging in peer questioning promotes the development of learners' critical thinking and problem-solving skills. However, the results of this study revealed that learners' reluctance to respond to peers' questions can hinder folding back. There were a few, yet significant, instances where learners were unwilling to engage with questions posed by their peers. This is evident in Vignette 4, where a learner giggles and hides their book instead of responding to a peer's question. Such behaviour highlights that they may have been uncertain about their answer or reluctant to share their thinking with their peers. Consequently, the misconception that 'p.a.' referred to 'per month' rather than 'per annum' remained hidden until the teacher intervened. Therefore, in the absence of the teacher's intervention the learner in the group may have continued with a misunderstanding of the concept simply because of the reluctance to answer the question asked by their peer. Thus, this reluctance to engage with peer questioning can hinder effective collaboration, thereby limiting opportunities for folding back. These limited opportunities for folding back leave room for incorrect algebraic reasoning to persist unchallenged. Consequently, this resistance constrains creative thinking and restricts learners' ability to develop a solid understanding of complex mathematical concepts. This finding supports Sujathan and Vinayakan's (2022) view that peer questioning only contributes to deeper understanding when learners are willing to explain and justify their ideas. Likewise, Yao and Manouchehri (2020b) argue that learners need opportunities to externalise their thinking so that misconceptions can be recognised and addressed.

5.5.3 Learners during peer interactions fail to reach a shared reasoning pathway within a folding back learning environment

Folding back provides learners with opportunities to develop a shared mathematical understanding (Chuene et al., 2023; Martin, 2018; Pirie & Kieren, 1994). However, the results of this study revealed that, during peer interactions, learners sometimes fail to reach a shared reasoning pathway in a folding-back learning environment. The confidence with which Grade 10 learners' approach mathematical tasks, particularly when operating at the formalising layer, can make it difficult for them to acknowledge limitations in their reasoning or to accept correction from their peers, often resulting in disagreement. This was evident in Vignette 22, where two learners working on the same mathematical problem were overly confident in their respective methods. Consequently, instead of collaboratively working towards a common mathematical meaning, they argued over who was correct (Vignette 22, lines 22.1 & 22.2) rather than on why a particular solution made mathematical sense. These findings suggest that the difficulty did not lie in the existence of different ideas, but rather in learners' inability to negotiate a common understanding from those ideas. Al-Mutairi and Marzouq (2025) argue that learners develop stronger algebraic reasoning when they justify and evaluate the relationships underpinning their solutions to others. As such, the inability to reach a shared reasoning pathway therefore constrains the development of learners' algebraic reasoning, as they are unable to fold back to earlier layers of understanding to reassess the algebraic structures underpinning their solutions.

Similarly, it is believed by Novikasari's (2020) that mathematical understanding is developed not only through obtaining correct answers, but also through making sense of the meaning and reasonableness of those answers. However, Vignette 9 (lines 9.4 & 9.5) demonstrate that learners are exposed to limited opportunities to deepen their understanding of algebraic relationships through dialogue, leading to a persistent fixation on individual reasoning. It was only after the teacher intervened through probing questions that the learners began to evaluate their solutions and recognise the shortcomings in their reasoning. This therefore suggests that the inability to reach a shared reasoning pathway alone does not necessarily promote folding back but rather, learners need to be supported for them to justify, assess and negotiate their solutions in order to fold back to earlier layers of understanding and develop a shared mathematical meaning.

5.6 LIMITATIONS

The results of the study should be interpreted in relation to the following limitations:

5.6.1 Data collection

Although the data collection methods employed in this autoethnographic study were fit for purpose, the inclusion of a pre-test and post-test could have been useful in measuring learners' achievement and in more directly evaluating the impact of folding back on their algebraic reasoning.

5.6.2 Small sample size

The sample size consisted of a small number of learners from one school, particularly in the semi-structured interviews, where only four learners participated. This limited sample size restricts the extent to which the findings can be generalised to a larger population. Involving a larger number of participants from different schools could have yielded a more diverse range of responses and perspectives on the folding back phenomenon.

5.7 RECOMMENDATIONS

Despite the limitations of the study, the following recommendations are proposed:

5.7.1 Recommendations for mathematics teachers

The results of the study revealed that both teacher and peer interventions prompt learners to fold back to various layers of understanding, thereby promoting their algebraic reasoning. As a result, it is recommended that mathematics teachers probe learners' thinking in a mathematics classroom by asking them questions that require them to explain, justify and evaluate their solutions rather than immediately correcting their errors. In reference to this study, questions like ‘What does this value represent? ‘, ‘Can you explain how you arrived at that answer?’ and ‘Would this still work if…?’ amongst others can assist learners to fold back to previous layers of understanding and reconstruct their understanding of a mathematical concept.

Furthermore, teachers should create collaborative opportunities for learners to probe each other’s thinking and justify their thought processes. This can be done through organising learners into smaller groups (as seen in this study). Working in these smaller groups creates closer interactions in which learners are encouraged to question one another’s thinking as they work towards a shared reasoning pathway. However, as much as learners will be working together, it is important for the teacher to be actively present to attend to learners’ uncertainty, their reluctance to engage with peers and disagreements that arise without justification. Such moments may lead to misconceptions that often remain unnoticed without the teacher’s intervention.

5.7.2 Recommendations for future research

Future research on folding back could focus on addressing the methodological gap, particularly by quantitatively measuring the effect of folding back on learners’ mathematics performance through the use of pre- and post-tests. It is my belief that such studies may provide extensive evidence regarding how folding back improves learners’ algebraic reasoning and mathematical performance.

Considering that this study was conducted in a multilingual classroom, future studies could explore how language may support or constrain the effectiveness of a folding back classroom. Such a study may provide insight into the role that learners’ home languages play in influencing their ability to share their mathematical ideas with other learners. In addition, future studies could extend the exploration of folding back to other mathematics topics where there is still limited literature (e.g., trigonometry or calculus) in order to determine if the findings in this study are transferable to other contexts.

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Education (TURCOMAT), 10(3), 588-616. https://doi.org/10.16949/turkbilmat.487243

ANNEXURE A: ETHICS CERTIFICATE

College of Education _ERC

NHREC Registration # : (if applicable)
Ref #: 7746
Name: Miss Malose Tlhako
Student #: 24730467
Staff #:

Date: 25/05/2025

Dear: Miss Malose Tlhako

Decision: Ethics Approval from 25/05/2025 to 25/05/2028

Researcher: Miss Malose Tlhako

15 Nyathi street, Rethabile Amandebult, 0363

Thabazimbi

24730467@mylife.unisa.ac.za 0670441293

Supervisor: Dr Koena Mabotja mabotjaks@gmail.com

Co-Supervisor:

Co-Researcher(s):

Email address:

AUTOETHNOGRAPHIC PERSPECTIVE OF PROMOTING GRADE 10 LEARNERS' ALGEBRAIC REASONING THROUGH FOLDING BACK

Qualification: Masters in Mathematics Education

Thank you for the application for research ethics clearance by the College of Education _ERC for the above mentioned research study Ethics approval is granted for three years.

The **low risk application** was **reviewed** by College of Education _ERC on 7 May 2025 in compliance with the Unisa Policy on Research Ethics and the Standard Operating Procedure on Research Ethics Risk Assessment.

The proposed research may now commence with the provisions that:

1. The researcher(s) will ensure that the research project adheres to the values and principles expressed in the UNISA Policy on Research Ethics.
2. Any adverse circumstance arising in the undertaking of the research project that is relevant to the ethicality of the study should be communicated in writing to the College of Education _ERC.
3. The researcher(s) will conduct the study according to the methods and procedures set out in the approved application.
4. Any changes that can affect the study-related risks for the research participants, particularly in terms of assurances made with regards to the protection of participants' privacy and the confidentiality of the data, should be reported to the Committee in writing, accompanied by a progress report.

Page 1 of 2

5. The researcher will ensure that the research project adheres to any applicable national legislation, professional codes of conduct, institutional guidelines and scientific standards relevant to the specific field of study. Adherence to the following South African legislation is important, if applicable: Protection of Personal Information Act, no 4 of 2013; Children's act no 38 of 2005 and the National Health Act, no 61 of 2003.

6. Only de-identified research data may be used for secondary research purposes in future on condition that the research objectives are similar to those of the original research. Secondary use of identifiable human research data requires additional ethics clearance.

7. No field work activities may continue after the expiry date (25/05/2028). Submission of a completed research ethics progress report will constitute an application for renewal, for Ethics Research Committee approval.

Additional Conditions

1. Disclosure of data to third parties is prohibited without explicit consent from Unisa.
2. De-identified data must be safely stored on password protected PCs.
3. Care should be taken by the researcher when publishing the results to protect the confidentiality and privacy of the university.
4. Adherence to the National Statement on Ethical Research and Publication practices, principle 7 referring to Social awareness, must be ensured: "Researchers and institutions must be sensitive to the potential impact of their research on society, marginal groups or individuals, and must consider these when weighing the benefits of the research against any harmful effects, with a view to minimising or avoiding the latter where possible." Unisa will not be liable for any failure to comply with this principle.
5. Kindly note that the College of Education _ERC requires the submission of regular progress reports to be submitted annually. Inline with section 7.2 of the Unisa Policy on Research Ethics (2024).

Note

The reference number 7746 should be clearly indicated on all forms of communication with the intended research participants, as well as with the Committee.

Kind regards,

Prof Justin Oswin August
Chair of College of Education _ERC
E-mail: augusjo@unisa.ac.za

Prof Mpine Makoe
Executive Dean College of Education
E-mail: qakisme@unisa.ac.za, magolmc@unisa.ac.za

ANNEXURE B: ETHICS CERTIFICATE (LDOE)

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

OFFICE OF THE PREMIER

Office of the Premier

Research and Development Directorate

Private Bag X9483, Polokwane, 0700, South Africa

Tel: (015) 230 9910, Email: mokobij@premier.limpopo.gov.za

LIMPOPO PROVINCIAL RESEARCH ETHICS

COMMITTEE CLEARANCE CERTIFICATE

REVIEW DATE: 02 JULY 2025

PROJECT NUMBER: LPREC/126/2025: PG

SUBJECT: AUTOETHNOGRAPHIC PERSPECTIVE OF PROMOTING GRADE 10 LEARNERS' ALGEBRAIC REASONING THROUGH FOLDING BACK

RESEARCHER: MA TLHAKO

Chairperson: Prof I Swarts

Swarts

Chairperson: Limpopo Provincial Research Ethics Committee

The Limpopo Provincial Research Ethics Committee (LPREC) is registered with National Health Research Council (NHREC) Registration Number **REC-111513-038.**

Note:

i. **This study is categorized as a Low Risk Level in accordance with risk level descriptors as enshrined in LPREC Standard Operating Procedures (SOPs)**

ii. **Should there be any amendment to the approved research proposal; the researcher(s) must re-submit the proposal to the ethics committee for review prior data collection.**

iii. **The researcher(s) must provide annual reporting to the committee as well as the relevant department and also provide the department with the final report/thesis.**

iv. **The researchers will be required to make presentations of the study findings and recommendations at the Provincial Research Conference/Departmental Research Day.**

v. **The ethical clearance certificate is valid for 12 months. Should the need to extend the period for data collection arise then the researcher should renew the certificate through LPREC secretariat. PLEASE QUOTE THE PROJECT NUMBER IN ALL ENQUIRIES**

ANNEXURE C: PERMISSION LETTER (LDOE)

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

DEPARTMENT OF
EDUCATION

CONFIDENTIAL

Ref: 2/2/2 Enq: Makola MC Tel No: 015 290 9448 E-mail:MakolaMC@edu.limpopo.gov.za

TLHAKO MA
15 NYATHI STREET, RETHABILE
AMANDEBULT
THABAZIMBI
0363

24730467@mylife.unisa.ac.za/ malose.alsin@icloud.com [067 044 1293]

RE: REQUEST FOR PERMISSION TO CONDUCT RESEARCH.

1. The above bear's reference.

2. The Department wishes to inform you that your request to undertake research titled: **"AUTOETHNOGRAPHIC PERSPECTIVE OF PROMOTING GRADE 10 LEARNERS' ALGEBRAIC REASONING THROUGH FOLDING BACK"** has been approved

3. The following conditions should be considered:

3.1 The research should not have any financial implications for Limpopo Department of Education.

3.2 Arrangements should be made with the Circuit Office and the School concerned.

3.3 The conduct of research should not in any how disrupt the academic programmes at the school(s).

3.4 The research should not be conducted during the time of Examinations, especially in the fourth term.

REQUEST FOR PERMISSION TO CONDUCT RESEARCH: TLHAKO MA Page 1

Cnr 113 Biccard & 24 Excelsior Street, POLOKWANE, 0700, Private Bag X 9489, Polokwane, 0700
Tel:015 290 7600/ 7702 Fax 086 218 0560

The heartland of Southern Africa-development is about people

3.5 During the study, applicable research ethics should be adhered to; in particular the principle of voluntary participation (the people involved should be respected and treated with dignity).

3.6 Upon completion of the research, the researcher shall share the final product of the research with the Department.

4 Additionally, you are expected to produce this letter at School(s)/Office(s) where you intend to conduct your research as evidence that permission has been granted for access to the research site(s).

5 The Department appreciates the contribution that you wish to make and wishes you success in your investigation.

Best wishes.

MC Makola PhD

21/08/2025

Date

Molope NM

Acting DDG: CORPORATE SERVICES

25/08/2025

Date

ANNEXURE D: SCHOOL PERMISSION LETTER

Miss Tlhako MA Email- 24730467@mylife.unisa.ac.za

Contacts-067 044 1293

Research Title: **Autoethnographic Perspective of Promoting Grade 10 Learners' Algebraic Reasoning through Folding Back**

The Principal,

Dear Sir/Madam,

Request permission to conduct research at your school.

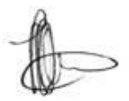
I, Malose Alsin Tlhako, am conducting a research project under the supervision of Dr. K.S. Mabotja at the University of South Africa (UNISA) in the Department of Mathematics, Science, and Technology Education. I would like to invite Grade 10 Mathematics learners at your school to participate in my study titled "**Autoethnographic Perspective of Promoting Grade 10 Learners' Algebraic Reasoning through Folding Back** ."

Data for this study will be collected through fieldwork, video recordings, reflective journaling, participation in learning activities, and semi-structured interviews. I request permission to video record the sessions to accurately capture classroom interactions and validate the results without bias. These recordings will only be used for research purposes and will not be shared publicly. If parents or guardians prefer that their child not be recorded, they will be informed that the child may still participate in other aspects of the study that do not involve video recording.

The aim of the study is to explore how to promote algebraic reasoning skills within a problem-based learning environment through the concept of folding back. To achieve this, learners will engage in meaningful mathematical interactions and utilise their algebraic reasoning to solve Analytical Geometry learning activities, which will enhance their reasoning skills. There are no known risks associated with participating in this study. Sessions will take place during the morning and afternoon and will not disrupt the normal school program or class timetable. The insights gained from this research may help both novice and experienced teachers in South Africa understand how to teach responsively in secondary school mathematics, particularly in challenging content areas.

Participation in this study is completely voluntary, and participants may withdraw at any time. All data collected will be kept strictly confidential. Student names and personal information will not appear in any published results. Data will be stored securely on a password-protected device and will be destroyed after five years, in accordance with ethical research guidelines. I will provide feedback on the research outcomes and findings to the participants.

Yours faithfully,



Tlhako, M.A.

Master's Student in Mathematics Education University of South Africa (UNISA)

ANNEXURE E: PERMISSION LETTER (PARENTS)

Miss Tlhako MA Email- 24730467@mylife.unisa.ac.za

Contacts-067 044 1293

Research Title: **Autoethnographic Perspective of Promoting Grade 10 Learners' Algebraic Reasoning through Folding Back.**

Dear Parents

Permission for your child's participation in research

I, Malose Alsin Tlhako (the researcher) am doing a Master of Education research project under the supervision of Dr. Mabotja, K.S. at the University of South Africa in the Department of Mathematics Education. I would like to invite the Grade 10 Mathematics learners at the school to participate in this research study titled: **Autoethnographic Perspective of Promoting Grade 10 Learners' Algebraic Reasoning through Folding Back**. Data will be collected through fieldwork, video recordings, reflective journaling, engaging in learning activities, and conducting semi-structured interviews. As a result, I request permission to video record the session as the recording will help me to capture what occurred in the classroom and to validate results without bias. These recordings will only be used for research purposes and will not be shared publicly. If you prefer that your child not be recorded, they may still participate in other aspects of the study that do not involve video recording. The aim is to study aims to explore the promotion of algebraic reasoning skills in a problem-based learning environment through folding back. To realise the purpose, learners need to engage in meaningful mathematical interactions and tap into algebraic reasoning to answer Analytical Geometry learning activities. There are no known risks associated with participating in this study. The study will take place during morning and afternoon sessions and will not disrupt normal school programs or classroom timetables. Any information regarding the research study related to the child's identity will always remain confidential and anonymous, and the child's informed data will not be shared with anyone and will be encrypted.

Participation in this study is completely voluntary. Your child may choose to withdraw from the study at any time without any impact on their term report marks. Refusing to participate will not affect their academic standing. Participants are free to exit the study whenever they wish.

Yours sincerely,

Tlhako, M.A.

Master's Student in Mathematics Education University of South Africa (UNISA)

ANNEXURE F: INFORMED CONSENT

UNISA university of south africa

Miss Tlhako MA Email- 24730467@mylife.unisa.ac.za

Contacts-067 044 1293

Research Title: **Autoethnographic perspective of promoting grade 10 learners' algebraic reasoning through folding back.**

Dear Parents

Parent/Guardian Informed Consent form

I, (Name and Surname) …………………………………………………… allow my daughter/son to form part of your study. I am aware that my child's participation in the study is voluntary. If for any reason I want my child to withdraw from the study, I have the right to stop him/her from participating without giving any explanation. I understand the purpose of the research study. I am aware that data will be used for the master's dissertation or research paper. The data gathered will be confidential and anonymous concerning my child's identity unless otherwise specified or indicated. In the case of video-recorded teaching episodes, interviews, and class activities, I have been promised that my child's identity and that of the school will be protected and will not be shared to any third party and data will be encrypted. I, therefore, agree to the recordings of……………………………………… (Interviews, fieldwork, class interaction).

I have read and understood the above information. I give my consent for my daughter/son to take part in the study.

Participants name and surname………………………….

Parent/Guardian signature……………………………………… Date………………

Researcher's signature………………………………… Date:

ANNEXURE G: MINOR ASSENT FORM

Miss Tlhako MA Email- 24730467@mylife.unisa.ac.za

Contacts-067 044 1293

Research Title: **Autoethnographic Perspective of Promoting Grade 10 Learners' Algebraic Reasoning through Folding Back**

I……………………………………………(Surname and names) understand that my parents/guardians have permitted me to participate in Miss Tlhako's MA study. I fully understand that if I take part in the study, I am allowed to withdraw at any time and nothing will happen to me.

My informed data will be kept confidential and encrypted. Anything else I say in the study will not be shared with my parents, teachers or anyone else.

I am willing to take part in the study……………….**(Yes/No)**

I am willing to be video recorded on the recording device ………… **(Yes/No)**

Signature…………………………… Date………………………..

ANNEXURE H: SEMI-STRUCTURED INTERVIEWS

1. How did working on the mathematical tasks Influence your algebraic thinking? Can you provide an example of a task that enhanced your skills?
2. While you were tapping into your previous knowledge to tackle the assigned mathematical tasks, did you come across any helpful information that assisted you in answering the mathematical questions presented?
3. How would you then describe your reflection most effectively?
4. What factors do you think influenced your algebraic reasoning skills? Consider both the questions you were able to solve, albeit with some difficulty and those that you found completely challenging.
5. What factors do you think influenced your algebraic reasoning skills? Consider both the questions you were able to solve, albeit with some difficulty and those that you found completely challenging.
6. Would you say that reflecting on or revisiting previously learnt concepts is a helpful way to enhance your algebraic reasoning? Why is that?
7. What are your thoughts on the classroom setup we had when it comes to learning mathematics?.
8. Which interventions—self-reflection, peer discussions, teacher guidance, or materials—were most effective in promoting your algebraic reasoning? Why?
9. Has your approach to solving algebraic problems changed since your experience of a Folding Back classroom? If so, how has it changed?

ANNEXURE I: MATHEMATICS LEARNING ACTIVITIES

GRADE 10 MATHEMATICS

TERM 3 TOPIC: FINANCE AND GROWTH

CONCEPT 1: SIMPLE AND COMPOUND INTEREST

Activity 1

1.1

R5 000 is invested in a bank. Calculate, using the appropriate formulae, the accumulated amount after three years and the total interest received, if the interest rate is:
(a) 3,75% per annum simple interest (b) 3,75% per annum compound interest

1.2

How long would it take an investment of R500 to increase by R350 if the interest rate is 4,3% p.a simple interest?

1.3

Dominic invests R5 410 in a savings account and receives R7 720 in total after some time. Determine how long the money was invested for if the interest he received was 6.2% p.a simple interest.

1.4

Jess invests a certain amount at 12% simple interest. How long does it take to double?

1.5

How long does it take for Baruti's investment of R37 800 to grow to R45 900 if invested at 10% p.a simple interest.

1.6

Mary invests a certain amount for 2 years at 15% p.a compound interest and receives R5000 after 2 years. How much did Mary put in?

Activity 2

1.1 Ellen decides to invest R500 in the bank. If the bank pays her 13% interest per year, how much money will she have after 9 years?

1.2 Rachel has just opened a small business and takes out a loan to provide for the initial start-up costs. She agrees to repay the loan four years later by means of a payment of R1 200 000. The bank charges her an interest rate of 18% per annum compounded annually. What was the amount of money she originally borrowed?

1.3 Thapelo invests R1800 for his upcoming matric dance ceremony, while in Grade 10. The bank offers him an interest rate of 13,5% p.a on simple interest. Calculate how much money he would have in his account given he does not repeats a phase.

1.4 Five years ago, a certain amount of money was invested in a bank. The value of the investment is currently R200 000. Calculate the original amount invested (P) if the interest rate was 5% per annum compound interest.

1.5 How long would it take for an investment of R40 000 to increase by R5 000 if the interest rate is 4,5% per annum simple interest?

1.6 A store is running a contracted deal for a mobile phone as depicted below:

Phone payable over 36 monthly installments of

Latest phone, latest trends, Get it NOW so you do not miss out.

Determine:

a) the total amount a customer would pay at the end of the 36 months period.
b) the cash price of the cellphone if the monthly instalments quoted on the advertisement are on simple interest agreement which offer an interest rate of 8,2% on the cash price of the cellphone.

Activity 3

a) Joseph invests R18 000 and it grows to R25 000 over a period of 2 years. Calculate the interest p.a to one decimal if the interest earned was simple interest.
b) Paul invests R35 000 and it accumulates to R55 000 over a period of five years. What simple interest rate would you need to secure?
c) Janet invests R80 000 and it accumulates to R90 000 over a period of two years. What simple interest rate will she need to receive in order to achieve this?

Activity 4

1.

1.1. Calculate the total amount to be paid for the cycling bicycle after 36 months.

1.2. Calculate the cash price of the bicycle if the interest is charged at 7,0% p.a simple interest.

1.3. After 3 years, the same model's cash price has increased to R25 644,12 due to inflation. Calculate the annual inflation rate over this period assuming the increase in price is compounded annually.

Activity 5

1. Covert $\frac{1}{3}$ to a percentage

2. What is the interest rate if Nosipho invested R2 000 at compound interest for a period of three years, and received R2 800?
3. Joseph invests R18 000 and it grows to R25 000 over a period of 2 years. Calculate the interest p.a to one decimal if the interest earned was compound interest.
4. Paul invests R35 000 and it accumulates to R55 000 over a period of five years. What compound interest rate would you need to secure?
5. Janet invests R80 000 and it accumulates to R90 000 over a period of two years. What compound interest rate will she need to receive in order to achieve thi

CONCEPT 2: REAL-LIFE APPLICATION (INFLATION, POPULATION GROWTH AND HIRE PURCHASE)

Activity 6

1.

1.1 The average salary of a domestic worker in South Africa in the year 2000 was R2 000. Assuming an annual average rate of inflation of 5,7%, would a salary of R3 000 have the same buying power in 2015?

1.2 University fees for a student stuying a bachelor's degree are, on average, about R34 000 per year. Assuming an average inflation rate of 5% per annum, what will the fees be in twenty years' time?

2. Forty years ago, John deposited R5 000 in a bank paying him 3% per annum compound interest. The average inflation rate over the forty years was 6%.

2. How much money will he have saved after forty years?

2. Calculate the buying power of R5 000 after forty years.

2. Comment on the value of John's savings after forty years.

3.

3.1 The population of a city in South Africa increased by 5,25% for the year 2017

At the beginning of 2017 the population of the city was 2560000.

Assuming that the population will continue to increase at a constant rate of 5,25% each year, estimate the population of the city at the beginning of 2020.

Give your answer correct to the nearest whole number

3.2

The number of black rhinos in Africa during 2012 was estimated at 5 487. If the average population growth rate of black rhinos is 4,9% per annum, calculate how many rhinos were there in Africa in 2007.

Activity 7

1.

Vanessa buys a stove costing R16 000. She takes out a twenty four month hire-purchase loan to make this purchase. The interest rate charged on the loan is 22% per annum simple interest. Calculate how much she will actually pay for the stove.

2.

Mike buys a mobile phone costing R8 000 on HP, pays a deposit of R800, and then pays 36 monthly payments of R344. Calculate the simple interest rate.

3.

Mrs. Reyem wants to buy herself a new laptop. After completing her monthly budget she realizes that she does not have the money to buy a new laptop right now.

Mrs. Reyem decides to buy a laptop on a hire purchase agreement from a computer store that advertises the following special:

Cash Price: **R4 999**
OR
Hire purchase: 12% Deposit and 24 equal monthly payments to the value of **R6 598,68**

3.1. Calculate how much Mrs. Reyem would pay for the deposit.

3.2. Calculate the total amount that Mrs. Reyem will pay for her new laptop on the hire purchase agreement.

4.

Patricia wants to buy a furniture suite for R38 000. She decides to take out a hire-purchase loan involving equal monthly payments over five years. The deposit is 20% and the simple interest rate charged per annum is 15%. Calculate:

(1) how much must be paid each month
(2) the amount of interest paid
(3) the actual amount paid for the furniture suite

Activity 8

a) David buys a computer which costs R17 000. He takes out a 36-month hire purchase agreement and the interest charged is 14% per annum simple interest. What will he actually pay for the laptop?

b) Shaun buys a smartphone on HP which costs R11 799,90. He will have to pay R639 per month for 24 months on hire purchase. No deposit will be required. Calculate the simple interest rate.

c) A tablet with WiFi costs R10 579. Mark buys a tablet on HP and agrees to pay a deposit of R1 500 and 36 monthly payments of R350. Calculate the the rate of simple interest.

d) Tyler purchases a sound system for R7,999 and pays an 11% deposit. The store charges Tyler 20% interest per year, with the loan payable over 4 years. Calculate the total amount that he will have to pay back?

Activity 9

1. A hire purchase contract for a sound system requires James to pay a deposit of R2 000 and to then make six monthly payments of R3 375. If the price of the sound system is R20 000, calculate the total simple interest paid and the rate of simple interest.

2. Belinda buys a flat screen plasma television costing R20 000 on hire purchase. She traded in an old television for R3 000 and paid a deposit of R 2000. The balance was paid by means of monthly instalments of R900 over two years. Calculate the total simple interest paid.

3. Jeremy wants to buy a motorbike. He can only afford to pay R3 000 per month. He wants to take out a hire purchase loan over 24 months at an interest rate of 12% per annum simple interest. Calculate the price of the computer that he can afford to buy.

4.

BRAND NEW HISENSE washing machine **ONLY** for R39845 cash price.

OR

Payable over a hire purchase agreement:

10% deposit and the value of the 36 monthly equal Instalments to the value of R44 467,02

4.1. Write down cost price of the displayed washing machine.

4.2. Given Mrs Mothipa takes the washing machine on the hire purchase agreement:

a) Calculate the total Mrs Mothipa would have to pay.

b) How much interest would she pay in total?

5. Ayanda wants to buy a new car and can afford to pay R4 899 per month. A car dealership offers her a payment plan over 72 months at a simple interest rate of 10,5% per annum. What is the price of the car she can afford to buy?

CONCEPT 3: HIRE PURCHASE (MONTHLY REPAYMENTS)

Activity 10

1. Senzo wishes to buy a study table on a hire purchase agreement and the terms of the agreement are as follows:
   - Study table price: R10 000
   - 15% deposit (Compulsory)
   - 48 Monthly payments
   - interest rate of 12% p.a charged on the balance.

a) Calculate the outstanding amount after Senzo pays the deposit
b) Calculate the total interest Senzo would incur is he take up the hire purchase agreement.
c) Determine the total amount Senzo ends up paying if he opts for buying the study table on hire purchase.

2. Gugu buys a double bed which costs R12 000 on hire purchase. She is charged a simple interest of 12%p.a. over six years.

2.1 Calculate the total amount she will pay for the double bed.

2.2 How much interest will she pay over this period?

2.3 Calculate her monthly instalment.

Activity 11

An iPhone 14 Pro Max is offered on a monthly payment plan as shown below:

a) If a customer was to take up this offer, calculate the amount they will pay in total for the phone over the 36 months period.
b) If the cash price of the iPhone 14 Pro Max is R31 560, calculate the total amount of interest paid over the 36-month period.
c) Determine the percentage increase from the cash price to the total amount paid on the monthly plan.

Activity 12

1. Reabetswe decides to buy a home theatre system which costs R9 500 on a hire purchase agreement.
    i. If Reabetswe pays a 25 % deposit, calculate the balance after Reabetswe has paid the deposit.
    ii. The store charges Reabetswe 20 % interest per year, with the loan payable over 4 years. Calculate the total amount that he will have to pay back.
    iii. Reabetswe is also required to pay a monthly insurance of R30 and an administration fee of RI2. How much will Reabetswe's monthly repayments be?

2. A fridge costs R1500. Unfortunately, you cannot afford to pay cash so you enter into a Hire Purchase agreement with the store. You choose to pay a 10% deposit, and the remaining amount via equal monthly instalments. Interest is charged at a rate of 28% p.a. Determine the monthly instalment if the loan is to be paid over 2 years.

3. Thabo can afford to pay a total of R500 per month for a bicycle. He will pay it off on a hire purchase agreement with the following terms from the shop:

- Compulsory Monthly Insurance of R53 per month.
- 15% deposit
- Interest charged at 20% per annum.
- He will pay it off over a 3-year period

If his Dad agrees to pay the deposit then what is the most expensive bike he can afford to buy?

Activity 13

1.

In June 2019, the pound to rand exchange rate was £1 = R18,18. Zola, travelled to the United Kingdom to watch some WWE wrestling matches. The total cost needed for the trip was £3 569. Convert this amount into rands.

2.

Sean wants to buy the latest DJ equipment, which has been advertised in a US catalogue for $4 000. He wants to order and pay for the equipment online. The current rand/dollar exchange rate is R12,56 to the US dollar. Calculate the cost in rands of the DJ equipment.

3.

Simone is on a trip to the UK to visit her mom. The current rand/pound exchange rate is R18,50 to the British pound. She has R40 000 to spend in the UK. How many pounds does she have to spend?

4.

Nathan wants to purchase electric drums from a music dealership in England. The drums cost £4 000 and Nathan has saved R75 000. The rand/pound exchange rate is R18,24 per one pound. Will he have enough money, in rands, to buy the drums? Show your reasoning.

5.

A married couple living in England, having saved £4 000, decided to have a holiday in South Africa. The airfare cost £1 100 and they converted the balance of their money into rand which they intended to spend in South Africa. They actually spent R23 000 in South Africa and converted their remaining rands back into pounds when they returned to England. The exchange rate on arrival in South Africa was R18,24 to the pound and R18,46 to the pound when they departed from South Africa.

(1) How much, in rands, had the couple planned to spend in South Africa?
(2) How many pounds did they take back to England?

Activity 14

a) Greg invests R9000 into an account. A year later, he deposits R7000. Three years after the initial investment, he withdraws R4000. Calculate the total amount that Greg will have after 5 years if the interest rate changes as follows:
   - Year 1 and 2: 8% p.a compound interest.
   - Year 3, 4 and 5: 9% p.a compound interest.

b)

R8 000 is deposited into a savings account. The interest rate for the first three years is 3% per annum compounded annually. Thereafter, the interest rate changes to 4% per annum compounded annually. Calculate the value of the investment at the end of the eighth year.

c)

A certain amount of money was invested ten years ago and is now worth R200 000. The interest rate during the first 6 years was 3,5% per annum compounded annually and for the remaining four years, the interest rate was 4,6% per annum compounded annually. How much money was invested ten years ago?

d)

Portia invests R500 000 in the share market. Over a fifteen-year period, the average interest rate was 6% per annum compounded annually for the first eight years and 7% per annum compounded annually for the remaining seven years. How much money will she have saved at the end of the fifteen-year period?

5.3 Calculate the total interest paid over a 24 month period if the cellphone is bought using this hire purchase agreement. (2)

5.4 The cost of the cellphone is subject to inflation and increases to a cash price of R5 100,00 after 2 years. Calculate the annual inflation rate as a percentage. (3)

5.5 The same cellphone is sold for £250 on amazon.uk.

Using the following exchange rate and calculate the price of the cellphone in rand on amazon.uk.

£1	:	R 22,32

(2)

[11]

QUESTION 4

4.1 Seven years ago, Mrs Grey decided to invest R18 000 in a bank account that paid simple interest at 4,5% p.a.

4.1.1 Calculate how much interest Mrs Grey has earned over the 7 years. (2)

4.1.2 Mrs Grey wants to buy a television set that costs R27 660,00 now. If the average rate of inflation over the last 5 years was 6,7% p.a., calculate the cost of the television set 5 years ago. (3)

4.1.3 At what rate of simple interest should Mrs Grey have invested her money 7 years ago if she intends buying the television set now using only her original investment of R18 000 and the interest earned over the last 7 years? (3)

4.2 On a certain day the exchange rate between the US dollar and South African rand is \$1 = R12,91. At the same time the exchange rate between the British pound and the South African rand is £1 = R16,52.

Calculate the exchange rate between the British pound and US dollar on that day. (2)

[10]

QUESTION 6

6.1 Amy needs to buy a new computer. The computer costs R7 990. Amy does not have the full amount of money available, so she has decided to enter into a hire purchase agreement. The terms of the agreement are as follows;

15% per annum simple interest
48 monthly payments
Monthly processing fee of R13,50

Determine the monthly amount payable, including interest and fees. (5)

6.2 On his 29th birthday, John invested R5 000 on the Johannesburg Stock Exchange. His shares increased in value by 20% compounded annually. How much would his investment be worth on his 60th birthday? (5)

[10]

QUESTION 4

4.1 Peter wants to buy a computer costing R7 950, on a hire-purchase agreement. The conditions of the agreement are:

- Peter must pay a deposit of 25% of the purchase price.
- Interest is charged at 15% per annum simple interest on the balance.
- He must also pay a compulsory monthly insurance premium of R70,75.
- The balance is to be settled in monthly instalments.

4.1.1 Calculate the balance after Peter pays the deposit. (2)

4.1.2 If the balance is to be paid off in 24 months, calculate Peter's total monthly instalment. (4)

4.2 The table below shows the cost of one British pound and one US dollar in South African rand.

COUNTRY	UNIT	EXCHANGE RATE
England	Pound (£)	R23,43
USA	Dollar ($)	R14,58

4.2.1 It costs £55 to fill a car with 80 litres of petrol in England. How much will it cost to fill up with the same quantity of petrol if you were paying in South African rand? (1)

4.2.2 An English visitor to the USA notices a car on sale for $5 500. A similar vehicle in England costs £3 500. In which country is the car more expensive? Justify your answer with relevant calculations. (3)

[10]

QUESTION 6

The Nkosi family of three from Dobsonville wish to attend the Soccer World Cup in Qatar in 2022. The cost of the tickets at present is 2 150 Qatari Rials per person, per game. The family would like to attend the last four games.

6.1 Calculate the total cost, in rands, of the tickets for the family if the exchange rate is:

One Rial = R4.012778 (3)

6.2 On 1st January 2019 Mr Nkosi invested R50 000 at a simple interest rate of 13%. Determine whether the value of the investment at the end of 2021 will cover the cost of the tickets. (3)

6.3 Would the family want a strong rand or a weak rand, when they purchase the tickets? Give a reason for your answer. (2)

6.4 The inflation rate over the last four years remained at a constant rate of 0,5%. Calculate the cost of one ticket, in Qatari Rials, four years ago. (3)

[11]

QUESTION 1

1.1 A newly married couple bought furniture on hire-purchase for R60 000.They paid a cash deposit of 20%. The balance will be paid off over 5 years at an interest rate of 22% p.a. They also must pay a monthly insurance fee of R120. This is added to their monthly repayments.

1.1.1 Calculate the amount paid as a deposit. (1)

1.1.2 Calculate their monthly repayments. (4)

1.2 Calculate how many years it will take for an investment, earning 7,5% p.a. simple interest, to double in value. (3)

1.3 The price of a 700g loaf of brown bread in 2015 was R11,50. The current price of a 700g loaf of brown bread in 2023 is R16,00. Determine the rate of inflation, as a percentage, if we assume that the inflation rate remained unchanged for this period. (3)

1.4 The following exchange rates are given:

€ 1 = R 20,05
£ 1 = R 23,40

Use the given information to answer the following questions:

1.4.1 How many rand (R) is €100? (1)

1.4.2 How many pounds (£) is R2300? (1)

1.4.3 How many euros (€) is £250? (2)

[15]

Question 4

4.2 Sipho bought a brand-new Ford Ranger in April 2015 on hire purchase at a cost of R379 000. He agreed on paying 15% deposit and took out a loan for the remaining balance at an interest rate of 22,5%.

4.2.1 How much deposit did Sipho pay? (1)

4.2.2 Hence, calculate the initial value of the loan. (1)

4.2.3 Calculate the value of the loan with interest in April 2019. (3)

4.2.4 Calculate the monthly instalments if he paid off the loan after the four-year period. (2)

4.3 A sum of money was invested 6 years ago, earning interest at a rate of 6,7% p.a. compounded annually. The investment is currently worth R 96 714,02. Calculate how much was originally invested 6 years ago. (3)

[11]

QUESTION 4

4.1 Sylvia wants to buy a Defy dishwasher which is priced at R9 899 by means of a hire purchase agreement.

The conditions of the hire purchase agreement are as follows:

- Sylvia must pay a 30% deposit of the purchase price
- Interest is charged at 12% per annum simple interest on the balance
- Compulsory monthly insurance premium of R65,30
- The balance must be paid in monthly instalments
- Account should be settled in 36 months

4.1.1 Calculate the balance after Sylvia has paid the deposit. (2)

4.1.2 Calculate her monthly instalment, if the settlement must be settled in 36 months. (5)

4.2 The table below shows the exchange rate of the British pound and the US dollar in South African rand.

COUNTRY	UNIT	EXCHANGE RATE
USA	Dollar ($)	R16,24
England	Pound (£)	R27,63

4.2.1 George, a visitor from England, saw an industrial textile machine on sale for $6 800. This machine is suitable for his business back at home. The cost for a similar machine in England is £4 600. Calculate in which country will it be cost saving for George to buy the machine. (3)

4.2.2 To install an outdoor swimming pool will cost you £800 in England. How much will it cost you to install a swimming pool of the same capacity in South African rand? (2)

[12]

Question 6

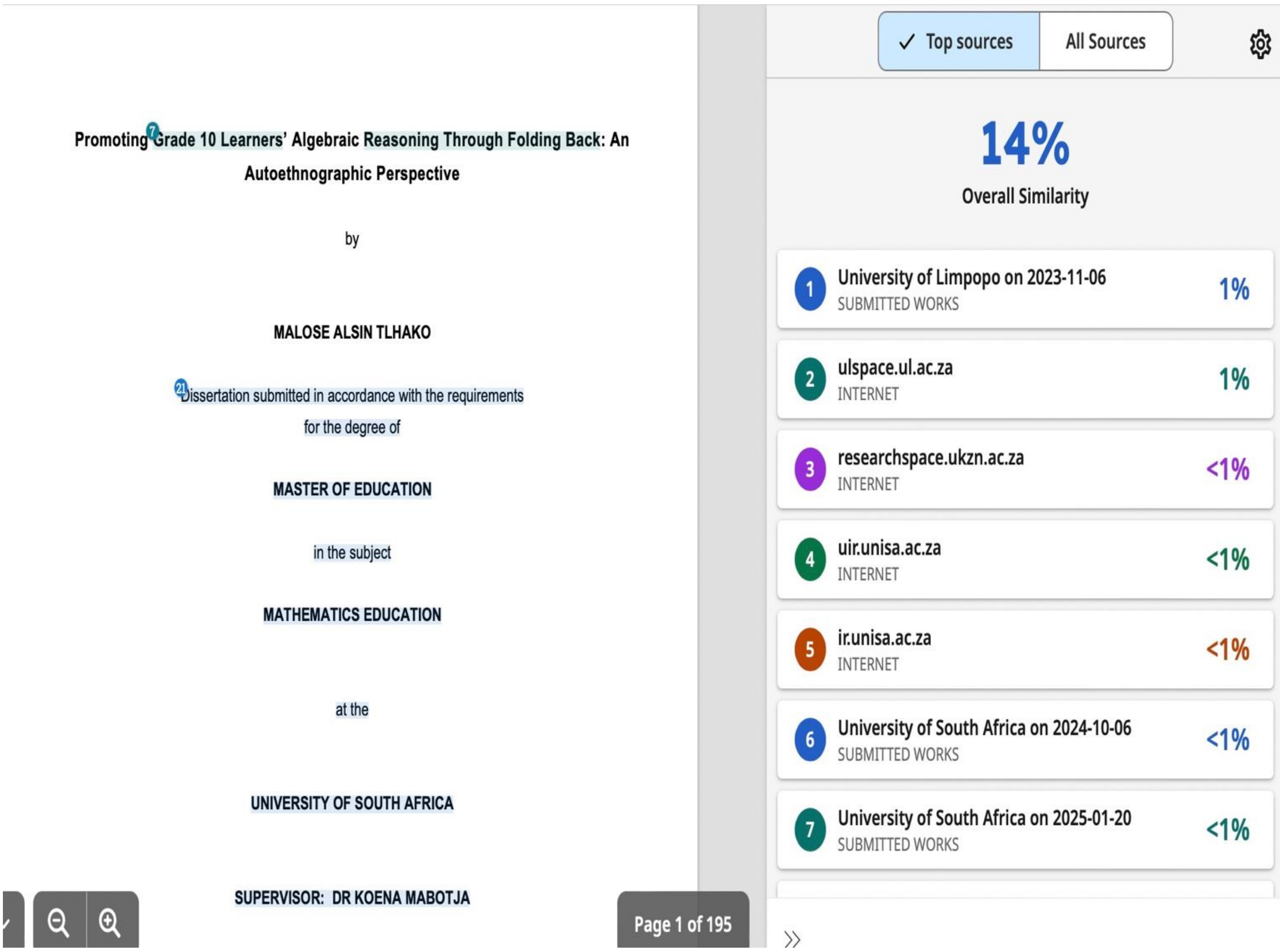
6.2 The population of a city in KwaZulu-Natal is 2 500 000 in the year 2020. Assuming that the population will continue to increase at a constant rate of 5,25% each year, estimate the population of the city at the beginning of 2024. (Give your answer correct to the nearest whole number.) (3)

6.3 If the current exchange rate is \$1 =R19,08 and £1 =R23,31, determine the exchange rate between dollar and pound. (2)

6.4 Brent crude oil costs \$93,78 a barrel. Calculate the cost in rands, of importing a barrel when the exchange rate is R19,08 to the dollar. (2)

[13]

ANNEXURE J: TURNITIN REPORT



ANNEXURE K: LANGUAGE EDITING

information giants

PHONE: 015 004 1103 / 081 284 9339 | **FAX:** 086 681 7699
EMAIL: info@informationgiants.co.za | **WEBSITE:** www.informationgiants.co.za

TO: **WHOM IT MAY CONCERN**

SUBJECT: **Language Editing**

DATE: Wednesday, 31 December 2025

ACKNOWLEDGMENT OF LANGUAGE EDITING

We hereby confirm the language editing of the following research project using the Windows 'tracking' system to reflect our comments and suggested corrections for the writer to action.

Project Title: **"Promoting Grade 10 Learners' Algebraic Reasoning Through Folding Back: An Autoethnographic Perspective"** submitted to us by **MALOSE ALSIN TLHAKO** has been duly edited for language. It is hoped that if all the editorial aspects suggested therein were considered, the target readers of the work would find the document decipherable.

For any enquiries relating to the above, please contact the office during working hours at 081 284 9339 or info@informationgiants.co.za.

Kind Regards,
Sheryl Lawrence

Sheryl

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Disclaimer:
Although we have made comments and suggested corrections, the responsibility for the quality of the final document lies with the writer in the first instance and not with our organisation as the editors.