TOPICS IN POINT-FREE RINGS OF CONTINUOUS FUNCTIONS

by

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Declaration

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I declare that *Topics in point-free rings of continuous functions* is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

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Abstract

Let L be a completely regular frame and $\mathcal{R}L$ be the ring of real-valued continuous functions on L. In this dissertation, we will study the maximal ideals and prime ideals of the ring $\mathcal{R}L$ which were described by Dube in [20] and [25]. We will also study the z-ideals and d-ideals of point-free function rings and how they are related. In the class of f-rings, sufficient conditions for the sum of z-ideals (resp. d-ideals) to be a z-ideal (resp. d-ideal) will be given. In [29], Dube and Ighedo show that, as in $C(X)$, the sum of two z-ideals in $\mathcal{R}L$ is a z-ideal. In [64], the authors study algebraic properties of quotient rings in the ring $C(X)$ of real-valued continuous functions on a Tychonoff space X. We show that this result is actually purely ring theoretic and thus deduce its extensions to the f-rings $\mathcal{R}L$ of real-valued continuous functions on a frame L. We will show that the maximal essential ideals of $\mathcal{R}L$ are precisely the $\boldsymbol{M}$-ideals indexed with the nowhere dense sublocales of βL (the Stone-Čech compactification of L). We will also show that the nowhere dense sublocales of βL are perfect if and only if every prime ideal in $\mathcal{R}L/\boldsymbol{M}^A$ is essential, where A is a closed sublocale of βL.

Keywords: Completely regular frame, Sublocale, Quotient rings, Classical rings of real-valued continuous functions, Ideal, Socle, Essential maximal ideal.

Contents

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Chapter 1

Introduction and preliminaries

The first chapter will be introductory, and consist of basic definitions from commutative rings. It is also within this chapter that we will fix notation and terminology. This is necessary because, unfortunately, terminology in commutative algebra is not standard. We will recall concepts that we will use throughout the dissertation such as: prime ideal, maximal ideal, radical ideal, ring homomorphism, the cozero map and so on. We will also give recount on the full construction of the ring $\mathcal{R}L$ in this chapter.

In the second chapter we study the maximal ideals and prime ideals of $\mathcal{R}L$ which were described by Dube in [20] and [25]. We will characterize maximal ideals and show that maximal ideals of $\mathcal{R}L$ are precisely of the form $\mathbf{M}^J$ where J is a point of βL. We will also show that minimal prime ideals of $\mathcal{R}L$ are precisely the ideals $\mathbf{O}^I$ for I maximal in βL.

In the third chapter we look into z-ideals and d-ideals. For any classical function ring the sum of two z-ideals is a z-ideal (see, [39] and [63]). In [30], it has indeed been shown by Dube and Ighedo that in $\mathcal{R}L$, the sum of two z-ideals is a z-ideal. Furthermore, it is shown by Henriksen and Smith in [44] that there are rings in which the sum of z-ideals is not a z-ideal. In this section we will discuss, mainly in the class of f-rings, sufficient conditions for the sum of z-ideals to be z-ideal. Most results in this regard were proven by Larson (see [52]).

There are special types of z-ideals that have received a great deal of attention both in rings and Riesz spaces. These are called d-ideals, and have been studied by many mathematicians such as Huijsman and de Pagter (see [46]). Other authors, such as Azarpanah and Mohamadian call these z°-ideals (as seen from [6]). We will give various properties and characterizations of d-ideals. We will also give sufficient conditions when their sums are also d-ideals. Rings in which d-ideals coincide with z-ideals will be characterized.

In the fourth chapter, we will discuss the characterizations of various forms of P- and F-frames. This will include almost P and almost F frames as well as quasi P- and quasi F- frames. These have been highly discussed by Dube in [43] as well as in [14] with the most recent work on quasi F frames by Dube and Matlabyane [32].

The last chapter in the dissertation will consist of quotient rings of $\mathcal{R}L$. Much has been written about quotient rings of $C(X)$, which has not been extended to an f-ring $\mathcal{R}L$.

1.1 Rings and ideals

A ring $(A, +, \cdot)$ is a set A such that:

1. $(A, +)$ is an abelian group.
2. Multiplication is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. The distributive laws hold: $a(b + c) = ab + ac \quad \text{and} \quad (a + b)c = ac + bc$.

A ring is commutative if for each $a, b \in A$, $ab = ba$. For any ring A, there is the additive identity (denoted as $\mathbf{0}$) and the multiplicative identity (denoted as $\mathbf{1}$). The multiplicative identity only exists if the A has a unit element such that for every $a \in A$, $1_A \cdot a = a \cdot 1_A = a$. An element is *invertible* if it has a multiplicative inverse. An element is called a *zero divisor* if $a \neq 0$ and there exists a non-zero element $b \in A$ such that $ab = 0$ or $ba = 0$. An element $a \in A$ is *nilpotent* if there exists some positive integer $n \in \mathbb{Z}^+$ such that $a^n = 0$ and a is *idempotent* if $a^2 = a$.

The *annihilator* of $S \subseteq A$ is the ideal $\mathrm{Ann}(S) = \{a \in A \mid as = 0 \text{ for every } s \in S\}$. The double annihilator of $S \subseteq A$ is the ideal $\mathrm{Ann}^2(S) = \{a \in A \mid ay = 0 \text{ for every } y \in \mathrm{Ann}(S)\}$. If S is a singleton, say $S = \{a\}$, we shall abbreviate $\mathrm{Ann}(\{a\})$ as $\mathrm{Ann}(a)$ and $\mathrm{Ann}^2(\{a\})$ as $\mathrm{Ann}^2(a)$. Furthermore $\mathrm{Ann}^3(a) = \mathrm{Ann}(a)$. A ring A is called:

- an *exchange ring* provided that, for any $a \in A$, there exists an idempotent $e \in A$ such that $a + e$ is invertible,

- a *regular ring* if for every $a \in A$ there exists $b \in A$ with $a = aba$, and
- a *reduced ring* if it has no nonzero nilpotent elements.

Let $c : A \to B$ be a ring homomorphism between two rings A and B. The kernel of c, denoted as $\ker(c)$, is $\ker(c) = \{a \in A \mid c(a) = 0_B\}$.

An *ideal* in a bounded distributive lattice L is a subset $J \subseteq L$ such that: $0 \in J$, $a, b \in J \Rightarrow a \vee b \in J$, and $b \leq a$ and $a \in J \Rightarrow b \in J$. An ideal $I = \{0\}$ is called the *zero ideal.* An ideal I of a ring A is:

1. *prime* if for all $a, b \in A$, $ab \in I$ implies $a \in I$ or $b \in I$,
2. *maximal* if there is no ideal C such that $I \subsetneq C \subsetneq A$,
3. *essential* if it meets every nonzero ideal non-trivially,
4. a *principal ideal* if it is generated by one element. We shall denote the ideal generated by an element a by $\langle a \rangle$, and
5. a *minimal ideal* if $I \neq \langle 0 \rangle$ and there is no ideal J with $\langle 0 \rangle \subsetneq J \subsetneq I$. A ring need not contain minimal ideals.

   Example 1.1.1. $\mathbb{Z}$ has no minimal ideals: Every nonzero ideal I in $\mathbb{Z}$ has the form $I = \langle n \rangle$ for some nonzero integer n, and $I = \langle n \rangle \supsetneq \langle 2n \rangle \neq \langle 0 \rangle$.

A prime ideal P containing an ideal I is said to be a *minimal prime* over I if there is no any prime ideal strictly contained in P that contains I. A ring ideal I of an f-ring A is an ℓ-ideal if $|x| \leq |y|$, $y \in I$ implies $x \in I$. Suppose A is a ring and I is an ℓ-ideal of A. The ℓ-ideal I is:

- *semiprime* if $a^2 \in I$ implies $a \in I$, and
- *prime* if $ab \in I$ implies $a \in I$ or $b \in I$.

1.2 f-rings

Recall that an *f-ring* is a lattice-ordered ring A such that for any $a, b \in A$ and any $c \geq 0$,

$$c(a \vee b) = (ca) \vee (cb).$$

Given an f-ring A and $x \in A$, we let

$$A^+ = \{a \in A \,:\, a \geq 0\}$$

and

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0 \quad \text{and} \quad |x| = x \vee (-x).$$

Squares are positive in f-rings. An f-ring has *bounded inversion* property if every element above 1 is invertible. Then, if A has bounded inversion, for any

$$a \in A, \quad \frac{1}{1+|a|} \text{ and } \frac{a}{1+|a|}$$

are bounded elements.

If A is an f-ring, then any $n \in \mathbb{N}$ can be viewed as belonging to A by writing

$$n = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}.$$

The *bounded part* of A, denoted A^*, is the sub-f-ring

$$A^* = \{a \in A \colon |a| \leq n \text{ for some } n \in \mathbb{N}\}.$$

An element of α of $\mathcal{R}L$ is bounded if there exist $p, q \in \mathbb{Q}$ such that $\alpha(p, q) = 1_L$.

1.3 Frames and their homomorphisms

A complete lattice L is called a *frame* if the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for every $a \in L$ and $S \subseteq L$. We shall denote the top element of L by 1_L and the bottom element by 0_L (some authors denote the top element and the bottom element by $\top$ and $\bot$ respectively), dropping the subscripts if L is clear from the context. An element x of L is said to be *rather below* an element y, written $x \prec y$, in case there is an element s, called a *separating element*, such that $x \wedge s = 0$ and $s \vee y = 1$. On the other hand, x is *completely below* y, written $x \prec\!\!\prec y$, if there are elements (a_r) indexed by the rational numbers $\mathbb{Q} \cap [0, 1]$ such that $x = a_0, a_1 = y$ and $a_r \prec a_s$ for $r < s$. A frame L is *regular* if

$$a = \bigvee \{x \in L \mid x \prec a\} \text{ for each } a \in L,$$

and *completely regular* if

$$a = \bigvee\{x \in L \mid x \prec\!\!\prec a\} \text{ for each } a \in L.$$

The *pseudocomplement* of an element a of L is the element

$$a^* = a \to 0 = \bigvee\{x \in L \mid x \wedge a = 0\}$$

and it is sometimes denoted by $a^\perp$. An element a of a frame L is called *dense* if $a^* = 0$, and *complemented* if $a \vee a^* = 1$.

A *frame homomorphism* is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. A frame homomorphism $h : L \to M$ is called *dense* if for any $a \in L$, $h(a) = 0_M$ implies $a = 0_L$. If h is a *dense onto* frame homomorhism then $h(a^*) = h(a)^*$ for every $a \in L$.

Given a frame L and an element $a \in L$, the sets

$${\uparrow}a = \{x \in L \mid x \geq a\} \qquad \text{and} \qquad {\downarrow}a = \{x \in L \mid x \leq a\}$$

are frames, and the maps

$$L \to {\uparrow}a \qquad \text{and} \qquad L \to {\downarrow}a,$$

given respectively by $x \mapsto a \vee x$ and $x \mapsto a \wedge x$, are onto homomorphisms. A frame L is said to be *normal* if for $a, b \in L$ we have that $a \vee b = 1$, then we can get $c, d \in L$ such that $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$.

A *point* of L is an element $p < 1$ such that $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. Points of a frame are also called *prime elements.* We denote the set of all points of L by $\mathrm{Pt}(L)$. The points of any regular frame L are precisely those elements which are maximal in the poset $L \smallsetminus \{1\}$, which means that those are elements which are maximal strictly below the top. For a given point in any regular frame, the point is either dense or complemented. A frame has enough points if every element is a meet of points above it. Frames that have enough points are said to be *spatial.* Note that if x is dense and $x \prec\!\!\prec y$, then $y = 1$. Thus, if $J \in \mathrm{Pt}(L)$, then it does not contain a dense element (lest it contains the top). An ideal J of L is *completely regular* if for every $x \in J$, there exists $y \in J$ such that $x \prec\!\!\prec y$.

1.4 The ring $\mathcal{R}L$ and the cozero map

Every completely regular frame L has associated with it the ring $\mathcal{R}L = (\mathbf{Frm}(\mathcal{R}L), L)$ of its continuous real functions. The *frame of reals* is the completely regular frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q), with $p, q \in \mathbb{Q}$, subject to the following relations:

1. $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
2. $(p, q) \vee (r, s) = (p, s) : p \leq r < q \leq s$,
3. $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$, and
4. $1 = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

The ring $\mathcal{R}L$ has as its elements frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, with operations induced by those of $\mathbb{Q}$ as an ℓ-ring. We denote the zero of this ring and its identity by $\mathbf{0}$ and $\mathbf{1}$, respectively.

For any Tychonoff space X, the two rings $C(X)$ and $R(\mathfrak{O}X)$ are isomorphic based on the fact that $\Sigma(\mathfrak{L}(\mathbb{R})) \cong \mathbb{R}$ and hence $Top(X, \mathbb{R}) \cong Frm(\mathfrak{L}(\mathbb{R}), \mathfrak{O}X)$ by the relation between the functors Σ and $\mathfrak{O}$, taking $a \in C(X)$ to

$$\tilde{a} : \mathfrak{L}(\mathbb{R}) \to \mathfrak{O}X$$

such that $\tilde{a}(p, q) = a^{-1}[\{x \in \mathbb{R} \mid p < x < q\}]$ (See[10]).

Lemma 1.4.1. *Every idempotent of $\mathcal{R}L$ is bounded.*

Proof. We use the fact that $\mathcal{R}L$ is an f-ring. Let μ be an idempotent in $\mathcal{R}L$. Then

$$\begin{aligned}\mu \wedge 1 &= \tfrac{1}{2}(1 + \mu - |1 - \mu|)\\ &= \tfrac{1}{2}(1 + \mu - 1 + \mu) \text{ since } 1 - \mu \geq 0\\ &= \mu,\end{aligned}$$

showing that $0 \leq \mu \leq 1$. Therefore μ is bounded. $\square$

Every $\mathcal{R}L$ is a reduced f-ring with bounded inversion.

A frame L and its ring of real-valued continuous functions $\mathcal{R}L$ are linked by the cozero map $\operatorname{coz}: \mathcal{R}L \to L$ defined by

$$\begin{aligned}\operatorname{coz}\varphi &= \bigvee\{\varphi(p,0) \vee \varphi(0,q) \mid p,q \in \mathbb{Q}\}\\ &= \varphi((-,0) \vee \varphi(0,-))\end{aligned}$$

where for any $r \in \mathbb{Q}$,

$$(-,r) = \bigvee\{(p,r) \mid p < r \in \mathbb{Q}\} \text{ and } (r,-) = \bigvee\{(r,q) \mid q > r \in \mathbb{Q}\}.$$

A *cozero element* of L is an element of the form $\operatorname{coz}\alpha$ for some $\alpha \in \mathcal{R}L$. The collection of all cozeros of L, denoted $\operatorname{Coz} L$, forms a sub-σ-frame of L, i.e., a sublattice which is closed under countable joins and satisfies the frame distributivity law for countable joins.

Proposition 1.4.2. *The following hold for any $\alpha, \beta \in \mathcal{R}L$.*

1. $\operatorname{coz}\alpha = \operatorname{coz}(|\alpha|) = \operatorname{coz}(\alpha^2)$.
2. $\operatorname{coz}\alpha = 0$ iff $\alpha = \mathbf{0}$.
3. $\operatorname{coz}\beta = 1$ iff β is invertible in $\mathcal{R}L$.
4. $\operatorname{coz}(\alpha\beta) = \operatorname{coz}\alpha \wedge \operatorname{coz}\beta$.
5. $\operatorname{coz}(\alpha + \beta) \leq \operatorname{coz}\alpha \vee \operatorname{coz}\beta$.
6. $\operatorname{coz}(\alpha^2 + \beta^2) = \operatorname{coz}\alpha \vee \operatorname{coz}\beta$.

A homomorphism $h : L \to M$ is said to be:

- *coz-onto* if, for every $c \in \operatorname{Coz} M$, there exists $d \in \operatorname{Coz} L$ such that $h(d) = c$;
- *coz-codense* if, for any $c \in \operatorname{Coz} L$, $h(c) = 1$ implies $c = 1$;
- *almost coz-codense* if, for any $c \in \operatorname{Coz} L$, $h(c) = 1$ implies there exists $d \in \operatorname{Coz} L$ such that $d \vee c = 1$ and $h(d) = 0$.

Note that a frame L is *realcompact* if every maximal ideal of $\mathrm{Coz}\, L$ that has join equal to the top has a countable subset whose join is the top. Realcompact completely regular frames form a coreflective subcategory of the category **CRFrm** of completely regular frames. The *universal Lindelöfication* of L, denoted by λL, is the frame of σ-ideals of $\mathrm{Coz}\, L$. The join map $\lambda L : \lambda L \to L$ is a dense onto frame homomorphism. The *Hewitt realcompactification of* L, denoted by υL, has the join map $\upsilon L : \upsilon L \to L$ which is a dense onto frame homomorphism. For any L we have

$$\mathrm{Coz}(\lambda L) = \mathrm{Coz}(\upsilon L) = \{[c] \mid c \in \mathrm{Coz}\, L\}, \text{ where } [c] = (\upsilon_L)_*(c)$$

for which we then have that each of the maps λL and υL are C-quotient maps, $\mathrm{Pt}(\lambda L) = \mathrm{Pt}(\upsilon L)$ and for any $a \in L, (\lambda L)_*(a) = (\upsilon L)_*(a) = [a]$.

We finalize the section with the following lemma from [25].

Lemma 1.4.3. *Any $\delta \in \mathcal{R}L$ is a nonzero divisor if and only if $\mathrm{coz}\, \delta$ is dense.*

Proof.

$$\begin{aligned}
\mathrm{coz}\, \delta \text{ is dense} &\iff (\mathrm{coz}\, \delta)^* = 0 \\
&\iff (\mathrm{coz}\, \delta)^* \leq 0 = (\mathrm{coz}\, \mathbf{1})^* \\
&\iff \mathrm{Ann}(\delta) \subseteq \mathrm{Ann}(\mathbf{1}) = \{\mathbf{0}\} \\
&\iff \mathrm{Ann}(\delta) = \{\mathbf{0}\} \\
&\iff \delta \text{ is a nonzero divisor.}
\end{aligned}$$

This concludes the proof. □

1.5 The Stone-Čech Compactification

An element c of a frame L is said to be *compact* if for any

$$S \subseteq L, c \leq \bigvee S \text{ implies } c \leq \bigvee T, \text{ for some finite } T \subseteq S.$$

If the top of L is compact, we say the frame itself is *compact.* The set of all completely regular ideals of L is a compact regular frame, denoted by βL, and the join map $\beta L \to L$

is the coreflection map from compact completely regular frames to L and its right adjoint is consequently denoted by r_L. Recall that a frame L is said to be *normal* if for $a, b \in L$ we have that $a \vee b = 1$, then we can get $c, d \in L$ such that $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$. If L is normal, then $r_L : L \to \beta L$ preserves finite joins.

Recall that an ideal I of $\operatorname{Coz} L$ is called *completely regular* if for every $u \in I$ there exists $v \in I$ such that $u \prec\!\!\prec v$. Then βL is the lattice of all completely regular ideals of $\operatorname{Coz} L$. The mapping

$$j_L \colon \beta L \to L \quad \text{defined by} \quad j_L(I) = \bigvee I$$

is dense onto, and is the coreflection map to L from compact completely regular frames. Its right adjoint is given by

$$r_L(a) = \{c \in \operatorname{Coz} L \mid c \prec\!\!\prec a\}.$$

Some other properties of the mapping $r_L \colon L \to \beta L$ that we shall use are:

(a) If $a \prec\!\!\prec b$ in L, then $r_L(a) \prec\!\!\prec r_L(b)$ in βL.

(b) If $a \wedge b = 0$, then $r_L(a \vee b) = r_L(a) \vee r_L(b)$.

(c) If $c, d \in \operatorname{Coz} L$, then $r_L(c \vee d) = r_L(c) \vee r_L(d)$.

(d) For any $I \in \beta L$, $I^* = r_L((\bigvee I)^*)$.

1.6 Sublocales

A frame is also called a *locale*, especially when frame homomorphisms are not considered as part of the discussion. Every frame is a Heyting algebra, with the Heyting implication explicitly given by

$$a \to b = \bigvee\{x \in L \mid a \wedge x \leq b\}.$$

A *sublocale* of a frame L is a subset $A \subseteq L$ such that:

- for every $A \subseteq S$, $\bigwedge A \in S$, and
- for every $a \in L$ and $s \in S$, $a \to s \in S$.

A sublocale is a frame in its own right, with meets (and hence the partial order) calculated in L. The lattice of all sublocales of L is denoted by $\mathcal{S}(L)$. The meet in this lattice is intersection, and the join of any collection $\{S_i \mid i \in I\} \subseteq \mathcal{S}(L)$ is given by

$$\bigvee_i S_i = \Big\{\bigwedge M \mid M \subseteq \bigcup_i S_i\Big\}.$$

Partially ordered by inclusion, $\mathcal{S}(L)$ is a *coframe*, which is to say for any $S \in \mathcal{S}(L)$ and any family $\{S_i \mid i \in I\}$ of sublocales, the distributive law below holds:

$$S \vee \bigwedge_i S_i = \bigwedge_i (S \vee S_i).$$

The smallest sublocale of L is $\{1\}$, and is usually denoted by O. It is called the *void sublocale*. If T and S are sublocales, we say T *misses* S, or T and S are *disjoint*, if $S \cap T = \mathsf{O}$. A sublocale of a frame is said to be *nowhere dense* [62] if it misses the smallest dense sublocale of the frame.

A sublocale of L is *complemented* if it has a complement in $\mathcal{S}(L)$. Complemented sublocales are *linear*, meaning that if C is a complemented sublocale, then

$$C \cap \bigvee\{S_i \mid i \in I\} = \bigvee\{C \cap S_i \mid i \in I\},$$

for any family $\{S_i \mid i \in I\}$ of sublocales. In fact, complemented sublocales are precisely the linear ones. Unlike the lattice of subspaces of a topological space, $\mathcal{S}(L)$ is not always a Boolean algebra. Thus, in general, not every sublocale has a complement. However, every sublocale S has a *supplement* (which is dual to pseudocomplement in frames), denoted $L \smallsetminus S$ or $S^{\#}$, and given by

$$L \smallsetminus S = \bigcap\{T \in \mathcal{S}(L) \mid T \vee S = L\} = \bigvee\{R \in \mathcal{S}(L) \mid R \cap S = \mathsf{O}\}.$$

The *open* and the *closed* sublocales corresponding to each $a \in L$ are, respectively, the sublocales

$$\mathfrak{o}_L(a) = \{a \to x \mid x \in L\} = \{x \mid x = a \to x\} \quad \text{and} \quad \mathfrak{c}_L(a) = {\uparrow}a = \{x \in L \mid x \geq a\}.$$

We shall at times drop the subscript if there is only one frame under discussion. If $a \in \operatorname{Coz} L$, we say $\mathfrak{o}(a)$ is a *cozero-sublocale*, and $\mathfrak{c}(a)$ is a *zero-sublocale*.

We shall use some of these properties of open and closed sublocales freely.

1. $\mathfrak{o}(0) = \mathfrak{c}(1) = \mathsf{O} \quad$ and $\quad \mathfrak{o}(1) = \mathfrak{c}(0) = L$.

2. $\mathfrak{c}(a) \subseteq \mathfrak{o}(b)$ if and only if $a \vee b = 1$.

3. $\mathfrak{o}(a) \subseteq \mathfrak{c}(b)$ if and only if $a \wedge b = 0$.

4. $\mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b) \quad$ and $\quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$.

5. $\bigvee_i \mathfrak{o}(a_i) = \mathfrak{o}\Big(\bigvee_i a_i\Big) \quad$ and $\quad \bigcap_i \mathfrak{c}(a_i) = \mathfrak{c}\Big(\bigvee_i a_i\Big)$.

The *closure* of a sublocale S of L, denoted $\overline{S}$ or $\mathrm{cl}_L(S)$, and its *interior*, denoted S° or $\mathrm{int}_L(S)$, are the sublocales

$$\overline{S} = \bigcap\{\mathfrak{c}(a) \mid S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}\Big(\bigwedge S\Big) \quad \text{and} \quad S^\circ = \bigvee\{\mathfrak{o}(a) \mid \mathfrak{o}(a) \subseteq S\} = \mathfrak{o}\Big(\bigwedge(L \smallsetminus S)\Big).$$

In particular, $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ and $\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$. A sublocale S of L is *dense* if $\overline{S} = L$. This is the case if and only if the bottom element of S is the bottom element of L. Every frame has the smallest dense sublocale, denoted $\mathfrak{B}L$, and called the *Booleanization* of L. As a set,

$$\mathfrak{B}L = \{a \in L \mid a = a^{**}\} = \{b^* \mid b \in L\},$$

and joins in $\mathfrak{B}L$ are given by

$$\bigvee^{\mathfrak{B}L} S = \Big(\bigvee S\Big)^{**}$$

for any $S \subseteq \mathfrak{B}L$. The mapping

$$\mathsf{b}_L \colon L \to \mathfrak{B}L \quad \text{given by} \quad \mathsf{b}_L(x) = x^{**}$$

is a dense onto frame homomorphism, whose right adjoint is the identical embedding $\mathfrak{B}L \rightarrowtail L$.

Let f be a mapping $f \colon L \to M$ and $f^* \colon M \to L$ be the left adjoint of f. Then $f \colon L \to M$ is called a *localic map* if for every $a \in L$, $b \in M$, and $S \subseteq L$:

(L1) $f(\bigwedge S) = \bigwedge f[S]$ (and, in particular, $f(1) = 1$),

(L2) $f(f^*(b) \to a)) = b \to f(a)$, and

(L3) $f(a) = 1$ implies $a = 1$.

It follows that f^* is a frame homomorphism, and if h is a frame homomorphism, then h_* is a localic map. A localic map $f\colon L \to M$ gives rise to two mappings

$$f[-]\colon \mathcal{S}(L) \to \mathcal{S}(M) \quad \text{and} \quad f_{-1}[-]\colon \mathcal{S}(M) \to \mathcal{S}(L)$$

given by

$$f[S] = \{f(x) \mid x \in S\} \quad \text{and} \quad f_{-1}[T] = \bigvee \{A \in \mathcal{S}(L) \mid A \subseteq f^{-1}[T]\}.$$

We have that $f[-]$ preserves all joins, and $f_{-1}[-]$ preserves all meets (recall that they are intersections) and all binary joins, which then makes the mapping

$$f_{-1}[-]\colon \mathcal{S}(M)^{\mathrm{op}} \to \mathcal{S}(L)^{\mathrm{op}}$$

a frame homomorphism whose right adjoint is the mapping $f[-]$.

Then, for any $b \in M$,

$$f_{-1}[\mathfrak{o}_M(b)] = \mathfrak{o}_L(f^*(b)) \quad \text{and} \quad f_{-1}[\mathfrak{c}_M(b)] = \mathfrak{c}_L(f^*(b)).$$

Chapter 2

On some ideals of $\mathcal{R}L$

2.1 Prime and maximal ideals

In this chapter we aim to study the maximal ideals and prime ideals of $\mathcal{R}L$ which were described by Dube in [20] and [25]. Before we do that we shall recall necessary terminology and notation. Recall that a *prime ideal* is an ideal I such that if $ab \in I$, then either $a \in I$ or $b \in I$. An ideal I is called *maximal* if it is a proper ideal and there is no proper ideal J such that $I \subsetneq J \subsetneq A$.

Let $h\colon \beta L \to L$ be a frame homomorphism. We shall denote by $\boldsymbol{M}^h$ and $\boldsymbol{O}^h$ by

$$\boldsymbol{M}^h = \{\varphi \in \mathcal{R}L \mid h(r_L(\operatorname{coz}\varphi)) = 0\} \text{ and } \boldsymbol{O}^h = \{\varphi \in \mathcal{R}L \mid h(r_L(\operatorname{coz}\varphi)^*) = 1\}.$$

In particular, if $J \in \beta L$ and $h\colon \beta L \to {\uparrow}J$ is the frame homomorphism given by $I \to I \vee J$, then we shall denote $\boldsymbol{M}^h$ and $\boldsymbol{O}^h$ by

$$\boldsymbol{M}^J = \{\varphi \in \mathcal{R}L \mid r_L(\operatorname{coz}\varphi) \subseteq J\} \quad \text{and} \quad \boldsymbol{O}^J = \{\varphi \in \mathcal{R}L \mid J \vee h(r_L(\operatorname{coz}\varphi)^*) = 1\}.$$

Because the map r_L commutes with pseudocomplements, and the rather below relation matches the completely below relation in normal frames, then

$$\boldsymbol{O}^J = \{\varphi \in \mathcal{R}L \mid r_L(\operatorname{coz}\varphi) \prec\!\!\prec J\}$$

and as a result,

$$\boldsymbol{O}^J = \{\varphi \in \mathcal{R}L \mid \operatorname{coz}\varphi \in J\}.$$

Proposition 2.1.1. *For any $J \in \beta L$, $\boldsymbol{O}^J \subseteq \boldsymbol{M}^J$.*

Proof. Suppose that $\varphi \in \mathbf{O}^J$, then $\operatorname{coz}\varphi \in J$. Then $r_L(\operatorname{coz}\varphi) \subseteq J$ and therefore $\varphi \in \mathbf{M}^J$, showing that $\mathbf{O}^J \subseteq \mathbf{M}^J$. $\square$

Before stating one of the most important result in this section, we shall need the following lemma from [25, Lemma 4.4].

Lemma 2.1.2. *Suppose* $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\delta$ *for some* $\gamma, \delta \in \mathcal{R}L$*. Then there exists* $\tau \in \mathcal{R}L$ *such that* $\gamma = \tau\delta$*.*

Proof. Since $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\delta$ by hypothesis, there exists $\tau \in \mathcal{R}L$ with $\operatorname{coz}\gamma \wedge \operatorname{coz}\tau = 0$ and $\operatorname{coz}\tau \vee \operatorname{coz}\delta = 1$. Then $\operatorname{coz}(\tau^2 + \delta^2) = 1$ since $\tau^2, \delta^2 \geq \mathbf{0}$ as squares are nonnegative in any f-ring. Consequently, $\tau^2 + \delta^2$ is invertible and so now we have that $\operatorname{coz}\gamma\tau = 0$, such that $\gamma\tau = 0$ and therefore,

$$\gamma = \gamma\frac{\tau^2 + \delta^2}{\tau^2 + \delta^2} = \frac{\gamma\tau^2 + \gamma\delta^2}{\tau^2 + \delta^2} = \delta\frac{\gamma\delta}{\tau^2 + \delta^2},$$

this concludes the proof. $\square$

Theorem 2.1.1. *If* Q *is a prime ideal, there is a unique* $I \in \mathrm{Pt}(\beta L)$ *such that* $\boldsymbol{O}^I \subseteq Q \subseteq \boldsymbol{M}^I$*.*

Proof. Let I be defined as $I = \bigvee\{r_L(\operatorname{coz}\delta) \mid \delta \in Q\}$. We then have the inclusion $Q \subseteq M^I$ from above. To show the other inclusion, that is $\boldsymbol{O}^I \subseteq Q$, let $\alpha \in \boldsymbol{O}^I$ such that $\operatorname{coz}\alpha \in I$ and then take $c \in \operatorname{Coz} L$ such that $\operatorname{coz}\alpha \prec\!\!\prec c \in I$. Let $\varphi \in \mathcal{R}L$ such that $\operatorname{coz}\varphi = c$ which has it that $\varphi \in Q$ such that $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\varphi$. So, there exists $\nu \in \mathcal{R}L$ by Lemma 2.1.2 such that $\alpha = \nu\varphi$ which means $\alpha \in Q$ since Q is an ideal. $\square$

A *2-boundary sublocale* is a sublocale of L such that it is of the form

$$\mathfrak{c}(c^*) \cap \mathfrak{c}(d^*) \text{ for some } c, d \in \operatorname{Coz} L \text{ with } c \wedge d = 0.$$

Note that, $\mathfrak{c}(c^*) \cap \mathfrak{c}(d^*) = \uparrow\!(c^* \vee d^*)$. We now characterize primeness of $\boldsymbol{O}^I$ when a frame L is normal as shown in [18].

Proposition 2.1.3. *The following are equivalent for a normal frame* L*.*

1. $\boldsymbol{O}^I$ *is prime for every* $I \in \mathrm{Pt}(\beta L)$ *with* $\bigvee I = 1$*.*

2. *Every 2-boundary sublocale of* L *is compact.*

Proof. (1) $\Rightarrow$ (2): Assume that (1) holds, and let $c \wedge d = 0$ in $\operatorname{Coz} L$. It suffices to show that $\uparrow(c^* \vee d^*)$ is compact, we use the compactness criterion quoted above. For brevity, write $a = c^* \vee d^*$. We prove this by contradiction, so suppose $\uparrow a$ is not compact. Then, there exists an $I \in \operatorname{Pt}(\beta(\uparrow a))$ such that $\bigvee I = 1$. For every $a \in L$, the frame homomorphism $\kappa_a : L \to \uparrow a$ is given by $x \mapsto a \vee x$. We take the homomorphism

$$\kappa_a : L \to \uparrow a$$

and write its Stone extension as

$$\bar{\kappa} : \beta L \to \beta(\uparrow a)$$

after suppressing a and see that $\bigvee \bar{\kappa}_*(I) = 1$ from the proof of [19, Proposition 4.8]. Now, $\bar{\kappa}_*(I) \in \operatorname{Pt}(\beta L)$ where

$$\bar{\kappa}_*(I) = \bigcup\{r_L(u) \mid u \in I\}.$$

By hypothesis, $\boldsymbol{O}^{\bar{\kappa}_*(I)}$ is prime in $\mathcal{R}L$ and therefore $\bar{\kappa}_*(I)$ is a prime ideal in $\operatorname{Coz} L$. We can assume that $c \in \bar{\kappa}_*(I)$ since $c \wedge d = 0$. This implies $c \prec\!\!\prec u$ for some $u \in I$. However, this means $c^* \vee u = 1$, and therefore, $(c^* \vee d^*) \vee u = 1$. Since I is an ideal in the frame $\uparrow(c^* \vee d^*)$, it contains the bottom of this frame($\uparrow(c^* \vee d^*)$), which then implies $1 \in I$. However, this is false since I is a point in $\beta(\uparrow a)$ thereby a contradiction which proves that $\uparrow(c^* \vee d^*)$ is compact.

(2) $\Rightarrow$ (1): Assume that (2) holds, and let $I \in \operatorname{Pt}(\beta L)$ be such that $\bigvee I = 1$. $\boldsymbol{O}^{\boldsymbol{I}}$ is a prime ideal since $\alpha\beta = \mathbf{0}$ given any two functions $\alpha, \beta \in \mathcal{R}L$ and if $\alpha \notin \boldsymbol{O}^{\boldsymbol{I}}$ then $\beta \in \boldsymbol{O}^{\boldsymbol{I}}$ as per the definition of a prime ideal. Let us set $a = \operatorname{coz}\alpha$ and $b = \operatorname{coz}\beta$. Then, a and b are cozero elements with $a \wedge b = 0$ and therefore by (2), $\uparrow(a^* \vee b^*)$ is compact, as then the cover $\{a^* \vee b^* \vee s \mid s \in I\}$ of $\uparrow(a^* \vee b^*)$ has a finite subcover. In light of I being an ideal, there is the implication that there exists some $c \in I$ such that $a^* \vee b^* \vee c = 1$ and since L is normal, there are cozero elements $u \leq a^*$ and $v \leq b^*$ such that $u \vee v \vee c = 1$ as seen in [8, Corollary 8.3.2]. Next we let $u = \operatorname{coz}\rho$ and $v = \operatorname{coz}\tau$ where $\rho, \tau \geq \mathbf{0}$ and $\rho, \tau \in \mathcal{R}L$. Since $a \wedge u = 0$, then see that $\alpha\rho = \mathbf{0}$ such that $\operatorname{coz}(\alpha\rho) = 0$ and since $\alpha \notin \boldsymbol{O}^{\boldsymbol{I}}$ and $\boldsymbol{O}^{\boldsymbol{I}} = O_{\boldsymbol{M}^I}$, it then follows that $\rho \in \boldsymbol{M}^{\boldsymbol{I}}$. Then the function $\gamma + \rho + \tau$ is invertible since $\operatorname{coz}(\gamma + \rho + \tau) = 1$, and therefore the function $\gamma + \rho + \tau \notin \boldsymbol{M}^{\boldsymbol{I}}$. Observe that $\gamma \in \boldsymbol{M}^{\boldsymbol{I}}$ because $c \in I$ implies $r_L(\operatorname{coz}\gamma) \subseteq I$. Thus, $\gamma + \rho \in \boldsymbol{M}^{\boldsymbol{I}}$, which then implies $\tau \notin \boldsymbol{M}^{\boldsymbol{I}}$. However, $\tau\beta = \mathbf{0}$ as $v \wedge b = 0$. Therefore, it follows that $\beta \in \boldsymbol{O}^{\boldsymbol{I}}$ since $\boldsymbol{O}^{\boldsymbol{I}}$ is the zero-component of $\boldsymbol{M}^{\boldsymbol{I}}$ and thus, $\boldsymbol{O}^{\boldsymbol{I}}$ is a prime ideal. $\square$

Next we show that the sum of two prime ideals in $\mathcal{R}L$ is prime. For use in the proof of the proposition below, we recall that an ideal I of an f-ring R is *square dominated* if

$$I = \{r \in R \mid |r| \leq a^2 \text{ for some } a \in R \text{ with } a^2 \in I\}$$

and we will first need the following lemma from [42].

Lemma 2.1.4. *A semiprime ℓ-ideal of an f-ring A that contains a prime ℓ-ideal is prime. In particular if P and Q are square-dominated semiprime ℓ-ideals of A and one of which is prime, then $P + Q$ is prime.*

Proposition 2.1.5. *The sum of two prime ideals in $\mathcal{R}L$ is a prime ideal.*

Proof. It is clear that an ℓ-ideal in an f-ring in which positive elements are squares (as is the case in $\mathcal{R}L$, see [21, Proposition 11] for more details) is square dominated. Since prime ideals in $\mathcal{R}L$ are ℓ-ideals, it follows from Lemma 2.1.4 that the sum of two prime ideals in $\mathcal{R}L$ is a prime ideal. $\square$

We now focus on the maximal ideals of the ring $\mathcal{R}L$. We recall from [25] the characterization of maximal ideals in $\mathcal{R}L$. Before presenting the main result, we require the following lemma and introduce the necessary notation. If A is an ideal of $\mathcal{R}L$, then $C[A]$ denotes the ideal of $\operatorname{Coz} L$ given by $C[A] = \{\operatorname{coz} \alpha \mid \alpha \in A\}$.

Lemma 2.1.6. *The ideal $\boldsymbol{M}^I$ is maximal if and only if I is a point of βL.*

Proof. ($\Leftarrow$): Let $c \in \operatorname{Coz} L$ such that $c \vee \operatorname{coz} \alpha \notin 1$ for each $\alpha \in \boldsymbol{M}^J$. We must show that $c \in C[\boldsymbol{M}^J]$. Take $\gamma \in \mathcal{R}L$ such that $c = \operatorname{coz} \gamma$, and suppose, for contradiction, that $c \notin C[\boldsymbol{M}^J]$. Then $\gamma \notin \boldsymbol{M}^J$, which implies that $r_L(\operatorname{coz} \gamma) * J$. Since J is a maximal element, this implies that $r_L(\operatorname{coz} \gamma) \vee J = 1$. Therefore, there exists $s \in J$ such that $\operatorname{coz} \gamma \vee s = 1$. Take a cozero element $u \in J$ such that $s \leq u$. Let $\mu \in \mathcal{R}L$ such that $u = \operatorname{coz} \mu$. Therefore $r_L(\operatorname{coz} \mu) \subseteq J$, and so $\mu \in \boldsymbol{M}^J$, when $\operatorname{coz} \mu \in C[\boldsymbol{M}^J]$; a contradiction since $c \vee \operatorname{coz} \mu = 1$.
($\Rightarrow$): We first verify that $\boldsymbol{M}^J = \boldsymbol{M}^K$ implies $J = K$. Let $a \in J$ and take $s \in \operatorname{Coz} L$ such that $s \in J$ and $a \prec\!\!\prec s$. Next, let $\sigma \in \mathcal{R}L$ such that $s = \operatorname{coz} \sigma$. Then $r_L(\operatorname{coz} \sigma) \subseteq J$, implying that $\sigma \in \boldsymbol{M}^J = \boldsymbol{M}^K$, and hence $r_L(\operatorname{coz} \sigma) \subseteq K$. This implies that $a \in K$, and therefore $J \subseteq K$. By symmetry, $K \subseteq J$. Now let $H \in \beta L$ such that $J \subseteq H$. Then $\boldsymbol{M}^J \subseteq \boldsymbol{M}^H$, which implies that

$\boldsymbol{M}^J = \boldsymbol{M}^J$ or $\boldsymbol{M}^J = \mathcal{R}L$ by maximality. The latter implies $H = \beta L$, and so J is a maximal element as required. $\square$

Theorem 2.1.2. *A subset Q of $\mathcal{R}L$ is a maximal ideal if and only if there exists a unique $J \in \mathrm{Pt}(L)$ such that $Q = \boldsymbol{M}^J$.*

Proof. Pick an element α of Q and let $J \in \mathrm{Pt}(\beta L)$ such that $\alpha \in \boldsymbol{M}^J$. Since $\boldsymbol{M}^J$ is a proper ideal, then α is a zero-divisor. Pick a nonzero $\gamma \in \mathcal{R}L$ such that $\alpha\gamma = \mathbf{0}$. Since $\gamma \neq \mathbf{0}$, $r_L(\operatorname{coz}\gamma) \neq 0$. By spatiality of βL it follows that there exists $I \in \mathrm{Pt}(\beta L)$ such that $r_L(\operatorname{coz}\gamma) \nsubseteq I$. Consequently $r_L(\operatorname{coz}\gamma) \vee I = 1$ by maximality of I. But $r_L(\operatorname{coz}\gamma) \wedge r_L(\operatorname{coz}\alpha) = 0$, so $r_L(\operatorname{coz}\alpha) \prec\!\!\prec I$, which implies that $\alpha \in Q$.

Conversely, let us define $I \in \beta L$ such that

$$I = \bigvee \{ r_L(\operatorname{coz}\delta) \mid \delta \in Q \},$$

then $I \neq 1$. By the argument in the proof of [25, Proposition 4.16] that did not use the property normality, the ideal Q contain some invertible element. Then, there exists some prime element of βL such that $I \subseteq J$ and since if $\alpha \in Q$, then $r_L(\operatorname{coz}\delta) \in I \subseteq J$, the ideal $\boldsymbol{M}^J$ contains Q. Maximality therefore implies that $Q = \boldsymbol{M}^J$ while the uniqueness of such a J is seen in the proof of Lemma 2.1.6 above. $\square$

We end this section by showing that the ring $\mathcal{R}L$ is a Gelfand ring. Recall that a ring A is Gelfand if $x + y = 1$ in A implies that $(1 + xr)(1 + ys) = 0$ for some $r, s \in A$. This is to mean that all prime ideals are contained in unique maximal ideals.

Proposition 2.1.7. *The ring $\mathcal{R}L$ is a Gelfand ring.*

Proof. Let P be a prime ideal of $\mathcal{R}L$ and have that $P \subseteq \mathbf{M}^J$ where $J \in \mathrm{Pt}(\beta L)$. Taking $\varphi \in \mathbf{O}^J$, then there is some $\gamma \notin \mathbf{M}^J$ where $\varphi\gamma = 0$ and so $\varphi \in P$ because P is prime and $\gamma \notin P$. As such, $\mathbf{M}^J$ is the unique maximal ideal containing P. $\square$

2.2 Pure ideals

Given a ring R, an ideal I of R is *pure* if for every $a \in I$, there is a $b \in I$ such that $a = ab$. Jenkins and McKnight[48] seem to have invented the m-operator which is defined on the lattice of ideals of R by

$$mI = \{a \in R \mid a = ab \text{ for some } b \in I\}.$$

The ideal mI need not be pure in general. However, when it comes to $\mathcal{R}L$, mI is pure for every ideal I. We use the following lemmas from [15, Lemma 3.3] and [19, Lemma 3.2] in the next proposition, which provides a better characterization of the pure part of an arbitrary ideal I of $\mathcal{R}L$.

Lemma 2.2.1. *If α and β are elements of $\mathcal{R}L$ such that $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\beta$, then α is a multiple of β.*

Lemma 2.2.2. *Let I be an ideal of $\mathcal{R}L$, then*

$$mI = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\gamma \text{ for some } \gamma \in I\}.$$

Proposition 2.2.3. *Given that I is an ideal of $\mathcal{R}L$, and setting $J = \bigvee\{r(\operatorname{coz}\alpha) \mid \alpha \in I\}$, we then have that*

$$\begin{aligned} mI = \boldsymbol{O}^J &= \bigcap\{\boldsymbol{O}^{\mathfrak{p}} \mid \mathfrak{p} \text{ is a point of } \beta L \text{ with } I \subseteq \boldsymbol{M}^{\mathfrak{p}}\} \\ &= \bigcap\{\boldsymbol{O}^{\mathfrak{p}} \mid \mathfrak{p} \in \beta L \text{ with } I \subseteq \boldsymbol{M}^{\mathfrak{p}}\}. \end{aligned}$$

Proof. We first show that $mI = \boldsymbol{O}^J$ by showing both inclusions so first, let $\varphi \in mI$ such that $\operatorname{coz}\varphi \prec\!\!\prec \operatorname{coz}\gamma$ for some $\gamma \in I$ as seen from Lemma 2.2.2. Hence $r(\operatorname{coz}\varphi) \prec r(\operatorname{coz}\gamma) \leq J$ such that $\varphi \in \boldsymbol{O}^{\mathfrak{p}}$ and therefore, $mI \subseteq \boldsymbol{O}^J$. For the reverse inclusion, let $\varphi \in \boldsymbol{O}^J$ and so $\operatorname{coz}\varphi \in \bigvee\{r(\operatorname{coz}\alpha) \mid \alpha \in I\}$, such that there are finitely many elements $\alpha_1, \ldots, \alpha_m$ of I and $\operatorname{coz}\varphi = a_1 \vee \cdots \vee a_m$ for some $a_i \prec\!\!\prec \operatorname{coz}\alpha_i$. Therefore, $\operatorname{coz}\varphi \prec\!\!\prec \operatorname{coz}(\alpha_a^2 + \cdots + \alpha_m^2)$ and we have that $\varphi \in mI$ implying that $\boldsymbol{O}^J \subseteq mI$, and hence $mI = \boldsymbol{O}^J$.

We now show the second equality. Since for any $\mathfrak{p} \in \beta L$, $\bigvee\{r(\operatorname{coz}\varphi) \mid \varphi \in \boldsymbol{M}^{\mathfrak{p}}\} = \mathfrak{p}$ then $m\boldsymbol{M}^{\mathfrak{p}} = \boldsymbol{O}^{\mathfrak{p}}$. Let $\varphi \in mI$ and $\mathfrak{p} \in \operatorname{Pt}(\beta L)$ such that $I \subseteq \boldsymbol{M}^{\mathfrak{p}}$. Then $\varphi \in m\boldsymbol{M}^{\mathfrak{p}} = \boldsymbol{O}^{\mathfrak{p}}$ showing that

$$mI \subseteq \bigcap\{\boldsymbol{O}^{\mathfrak{p}} \mid \mathfrak{p} \text{ is a point of } \beta L \text{ with } \mathfrak{p} \subseteq \boldsymbol{M}^{\mathfrak{p}}\}.$$

We prove the other containment by contradiction. First, let γ be in the stated intersection and suppose, for contradiction, that $\gamma \notin mI$. Therefore $\gamma \notin \boldsymbol{O}^J$. Consequently, $r(\operatorname{coz}\gamma)^* \vee J \neq 1_{\beta L}$. There is a point $\mathfrak{p} \in \beta L$ such that $r(\operatorname{coz}\gamma)^* \vee J \leq \mathfrak{p}$ because βL has enough points. Notice that

$I \subseteq \boldsymbol{M}^{\mathfrak{p}}$, for if $\alpha \in I$, then $r(\operatorname{coz} \alpha) \leq J \leq \mathfrak{p}$. But now we cannot have $\gamma \in \boldsymbol{O}^{\mathfrak{p}}$, for that would imply $r(\operatorname{coz} \gamma)^* \vee \mathfrak{p} = 1_{\beta L}$, when we would have $\mathfrak{p} = 1_{\beta L}$ since $r(\operatorname{coz} \gamma)^* \leq \mathfrak{p}$. Therefore $\gamma \in mI$ showing that the second equality holds from which we have

$$mI \subseteq \bigcap \{\boldsymbol{O}^{\mathfrak{p}} \mid \mathfrak{p} \in \beta L \text{ with } I \subseteq \boldsymbol{M}^{\mathfrak{p}}\},$$

and

$$\{\boldsymbol{O}^{\mathfrak{p}} \mid \mathfrak{p} \in \beta L \text{ with } I \subseteq \boldsymbol{M}^{\mathfrak{p}}\} \subseteq \{\boldsymbol{O}^{\mathfrak{p}} \mid \mathfrak{p} \text{ is a point of } \beta L \text{ with } I \subseteq \boldsymbol{M}^{\mathfrak{p}}\} = mI,$$

because every element of βL is below some point of βL. □

Characterizations of pure ideals

In this section, we examine the characterizations of pure ideals of $\mathcal{R}L$, using the references [15] and [19] as a foundation. We first present several lemmas that will serve as a basis for establishing Theorem 2.2.1.

Lemma 2.2.4. *Let I be an ideal of $\mathcal{R}L$. Then I is pure if and only if for every $\alpha \in I$, there exists $\beta \in I$ such that* $\operatorname{coz} \alpha \prec\!\!\prec \operatorname{coz} \beta$.

Proof. ($\Rightarrow$): Let I be pure and $\alpha \in I$. Take $\varphi \in I$ such that $\alpha = \alpha\varphi$. Then $\alpha(1-\varphi) = 0$. Thus,

$$\operatorname{coz} \alpha \wedge \operatorname{coz}(1-\varphi) = 0 \text{ and } \operatorname{coz}(1-\varphi) \vee \operatorname{coz} \varphi = 1.$$

This shows that $\operatorname{coz} \alpha \prec \operatorname{coz} \varphi \in \operatorname{Coz} L$, and hence $\operatorname{coz} \alpha \prec\!\!\prec \operatorname{coz} \varphi$.

($\Leftarrow$): Let us suppose $\operatorname{coz} \alpha \prec\!\!\prec \operatorname{coz} \beta$ for some $\alpha, \beta \in \mathcal{R}L$ and there exists $\delta \in \mathcal{R}L$ that has $\operatorname{coz} \alpha \wedge \operatorname{coz} \delta = 0$ and $\operatorname{coz} \delta \vee \operatorname{coz} \beta = 1$. As such, $\operatorname{coz}(\delta^2 + \beta^2) = 1$ since $\delta^2, \beta^2 \geq 0$ in any f-ring and therefore $\delta^2 + \beta^2$ is invertible. We then have $\alpha = \alpha \cdot \beta^2\varphi,$ for some $\varphi \in \mathcal{R}L$. Since $\beta \in I$, $\beta^2\varphi \in I$, proving the same. □

Lemma 2.2.5. *Given I be an ideal of $\mathcal{R}L$, then $m(mI) = mI$ and therefore mI is pure.*

Proof. Let $\varphi \in mI$ and take $\gamma \in I$ such that $\operatorname{coz} \varphi \prec\!\!\prec \operatorname{coz} \gamma$. We then let $\delta \in \mathcal{R}L$ where $\operatorname{coz} \varphi \prec\!\!\prec \operatorname{coz} \delta \prec\!\!\prec \operatorname{coz} \gamma$. By Lemma 2.2.1, we then have $\delta \in I$ and therefore it is in mI. As a result, $\varphi \in m(mI)$ indicating that $mI \subseteq m(mI)$ and therefore $m(mI) = mI$. Furthermore, if $\varphi \in mI$, we take $\alpha \in m(mI)$ such that $\alpha = \alpha\beta$ for some $\beta \in mI$ therefore showing that mI is pure. □

Lemma 2.2.6. *The pure ideals of $\mathcal{R}L$ are precisely the ideals O^I, for $I \in \beta L$.*

Proof. ($\Leftarrow$): For any $\mathfrak{p} \in \beta L, \boldsymbol{O}^{\mathfrak{p}}$ is a pure because if $\alpha \in \boldsymbol{O}^{\mathfrak{p}}$, then $\operatorname{coz}\alpha \in \mathfrak{p}$, and therefore, by the completely regularity of $\mathfrak{p}$, there is a $\delta \in \mathcal{R}L$ such that $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\delta \in \mathfrak{p}$. But now $\delta \in \boldsymbol{O}^{\mathfrak{p}}$.

($\Rightarrow$): Let I be a pure ideal of $\mathcal{R}L$, and put $\mathfrak{p} = \bigvee\{r(\operatorname{coz}\alpha) \mid \alpha \in I\}$. We claim that $I = \boldsymbol{O}^{\mathfrak{p}}$. Let $\gamma \in I$. Since I is pure, there exists $\delta \in I$ such that $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\delta$. Consequently, $r(\operatorname{coz}\gamma) \prec\!\!\prec r(\operatorname{coz}\delta) \leq \mathfrak{p}$. This shows that $\gamma \in \boldsymbol{O}^{\mathfrak{p}}$, and hence $I \subseteq \boldsymbol{O}^{\mathfrak{p}}$. Now let $\gamma \in \boldsymbol{O}^{\mathfrak{p}}$. Then $r(\operatorname{coz}\gamma) \prec\!\!\prec \bigvee_{\alpha\in I} r(\operatorname{coz}\alpha)$, implying that $\gamma \in I$ thereby proving the $\supseteq$ inclusion hence equality. $\square$

Corollary 2.2.7. *I is pure if and only if $mI = I$.*

Proof. ($\Rightarrow$): Suppose I is pure and $\alpha \in I$, then $\alpha = \alpha\beta$ for some $\beta \in I$ and therefore $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\beta$ as seen in Lemma 2.2.1, such that $\alpha \in mI$, and hence $I \subseteq mI$.

($\Leftarrow$): Suppose $I = mI$, then I is pure since if $\alpha \in mI$, then $\alpha \in m(mI)$ and therefore $\alpha = \alpha\beta$ for some $\beta \in mI$. $\square$

Theorem 2.2.1. *For any ideal I of $\mathcal{R}L$, the following hold:*

1. *$m(mI) = mI$.*
2. *mI is pure.*
3. *I is pure if and only if $mI = I$.*
4. *I is pure if and only if for each $\alpha \in I$, there exists $\beta \in I$ such that $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\beta$.*

We note the following proposition as we conclude this section.

Proposition 2.2.8. *In $\mathcal{R}L$, given an ideal I, I and its pure part mI have the same annihilator.*

Proof. Given an element $\varphi \in I$, we can select a sequence (c_n) of cozero elements such that $c_n \prec\!\!\prec c_{n+1}$ for every n and $\operatorname{coz}\varphi = \bigvee(c_n)$. For each n, take $\delta_n \in \mathcal{R}L$ such that $c_n = \operatorname{coz}\delta_n$ and since $\operatorname{coz}\delta_n \prec\!\!\prec \operatorname{coz}\varphi$, each $\delta_n \in I$, and hence in mI. Now, since $mI \subseteq I$, then $\operatorname{Ann} I \subseteq (mI)$. On the other hand, let $\gamma \in \operatorname{Ann}(mI)$ and $\varphi \in I$. We then select elements $\varphi_n \in mI$ such that $\operatorname{coz}\varphi = \bigvee \operatorname{coz}\varphi_n$. Then $\operatorname{coz}\gamma\varphi = \operatorname{coz}\gamma \wedge \operatorname{coz}\varphi = \bigvee_n(\operatorname{coz}\gamma \wedge \operatorname{coz}\varphi_n) = \bigvee \operatorname{coz}\gamma\varphi_n = 0$, indicating $\gamma\varphi = 0$ and therefore $\gamma \in \operatorname{Ann} I$. From this we see that $\operatorname{Ann}(mI) \subseteq \operatorname{Ann} I$ and hence $\operatorname{Ann} I = \operatorname{Ann}(mI)$. $\square$

Given two ideals with identical pure parts then their annihilators are identical. The converse is not true. We can illustrate this with an example as such. For any $I \in \beta L, \operatorname{Ann}(\boldsymbol{O}^{I}) = \boldsymbol{M}^{I^*}$. This is so since:

$$\begin{aligned}
\delta \in \boldsymbol{M}^{I^*} &\iff r(\operatorname{coz}\delta) \subseteq I^* \\
&\iff (\operatorname{coz}\delta) \wedge I = 0_{\beta L} \\
&\iff \operatorname{coz}\delta \wedge a = 0 \text{ for every } a \in I \\
&\iff \operatorname{coz}\delta \wedge b = 0 \text{ for every cozero element } b \in I \\
&\iff \operatorname{coz}\delta \wedge \operatorname{coz}\gamma = 0 \text{ for every } \gamma \in \boldsymbol{O}^{I} \\
&\iff \delta \in \operatorname{Ann}(\boldsymbol{O}^{I}).
\end{aligned}$$

Example 2.2.9. Let X be a Tychonoff space and have that βX has more than one element not belonging to X. Moreover, let a and b be distinct elements of $\beta X \smallsetminus X$. Then

$$X \cap (\beta X \smallsetminus \{a\}) = X \cap (\beta X \smallsetminus \{b\}).$$

Because $\beta(\mathfrak{O}X)$ is isomorphic to $\mathfrak{O}(\beta X)$, I and J are distinct points of $\beta(\mathfrak{O}X)$ with $\bigvee I = \bigvee J$. Because $I^* = r((\bigvee I)^*) = J^*$, $\operatorname{Ann}(\boldsymbol{O}^{I}) = \operatorname{Ann}(\boldsymbol{M}^{J})$. However, $m\boldsymbol{O}^{I} = \boldsymbol{O}^{I} \neq \boldsymbol{O}^{J} = m\boldsymbol{O}^{J}$.

2.3 Fixed ideals

In this section we study the fixed maximal ideals of $\mathcal{R}L$ and $\mathcal{R}^*L$ which were introduced in [16]. We start by describing all maximal ideals of $\mathcal{R}^*L$ in a manner similar to the description of maximal ideals of $\mathcal{R}L$.

Definition 2.3.1. An ideal Q of $\mathcal{R}L$ (or of $\mathcal{R}^*L$) is *fixed* if

$$\bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\} < 1.$$

Definition 2.3.2. For each $a \in L$ with $a < 1$, define the subset $\mathbf{M}_a$ of $\mathcal{R}L$ by

$$\mathbf{M}_a = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \leq a\}.$$

Clearly, $\mathbf{M}_a$ is an ideal and, in fact $\mathbf{M}_a = \mathbf{M}^{r_L(a)}$. Since $r_L(a) = r_L(b)$ implies $a = b$, it follows that $\mathbf{M}_a = \mathbf{M}_b$ implies $a = b$. Furthermore, $\mathbf{M}_a$ is fixed.

The following lemma from [24] helps us to identify all fixed maximal ideals of $\mathcal{R}L$.

Lemma 2.3.3. *For any $I \in \beta L$,*

$$\bigvee\{\operatorname{coz}\alpha \mid \alpha \in \boldsymbol{M}^I\} = \bigvee I.$$

Hence, the ideal $\boldsymbol{M}^I$ is fixed if and only if $\bigvee I < 1$.

Proposition 2.3.4. *The fixed maximal ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}_p$ for $p \in \mathrm{Pt}(L)$ and these ideals are distinct for distinct points.*

Proof. Since, for any $p \in \mathrm{Pt}(L)$, $\boldsymbol{M}_p = \boldsymbol{M}^{r_L(p)}$, and $r_L(p)$ is a point of βL with $p < 1$, it follows that $\boldsymbol{M}_p$ is a fixed maximal ideal. Conversely, let Q be a fixed maximal ideal of $\mathcal{R}L$. Then, from the discussion above,

$$Q = \boldsymbol{M}^I \text{ for some point of } \beta L \text{ with } \bigvee I < 1.$$

Write $p = \bigvee I$, and note that $p \in \mathrm{Pt}(L)$. We need show that $\boldsymbol{M}^I = \boldsymbol{M}_p$. If $\alpha \in \boldsymbol{M}^I$, then $r_L(\operatorname{coz}\alpha) \subseteq I$. Taking joins yields $\operatorname{coz}\alpha \leq \bigvee I = p$, such that $\boldsymbol{M}^I \subseteq \boldsymbol{M}_p$. Since $\boldsymbol{M}^I$ is a maximal ideal and $\boldsymbol{M}_p$ is a proper ideal, we have $\boldsymbol{M}^I = \boldsymbol{M}_p$. □

We define a *character* of a frame L as a frame homomorphism $L \to \mathbf{2}$, where $\mathbf{2}$ denotes the two-element frame.

Corollary 2.3.5. *The fixed maximal ideals of $\mathcal{R}L$ are precisely the sets* $\ker(\mathcal{R}\zeta)$ *for the characters ζ of L. They are different characters.*

Proof. Let $\boldsymbol{M}_p$ be a fixed maximal ideal of $\mathcal{R}L$. Let $\zeta : L \to \mathbf{2}$ be a character of L determined by p. For any $\alpha \in \mathcal{R}L$, $\alpha \in \boldsymbol{M}_p$ if and only if $\alpha \in \ker(\mathcal{R}\zeta)$. It follows that $\boldsymbol{M}_p = \ker(\mathcal{R}\zeta)$. On the other hand, let $\eta : L \to \mathbf{2}$ be a character of L. Then $\eta_*(0)$ is a point of L. Also, for any $\gamma \in \mathcal{R}L$ is in $\ker(\mathcal{R}\zeta)$ if and only if $\operatorname{coz}\gamma \leq \eta_*(0)$, such that $\ker(\mathcal{R}\eta) = \boldsymbol{M}_{\eta_*(0)}$. □

Definition 2.3.6. For each $I \in \beta L$ with $I < 1_{\beta L}$, the ideal $\boldsymbol{M}^{*I}$ of $\mathcal{R}^*L$ is defined as

$$\boldsymbol{M}^{*I} = \{\alpha \in \mathcal{R}^*L \mid \operatorname{coz}(\alpha^\beta) \subseteq I\}.$$

We denote the ring isomorphism by $\boldsymbol{t}$ where

$$\boldsymbol{t}_L : \mathcal{R}(\beta L) \to \mathcal{R}^* L \text{ given by } \boldsymbol{t}_L(\alpha) = j_L\alpha,$$

the inverse of which we will denote by $\varphi \mapsto \varphi^\beta$.

Since a completely regular frame L is compact if and only if every maximal ideal of $\mathcal{R}L$ is fixed as seen in [19, Lemma 4.7], then for any $I \in \mathrm{Pt}(\beta L)$, $\boldsymbol{M}_I$ is the fixed maximal of $\mathcal{R}(\beta L)$ associated with the point I as per Definition 2.3.2.

We have the following proposition from [16, Proposition 3.8].

Proposition 2.3.7. *Maximal ideals of $\mathcal{R}^*L$ are precisely the ideals $\boldsymbol{M}^{*I}$, for $I \in \mathrm{Pt}(\beta L)$.*

Proof. Consider the isomorphism $\boldsymbol{t}_L : \mathcal{R}(\beta L) \to \mathcal{R}^*L$ described above. Maximal ideals of $\mathcal{R}^*L$ are precisely the maximal ideals of $\mathcal{R}(\beta L)$ under $\boldsymbol{t}_L$. Since βL is compact, its maximal ideals are all fixed, and are therefore precisely the ideals $\boldsymbol{M}_I$, for $I \in \mathrm{Pt}(\beta L)$. Therefore it suffices to show that, for each $I \in \mathrm{Pt}(\beta L)$, $\boldsymbol{M}^{*I} = \boldsymbol{t}_L[\boldsymbol{M}_I]$. Since $\boldsymbol{M}^{*I}$ is a proper ideal, it suffices, by the maximality of $\boldsymbol{t}_L[\boldsymbol{M}_I]$, to show that $\boldsymbol{t}_L[\boldsymbol{M}^I] \subseteq \boldsymbol{M}^{*I}$. Take $\alpha \in \boldsymbol{M}_I$, such that $\mathrm{coz}\,\alpha \subseteq I$. Now

$$\alpha = \boldsymbol{t}_L^{-1}(\boldsymbol{t}_L(\alpha)) = (j_L(\alpha))^\beta,$$

and therefore $\mathrm{coz}((j_L\alpha)^\beta) \subseteq I$. Consequently, $j_L\alpha \in \boldsymbol{M}^{*I}$, by definition. Thus $\boldsymbol{t}_L(\alpha) \in \boldsymbol{M}^{*I}$. The latter part of the proposition follows because $I \neq J$ in $\mathrm{Pt}(\beta L)$ implies $\boldsymbol{M}_I \neq \boldsymbol{M}_J$, as observed earlier. □

These results yield a spatiality criterion in terms of fixed maximal ideals. It is shown in [13] that a regular frame is spatial if and only every element different from the top is below some point of the frame. Using this characterization, we deduce from the result above the following corollary.

Corollary 2.3.8. *The following are equivalent for a completely regular frame L.*

1. *L is spatial.*
2. *Each fixed ideal of $\mathcal{R}L$ is contained in a fixed maximal ideal.*
3. *Each fixed ideal of $\mathcal{R}L$ is contained in $\ker(\mathcal{R}\zeta)$ for some character $\zeta : L \to \boldsymbol{2}$.*

In fact, for any spatial frame L, maximal ideals of $\mathcal{R}L$ are characterizable in terms of kernels of ring homomorphisms induced by characters of L. Namely:

Corollary 2.3.9. *If L is spatial, then an ideal of $\mathcal{R}L$ is fixed if and only if it is contained in* $\ker(\mathcal{R}\zeta)$ *for some character $\zeta : L \to \boldsymbol{2}$ of L.*

We now know what the maximal ideals of $\mathcal{R}^*L$ looks like. We identify the fixed ones among them.

Proposition 2.3.10. *For any $I \in \mathrm{Pt}(\beta L)$, the maximal ideal $\boldsymbol{M}^{*I}$ is fixed if and only if $\bigvee I < 1$.*

Proof. Suppose $\boldsymbol{M}^{*I}$ is fixed and put $s = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in \boldsymbol{M}^{*I}\}$. Then $s < 1$. Let $x \in I$, since I is a completely regular ideal, there is a cozero element c such that $x \leq c \in I$. Take a $\gamma \in \mathcal{R}^*L$ such that $c = \operatorname{coz}\gamma$. Now,

$$\bigvee \operatorname{coz}(\gamma^\beta) = \operatorname{coz}\gamma = c \in I.$$

Thus, for each $d \in \operatorname{coz}(\gamma^\beta)$, $d \leq c \in I$, and therefore $d \in I$. This shows that $\operatorname{coz}(\gamma^\beta) \subseteq I$, and therefore $\gamma \in \boldsymbol{M}^{*I}$, when $x \leq c \leq s$. As x is arbitrary , it follows that $\bigvee I \leq s < 1$.
Conversely, suppose $\bigvee I < 1$ and let $\varphi \in \boldsymbol{M}^{*I}$. Then $\operatorname{coz}(\varphi^\beta) \subseteq I$, and therefore

$$\operatorname{coz}\varphi = \bigvee \operatorname{coz}(\varphi^\beta) \leq \bigvee I < 1.$$

Consequently,

$$\bigvee\{\operatorname{coz}\alpha \mid \alpha \in \boldsymbol{M}^{*I}\} \leq \bigvee I < I,$$

and so $\boldsymbol{M}^{*I}$ is fixed. ∎

As before, we have now described maximal ideals of $\mathcal{R}^*L$, but by a condition (namely, $\operatorname{coz}(\varphi^\beta) \subseteq I$) which involves the Stone-Čech compactification of L denoted by βL. To replace this condition with one at the level of L itself, we get the definition below.

Definition 2.3.11. For each $a \in L$ with $a < 1$, let $\boldsymbol{M}^*_a$ be a subset of $\mathcal{R}^*L$ defined by

$$\boldsymbol{M}^*_a = \{\alpha \in \mathcal{R}^*L \mid \operatorname{coz}\alpha \leq a\}.$$

Clearly, $\boldsymbol{M}^*_a$ is a fixed ideal of $\mathcal{R}^*L$, and $\boldsymbol{M}^*_a = \boldsymbol{M}_a \cap \mathcal{R}^*L$. We will show that the fixed maximal ideals of $\mathcal{R}^*L$ are precisely the ideals $\boldsymbol{M}^*_p$ given a point $p \in \mathrm{Pt}(L)$. We shall need the following lemma.

Lemma 2.3.12. *Let I be a point of βL with $\bigvee < 1$. Also let $\varphi \in \mathcal{R}^*L$, then $\operatorname{coz}(\varphi^\beta) \subseteq I$ if and only if $\operatorname{coz}\varphi \leq \bigvee I$.*

Proof. ($\Rightarrow$): Since $\operatorname{coz}\varphi = \bigvee \operatorname{coz}(\varphi^\beta) \subseteq I$, then

$$\operatorname{coz}\varphi = \bigvee \operatorname{coz}(\varphi^\beta) \leq \bigvee I.$$

($\Leftarrow$): If not, then $\operatorname{coz}(\varphi^\beta) \vee I = 1_{\beta L}$ since I is a point of βL. Therefore,

$$\bigvee \operatorname{coz}(\varphi^\beta) \vee \bigvee I = 1,$$

that is, $\operatorname{coz}\varphi \vee \bigvee I = 1$, and hence $\bigvee I = 1$ since the present hypothesis is that $\operatorname{coz}\varphi \leq \bigvee I$, contradicting the initial assumption. $\square$

Proposition 2.3.13. *The fixed maximal ideals of $\mathcal{R}^*L$ are precisely the ideals $\boldsymbol{M}^*_p$, for $p \in \mathrm{Pt}(L)$.*

Proof. We need to show that each of the ideals $\boldsymbol{M}^*_p$ is of the form $\boldsymbol{M}^{*I}$ for a given $I \in \mathrm{Pt}(\beta L)$ with $\bigvee I < 1$. Since p is a point of L, then $r_L(p)$ is a point of βL with $\bigvee r_L(p) = p < 1$. Therefore, for any $\varphi \in \mathcal{R}^*L$ we have that $\operatorname{coz}\varphi \leq p$ if and only if $\operatorname{coz}(\varphi^\beta) \leq r_L(p)$, the consequence of which is that

$$\boldsymbol{M}^*_p = \{\varphi \in \mathcal{R}^*L \mid \operatorname{coz}\varphi \leq p\} = \{\varphi \in \mathcal{R}^*L \mid \operatorname{coz}(\varphi^\beta) \leq r_L(p)\} = \boldsymbol{M}^{r_L(p)}.$$

Now we need to show that for each $I \in \mathrm{Pt}(\beta L)$ with $\bigvee I < 1$, $\boldsymbol{M}^{*I} = \boldsymbol{M}^*_p$, for some point p of L. Let

$$\boldsymbol{M}^{*I} = \{\varphi \in \mathcal{R}^*L \mid \operatorname{coz}(\varphi^\beta) \leq I\} = \{\varphi \in \mathcal{R}^*L \mid \operatorname{coz}(\varphi) \leq \bigvee I\} = M^*_{\bigvee I}.$$

Since for any $a \in L$,

$$a = \bigvee\{\operatorname{coz}\varphi \mid \varphi \in \mathcal{R}^*L\} \text{ and } \operatorname{coz}\varphi \leq a,$$

then we see that for any distinct point then we obtain a distinct ideal. $\square$

We now look at extension and contraction maximal ideals. Recall that if $\varphi\colon A \to B$ is a ring homomorphism, then for any ideal I of B, the set $\varphi^{-1}[I]$ is an ideal of A called *the contraction of I along φ* and is denoted by I^c. As such, we can the denote the extension and contraction of a maximal ideal in $\mathcal{R}L$ as $(\boldsymbol{M}^I)^e$ and $(\boldsymbol{M}^I)^c$ respectively.

Lemma 2.3.14. *The following result holds for any $I \in \mathrm{Pt}(\beta L)$.*

1. $(\boldsymbol{M}^I)^c \subseteq \boldsymbol{M}^{*I}$.

2. $\boldsymbol{M}^I \subseteq (\boldsymbol{M}^{*I})^e$.

Proof. (1). Let $\alpha \in (\boldsymbol{M}^I)^c = \boldsymbol{M}^I \cap \mathcal{R}^*L$. Then $\alpha \in \mathcal{R}^*L$ and $r_L(\operatorname{coz}\alpha) \subseteq I$. Since $\alpha \in \mathcal{R}^*L$, the element α^β of $\mathcal{R}(\beta L)$ is defined, and we have $\alpha = j_L\alpha^\beta$, where $j_L : \beta L \to L$ such that

$$\operatorname{coz}\alpha = \operatorname{coz}(j_L\alpha^\beta) = j_L(\operatorname{coz}\alpha^\beta).$$

Since r_L is the right adjoint of j_L, this implies $\operatorname{coz}\alpha^\beta \leq r_L(\operatorname{coz}\alpha) \subseteq I$. Then $\alpha \in \boldsymbol{M}^{*I}$ and therefore $(\boldsymbol{M}^I)^c \subseteq \boldsymbol{M}^{*I}$.

(2). We take an $\alpha \in \boldsymbol{M}^I$, and have $\alpha = \dfrac{\alpha}{\mathbf{1}+\alpha^2} \cdot (\mathbf{1}+\alpha^2)$. We need to show that $\dfrac{\alpha}{\mathbf{1}+\alpha^2}$ is bounded. The element $\left(\dfrac{\alpha}{\mathbf{1}+\alpha^2}\right)^\beta$ of $\mathcal{R}(\beta L)$ exists. We then have that

$$\begin{aligned} j_L\left(\operatorname{coz}\left(\frac{\alpha}{\mathbf{1}+\alpha^2}\right)^\beta\right) &= \operatorname{coz}\left(j_L \cdot \left(\frac{\alpha}{\mathbf{1}+\alpha^2}\right)^\beta\right) \\ &= \operatorname{coz}\left(\frac{\alpha}{\mathbf{1}+\alpha^2}\right) \\ &= \operatorname{coz}\alpha \end{aligned}$$

which implies that

$$\operatorname{coz}\left(\tfrac{\alpha}{\mathbf{1}+\alpha^2}\right)^\beta \subseteq r_L(\operatorname{coz}\alpha) \subseteq I,$$

and since $\alpha \in \boldsymbol{M}^I$ then $\alpha \in (\boldsymbol{M}^{*I})^e$, and therefore $\boldsymbol{M}^I \subseteq (\boldsymbol{M}^{*I})^e$. □

We can now summarize the result above, concerning fixed maximal ideals, as follows.

Corollary 2.3.15. *The fixed maximal ideals of $\mathcal{R}^*L$ are precisely the contractions of the fixed maximal ideals $\mathcal{R}L$, and conversely, the fixed maximal ideals of $\mathcal{R}L$ are precisely the extensions of the fixed maximal ideals of $\mathcal{R}^*L$. Furthermore, the contraction and extensions maps are bijections, and inverses of each other when restricted to the sets of fixed maximal ideals.*

A frame L is *pseudocompact* if $\mathcal{R}^*L = \mathcal{R}L$ while an ideal I of L is said to be *σ-proper* if for any countable $S \subseteq I, \bigvee S < 1$. A completely regular frame L is pseudocompact if and only if every point of βL is σ-proper, as shown in [16].

Proposition 2.3.16. *The following are equivalent for a frame L.*

1. *Every maximal ideal of $\mathcal{R}L$ contracts to a maximal ideal of $\mathcal{R}^*L$.*

2. *Every maximal ideal of $\mathcal{R}^*L$ extends to a maximal ideal of $\mathcal{R}L$.*

3. *L is pseudocompact.*

Proof. (1) $\Rightarrow$ (2): Consider any maximal ideal $\boldsymbol{M}^{*I}$ of $\mathcal{R}^*L$. Since $(\boldsymbol{M}^I)^c \subseteq \boldsymbol{M}^{*I}$, and since $(\boldsymbol{M}^I)^c$ is a maximal ideal if we assume (1) then it follows that $(\boldsymbol{M}^I)^c = \boldsymbol{M}^{*I}$. Therefore $(\boldsymbol{M}^{*I})^e = (\boldsymbol{M}^I)^{ce} \subseteq \boldsymbol{M}^I$, which implies $(\boldsymbol{M}^{*I})^e = \boldsymbol{M}^I$. Thus (2) implied by (1).

(2) $\Rightarrow$ (3): Let $I \in \mathrm{Pt}(\beta L)$. By (2), $(\boldsymbol{M}^{*I})^e$ is a proper ideal. Since $\boldsymbol{M}^{*I} \subseteq (\boldsymbol{M}^{*I})^e$, we have that $\boldsymbol{M}^I = (\boldsymbol{M}^{*I})^e$, and hence I is σ-proper. Therefore L is pseudocompact.

(3) $\Rightarrow$ (1): Given that L is pseudocompact then $\mathcal{R}L = \mathcal{R}^*L$. As such, every maximal ideal of $\mathcal{R}L$ contracts to a maximal ideal in $\mathcal{R}^*L$ as they are equal. $\square$

Chapter 3

z-Ideals and d-ideals of $\mathcal{R}L$

3.1 Characterizations of z-ideals

Within the context of rings of continuous functions, the algebraic definition of z-ideals was brought about by Kohls in [51] and put out by Gillman and Jerison in their book *Rings of Continuous Functions* [39]. Mason in [56] studied the z-ideals in a commutative associative ring with identity and later, Dube extended the concept to point-free rings showing that the algebraic definition corresponds with the topological definition with regards to the cozero map in [14].

The set of all maximal ideals of a commutative ring R is denoted by $\mathrm{Max}(R)$. We define for any element a in the ring R, the set

$$\mathfrak{M}(a) = \{M \in \mathrm{Max}(R) \mid a \in M\}.$$

Definition 3.1.1. A *z-ideal* is an ideal I of a ring R where for any $a, b \in R$, $a \in I$ and $\mathfrak{M}(a) = \mathfrak{M}(b)$ implies $b \in I$. Equivalently, this means if $a \in I$ and $\mathfrak{M}(b) \subseteq \mathfrak{M}(a)$ then $b \in I$.

We recall that the *Jacobson radical* of a ring A, denoted by $\mathrm{Jac}(A)$, is the intersection of all maximal ideals of A. If A has zero Jacobson radical, then A is reduced because the ideal of nilpotent elements of A is the intersection of all prime ideals of A, and is therefore zero as well, which then says A has no nonzero nilpotent element. A minimal ideal of a reduced ring is generated by an idempotent element. Examples of z-ideals of a commutative ring are given in [56]. Some examples of z-ideals are as follows. In a ring with zero Jacobson radical, every minimal prime ideal and every maximal ideal is a z-ideal. The intersection of maximal ideals is a z-ideal. The proofs for the above mentioned examples can be found in [67]. Now let us check that the ideal $\boldsymbol{M}^J$ of $\mathcal{R}L$ is a z-ideal.

Example 3.1.2. Every $\boldsymbol{M}^J$ is a z-ideal.

Proof. Given $\gamma, \delta \in \boldsymbol{M}^h$ where $\boldsymbol{M}^h = \{\varphi \in \mathcal{R}L \mid h(r_L(\operatorname{coz}\varphi)) = 0\}$, then

$$(\operatorname{coz}\gamma \vee \operatorname{coz}\delta) \leq (\operatorname{coz}(\gamma+\delta)),$$

that is, $(\operatorname{coz}\gamma) \wedge (\operatorname{coz}\delta) \leq (\operatorname{coz}(\gamma+\delta))$, and therefore $h(r_L(\operatorname{coz}(\gamma+\delta))) = 0$, showing that $\gamma+\delta \in \boldsymbol{M}^h$. Similarly, if $\varphi \in \boldsymbol{M}^h$ and $\alpha \in \mathcal{R}L$, then $h(r_L((\operatorname{coz}\alpha\varphi))) = 0$ because

$$(\operatorname{coz}\alpha\varphi) = (\operatorname{coz}\alpha \wedge \operatorname{coz}\varphi) \geq (\operatorname{coz}\varphi).$$

So $\boldsymbol{M}^h$ is an ideal. Furthermore, if $\gamma \in \boldsymbol{M}^h$ and $\delta \in \mathcal{R}L$ such that $\operatorname{coz}\delta \leq (\operatorname{coz}\gamma)$, then $0 = h(r_L((\operatorname{coz}\gamma))) \leq h(r_L((\operatorname{coz}\delta)))$, showing that $\boldsymbol{M}^h$ is a z-ideal. $\square$

Note that for any ideal I of a ring R, we define

$$\mathfrak{M}(I) = \{M \in \operatorname{Max}(R) \mid M \subseteq I\}.$$

As such we have $M(I) = \bigcap \mathfrak{M}(I)$. We now characterize z-ideals of $\mathcal{R}L$. But first we present the following lemma from [14, Lemma 3.7].

Lemma 3.1.3. *Given that I is a z-ideal of $\mathcal{R}L$, then*

$$M(I) = \bigcap \mathfrak{M}(I) = \{\kappa \in \mathcal{R}L \mid r(\operatorname{coz}\kappa) \leq \bigvee_{\alpha \in I} r(\operatorname{coz}\alpha)\}.$$

Therefore, for each $\eta \in \mathcal{R}L$, $\bigcap \mathfrak{M}(\eta) = \{\kappa \in \mathcal{R}L \mid \operatorname{coz}\kappa \leq \operatorname{coz}\eta\}$.

Proof. Taking $\kappa \in \mathcal{R}L$ such that $r(\operatorname{coz}\kappa) \leq \bigvee_{\alpha\in I} r(\operatorname{coz}\alpha)$ and letting $J \in \operatorname{Pt}(\beta L)$ to have $I \subseteq \boldsymbol{M}^J$ then we must show that $\kappa \in \boldsymbol{M}^J$, that is, $r(\operatorname{coz}\kappa) \subseteq J$. So let $x \in r(\operatorname{coz}\kappa)$. The join $\bigvee\{r(\operatorname{coz}\alpha) \mid \alpha \in I\}$ is directed, for if $\alpha_1, \alpha_2 \in I$, then $\alpha_1^2 + \alpha_2^2 \in I$ and

$$r(\operatorname{coz}\alpha_1) \vee r(\operatorname{coz}\alpha_2) \leq r(\operatorname{coz}(\alpha_1^2 + \alpha_2^2)).$$

Thus

$$\bigvee_{\alpha \in I} r(\operatorname{coz} \alpha) = \bigcup_{\alpha \in I} r(\operatorname{coz} \alpha).$$

So, since $r(\operatorname{coz} \kappa) \subseteq \bigcup \{r(\operatorname{coz} \alpha) \mid \alpha \in I\}$, there exists $\alpha \in I$ such that $x \in r(\operatorname{coz} \alpha)$. But now if α is in I, then α is in $\boldsymbol{M}^J$, and hence $r(\operatorname{coz} \alpha) \subseteq J$. Consequently, $x \in J$, and hence $r(\operatorname{coz} \kappa) \subseteq J$, proving the inclusion $\supseteq$ in the first claimed equality. Now let $\kappa \in \mathfrak{M}(I)$ and suppose, by way of contradiction, that $r(\operatorname{coz} \kappa) \nleq \bigvee_{\alpha \in I} r(\operatorname{coz} \alpha)$. Let $H \in \operatorname{Pt}(\beta L)$ such that $I \subseteq \boldsymbol{M}^H$. Then $r(\operatorname{coz} \kappa) \subseteq H$, since $\kappa \in \boldsymbol{M}^H$. So, by what we have supposed, $H \nleq \bigvee_{\alpha \in I} r(\operatorname{coz} \alpha)$. Therefore $H \vee \bigvee_{\alpha \in I} r(\operatorname{coz} \alpha) = 1_{\beta L}$, since H is a maximal element of βL. By compactness (and keeping in mind that the join $\bigvee_{\alpha \in I} r(\operatorname{coz} \alpha)$ is directed), there exists $\tau \in I$ such that $H \vee r(\operatorname{coz} \tau) = 1_{\beta L}$. But this implies that $r(\operatorname{coz} \tau) \nsubseteq H$, contrary to the fact that $\tau \in \boldsymbol{M}^H$. So the reverse inclusion also holds. The second part of the lemma holds since $\bigvee_{\alpha \in \langle \eta \rangle} r(\operatorname{coz} \alpha) = r(\operatorname{coz} \eta)$ and $r(\operatorname{coz} \kappa) \leq r(\operatorname{coz} \eta)$ if and only if $\operatorname{coz} \kappa \leq \operatorname{coz} \eta$. $\square$

We use the following theorem from [30, Lemma 3.1].

Theorem 3.1.1. *Given an ideal I of $\mathcal{R}L$, the following are equivalent.*

1. *I is a z-ideal.*

2. *For any $\delta, \gamma \in \mathcal{R}L$, $\gamma \in I$ and $\operatorname{coz} \gamma = \operatorname{coz} \delta$ imply $\delta \in I$.*

3. *For any $\delta, \gamma \in \mathcal{R}L$, $\gamma \in I$ and $\operatorname{coz} \delta \leq \operatorname{coz} \gamma$ imply $\delta \in I$.*

4. *$I = \bigcup \{\boldsymbol{M}_{\operatorname{coz} \gamma} \mid \gamma \in I\}$.*

Proof. (1) $\Leftrightarrow$ (2): Suppose that (2) holds, and assume that $\operatorname{coz} \gamma = \operatorname{coz} \delta$ and $\delta \in I$. Then, by Lemma 3.1.3, $\gamma \in \bigcap \mathfrak{M}(\delta)$. But this implies that $\mathfrak{M}(\delta) \subseteq \mathfrak{M}(\gamma)$, and consequently that $\gamma \in I$ hypothetically and so I is a z-ideal.

Suppose that (1) holds, and let $\boldsymbol{M}^I$ be a maximal ideal of $\mathcal{R}L$ containing γ where $I \in \operatorname{Pt}(\delta L)$. Then $r_L(\operatorname{coz} \gamma) \subseteq I$ which implies that $r_L(\operatorname{coz} \delta) \subseteq I$, and we then get the implication that $\delta \in \boldsymbol{M}^I$ and $\delta \in I$ following that $\mathfrak{M}(\gamma) = \mathfrak{M}(\delta)$ and therefore I is a z-ideal.

(2) $\Rightarrow$ (3): Assume $\gamma \in I$ and $\operatorname{coz} \delta \leq \operatorname{coz} \gamma$. Then we have $\operatorname{coz} \delta = \operatorname{coz} \gamma \wedge \operatorname{coz} \delta = \operatorname{coz}(\gamma\delta)$. Because $\gamma \in I$ where I is an ideal, then $\gamma\delta \in I$ and therefore, by hypothesis, $\delta \in I$.

(3) $\Rightarrow$ (4): It is obvious that $I \subseteq \bigcup\{\boldsymbol{M}_{\operatorname{coz}\gamma} \mid \gamma \in I\}$ because, for any $\alpha \in \mathcal{R}L$, $\alpha \in \boldsymbol{M}_{\operatorname{coz}\alpha}$.

To show the converse, let $\gamma \in I$ and suppose that any $\delta \in \boldsymbol{M}_{\operatorname{coz}\gamma}$. This means $\operatorname{coz}\delta \leq \operatorname{coz}\gamma$, such that, by hypothesis, $\delta \in I$, showing that $\boldsymbol{M}_{\operatorname{coz}\gamma} \subseteq I$, hence the desired inclusion.

(4) $\Rightarrow$ (2): Suppose that $\gamma \in I$ and $\delta \in \mathcal{R}L$ such that $\operatorname{coz}\gamma = \operatorname{coz}\delta$. Then

$$\delta \in \boldsymbol{M}_{\operatorname{coz}\delta} = \boldsymbol{M}_{\operatorname{coz}\gamma} \subseteq I,$$

and so as a consequence (2) holds. $\square$

The z-ideal denoted I_z, defined by

$$I_z = \bigcup\{\boldsymbol{M}_{\operatorname{coz}\eta} \mid \eta \in I\},$$

for any ideal I of $\mathcal{R}L$, I_z is the smallest z-ideal containing I. Suppose that K is a z-ideal which contains I, then for any $\eta \in I$ and $\mu \in \boldsymbol{M}_{\operatorname{coz}\eta}$, $\operatorname{coz}\mu \leq \operatorname{coz}\ \eta$, and hence $\mu \in K$ since $\eta \in K$ and K is a z-ideal. Therefore $I_z \subseteq K$.

3.2 Sums of z-ideals

In this section, we use the results that were proven by Ighedo and Dube in [29] to show that the sum of two z-ideal in the ring $\mathcal{R}L$ is a z-ideal. We will also give sufficient conditions when the summation holds and most results in this regard were proven in [52].

Let $f \in \mathcal{R}L$ be a bounded function, with a unique function $f^\beta \in R(\beta L)$ such that $f = j_L \cdot f^\beta$. Then, the mapping $f \mapsto f^\beta$ is a ring isomorphism $\mathcal{R}^*L \to \mathcal{R}(\beta L)$ and the set-theoretic complement of Σ_a (for a in the spectrum of a frame M) will be denoted by $\Sigma_a^{'}$ such that,

$$\Sigma_a^{'} = \{p \in \operatorname{Pt}(M) \mid a \leq p\}$$

and then $\Sigma f^\beta : \Sigma\beta L \to \Sigma\mathfrak{O}\mathbb{R}$ is the continuous function defined by

$$\Sigma f^\beta(I) = f_*^\beta(I) = \mathbb{R} \smallsetminus \{p_I\},$$

for a given $p_I \in \mathbb{R}$ unique to each I. We then define a continuous map $\hat{f} : \Sigma\beta L \to \mathbb{R}$ by $\hat{f}(I) = p_I$ and the zero-set of $\hat{f}$ as $Z(\hat{f})$.

Lemma 3.2.1. *Given a bounded function $f \in \mathcal{R}L$, then*

$$\Sigma'_{r_L(a)} \subseteq Z(\hat{f}) \Rightarrow \operatorname{coz} f \leq a, \text{ for any } a \in L.$$

Proof. Assume $r_L(a) \leq I$ for $I \in \mathrm{Pt}(\beta L)$. This implies $\hat{f}(I) = 0$ meaning $f^{\beta}_{*}(I) = \mathbb{R} \smallsetminus \{0\}$. As such,

$$\operatorname{coz}(f^{\beta}) = f^{\beta}(\mathbb{R} \smallsetminus \{0\}) = f^{\beta}(f^{\beta}_{*}(I)) \leq I.$$

Because βL is spatial, then $r_L(a) = \bigwedge\{J \in \mathrm{Pt}(\beta L) \mid r_L(a) \leq J\}$, and as such $\operatorname{coz}(f^{\beta}) \leq r_L(a)$. As a consequence,

$$\operatorname{coz} f = \operatorname{coz}(j_L f^{\beta}) = j_L(\operatorname{coz}(f^{\beta})) = \bigvee \operatorname{coz}(f^{\beta}) \leq \bigvee r_L(a) = a,$$

showing that $\operatorname{coz}(f^{\beta}) \leq I$. $\square$

Proposition 3.2.2. *The sum of two z-ideals in $\mathcal{R}L$ is a z-ideal.*

Proof. Let $g_1 \in Z_1$ and $g_2 \in Z_2$ where Z_1, Z_2 are z-ideals in $\mathcal{R}L$. Take a $h \in \mathcal{R}L$ such that $\operatorname{coz} h \leq \operatorname{coz}(g_1 + g_2)$. We need show that $h \in Z_1 + Z_2$. Without loss of generality, then we can say that h is bounded since the function $g = \frac{h}{1+|h|}$ is bounded and $\operatorname{coz} g = \operatorname{coz} h$. We then write $s_i = \operatorname{coz}(g_i)$ for brevity. Then $r_L(\operatorname{coz} h) \leq r_L(\operatorname{coz} s_1) \vee r_L(\operatorname{coz} s_2)$ because r_L preserves binary joins of cozero elements. We now define a continuous map $\hat{h} : \Sigma\beta L \Rightarrow \mathbb{R}$ and define a map

$$t : \Sigma'_{r_L(s_1)} \cup \Sigma'_{r_L(s_2)} \to \mathbb{R}$$

as below:

$$t(I) = \begin{cases} 0 & \text{if } I \in \Sigma'_{r_L(s_1)}, \\ \hat{h}(I) & \text{if } I \in \Sigma'_{r_L(s_2)}. \end{cases}$$

For t to be well defined, we need to check that if $I \in \Sigma'_{r_L(s_1)} \cap \Sigma'_{r_L(s_2)}$, then $\hat{h}(I) = 0$. Observe that if $I \in \Sigma'_{r_L(s_1)} \cup \Sigma'_{r_L(s_2)}$ then $r_L(s_1) \vee r_L(s_2) \leq I$. Then,

$$\bigvee h^{\beta}(\mathbb{R} \smallsetminus \{0\}) = j_L h^{\beta}(\mathbb{R} \smallsetminus \{0\}) = h(\mathbb{R} \smallsetminus \{0\}) = \operatorname{coz} h,$$

meaning

$$h^{\beta}(\mathbb{R} \smallsetminus \{0\}) \leq r_L(\operatorname{coz} h) \leq r_L(s_1) \vee r_L(s_2) \leq I,$$

to have $\mathbb{R} \smallsetminus \{0\} \leq h^{\beta}_{*}(I)$, and therefore $h^{\beta}_{*}(I) = \mathbb{R} \smallsetminus \{0\}$ because $\mathbb{R} \smallsetminus \{0\}$ and $h^{\beta}_{*}(I)$ are both points in $\mathfrak{O}\mathbb{R}$. As such, $\Sigma h^{\beta}(I) = \mathbb{R} \smallsetminus \{0\}$ when $\hat{h}(I) = 0$, as needed. Then t is a continuous function on the closed set $\Sigma'_{r_L(s_1)} \cup \Sigma'_{r_L(s_2)}$ of the normal space $\Sigma\beta L$. Also, t has an extension to a continuous function $\bar{t} : \Sigma\beta L \to \mathbb{R}$ which is zero everywhere where t is zero by Tietze's extension theorem. As such $\Sigma'_{r_L(c_1)} \subseteq Z(\bar{t})$. Let $\lambda : \mathfrak{O}\mathbb{R} \to \beta L$ be the frame homomorphism $\lambda = \eta \cdot \mathfrak{O}\bar{t}$, where $\eta : \mathfrak{O}\Sigma\beta L \to \beta L$ is the isomorphism given by $\Sigma_J \mapsto J$ for every $J \in \beta L$. The homomorphism $j_L\lambda : \mathfrak{O}\mathbb{R} \to L$ has the property

$$\operatorname{coz}(j_L\lambda) = \bigvee \operatorname{coz}\lambda \leq r_L(s_1) = s_1 = \operatorname{coz}(g_1),$$

because $\operatorname{coz}\lambda \leq r_L(s_1)$ as $\Sigma'_{r_L(s_1)} \subseteq Z(\bar{t})$. Then $j_L\lambda \in Z_1$ because Z_1 is a z-ideal in $\mathcal{R}L$. Now arguing that $h - j_L\lambda \in Z_2$, we have $g = j_L\lambda$ for brevity. By Lemma 3.2.1, it suffices to show $\Sigma'_{r_L(s_2)} \subseteq Z(\widehat{h-g})$. We claim that if $I \in \Sigma'_{r_L(s_2)}$, then $g^\beta_*(I) = h^\beta_*(I)$. See that

$$g^\beta_*(I) = \lambda_*(I) = (\mathfrak{O}\bar{t})_*\eta_*(I) = (\mathfrak{O}\bar{t})_*(\Sigma_I) = (\mathfrak{O}\bar{t}) * [\Sigma\beta L \smallsetminus \{I\}].$$

Because $\bar{t}(I) = p_I$, as $I \in \Sigma'_{r_L(s_2)}$, then

$$\bar{t}^{-1}[\mathbb{R} \smallsetminus \{p_I\}] = \{J \in \Sigma\beta L \mid \bar{t}(J) \in \mathbb{R} \smallsetminus \{p_I\}\} \subseteq \Sigma\beta L \smallsetminus \{I\},$$

having, $\mathfrak{O}\bar{t}(h^\beta_*(I)) \subseteq \Sigma\beta L \smallsetminus \{I\}$, and therefore

$$h^\beta_*(I) \leq (\mathfrak{O}\bar{t})_*(\Sigma\beta L \smallsetminus \{I\}) = \lambda_*(I) = g^\beta_*(I),$$

showing the claimed equality because they are both points in $\mathfrak{O}\mathbb{R}$. We then show $(h-g)^\beta_*(I) = \mathbb{R} \smallsetminus \{0\}$. Pick $y \in \mathbb{R}$ to have $h^\beta_*(I) = g^\beta_*(I) = \mathbb{R} \smallsetminus \{y\}$. By [8, Lemma 3.2.1],

$$\operatorname{coz}(h^\beta - y) = h^\beta(\mathbb{R} \smallsetminus \{y\}) = h^\beta h^\beta_*(I) \leq I.$$

Also, $\operatorname{coz}(y - k^\beta) \leq I$ so

$$\begin{aligned}(h-g)^\beta(\mathbb{R} \smallsetminus \{0\}) = \operatorname{coz}((h-g)^\beta) &= \operatorname{coz}(h^\beta - g^\beta) \\ &= \operatorname{coz}((h^\beta - y) + (y - g^\beta)) \\ &\leq I \vee I = I\end{aligned}$$

This means $\mathbb{R} \smallsetminus \{0\} \leq (h-g)_*(I)$, and so equality as they are both points in $\mathfrak{O}\mathbb{R}$. Therefore, $(h-g)(I) = 0$, and so $\Sigma'_{r_L(s_2)} \subseteq Z(h-g)$, when $\operatorname{coz}(h-g) \leq s_2$, implying $h - g \in Z_2$. As a consequence, $h \in Z_1 + Z_2$, proving $Z_1 + Z_2$ is a z-ideal. $\square$

For any $a \in R$, let

$$M(a) = \bigcap \{Q \in \operatorname{Max}(R) \mid a \in Q\}.$$

This is referred as a *basic z-ideal.* In the lattice of z-ideals, $\operatorname{ZId}(A)$ is a sublattice of the lattice of all ideals of A if and only if the sum of any two z-ideals is itself a z-ideal as $I \vee J = (I+J)_z$ and $I \wedge J = I \cap J$. We will also have another z-ideal denoted J_z, for which

$$J_z = \bigcup \{\boldsymbol{M}_{\operatorname{coz}\eta} \mid \eta \in J\},$$

for any ideal J of $\mathcal{R}L$ and note that J_z is the smallest z-ideal that contains J. Recall that the set of all basic z-ideals of the ring R is $\{M_a \mid a \in R\}$. Ahmadi and Taherifar in [2] characterized rings in which the sum of two z-ideals is a z-ideal as in the proposition that follows but first we need the lemma below.

Lemma 3.2.3. *The following statements hold.*

1. *If P is minimal in the class of prime ideals containing a z-ideal I, then P is a z-ideal.*

2. *If I, J are two ideals in R, then $(I \cap J)_z = I_z \cap J_z$.*

Proposition 3.2.4. *The following statements are equivalent.*

1. *The sum of two z-ideals in R is a z-ideal.*

2. *For every ideal I, J of R, $(I_z + J_z)_z = I_z + J_z$.*

3. *The lattice $\operatorname{ZId}(R)$ is a sublattice of ideals of R.*

4. *For every two families $\{M_a \mid a \in S\}$ and $\{M_b \mid a \in K\}$ of basic z-ideals,*

$$\left(\sum_{a\in S} M_a + \sum_{b \in K} M_b\right)_z = \left(\sum_{a\in S} M_a\right)_z + \left(\sum_{b\in K} M_b\right)_z.$$

Proof. (1) $\Rightarrow$ (2): Consider two ideals I, J of the ring R and suppose that their sum $I_z + J_z$ is a z-ideal of R such that 1 holds. Then, $(I_z + J_z)_z = I_z + J_z$.
(2) $\Rightarrow$ (3): Suppose that (2) holds. Then,

$$\begin{aligned} I + J &= I_z + J_z \\ &= (I_z + J_z)_z \end{aligned}$$

thereby having $I + J$ as a z-ideal for two z-ideals I, J of R.

(3) $\Rightarrow$ (4): Suppose that (3) holds. By hypothesis, $(\Sigma_{a\in S}M_a)_z + (\Sigma_{b\in K}M_b)_z \in \mathrm{ZId}(R)$ and $(\Sigma_{a\in S}M_a) + (\Sigma_{b\in K}M_b) \subseteq (\Sigma_{a\in S}M_a)_z + (\Sigma_{b\in K}M_b)_z$. Also, $(\Sigma_{a\in S}M_a)_z + (\Sigma_{b\in K}M_b)_z \in \mathrm{ZId}(R)$ and $(\Sigma_{a\in S}M_a)_z + (\Sigma_{b\in K}M_b)_z \subseteq (\Sigma_{a\in S}M_a + \Sigma_{b\in K}M_b)_z$.

(4) $\Rightarrow$ (1): Consider two ideals I, J of R such that $I = \Sigma_{a\in I}M_a$ and $J = \Sigma_{b\in J}M_b$ and suppose that (4) holds. Then,

$$\begin{aligned} I + J &= \left(\sum_{a\in I} M_a\right) + \left(\sum_{b\in J} M_b\right) \\ &= \left(\sum_{a\in I} M_a\right)_z + \left(\sum_{b\in J} M_b\right)_z \\ &= \left(\sum_{a\in I} M_a + \sum_{b\in J} M_b\right)_z . \end{aligned}$$

□

We use the following corollaries to extend the results from Mason's proof in [55] that in $C(X)$, the sum of a z-ideal and a prime ideal which are not in a chain is a prime z-ideal.

Corollary 3.2.5. 1. *Let I and J be two ideals in $C(X)$. If J is a z-ideal and $J \subseteq \sqrt{I}$, then $J \subseteq I$.*

2. *If I is an ideal in $C(X)$, then $\sqrt{I}$ is a z-ideal if and only if I is also a z-ideal.*

We then have the proposition below from [1] with comprehensive proof from [6].

Proposition 3.2.6. *The sum of a primary ideal and a z-ideal in $\mathcal{R}L$ which are not in a chain is a prime z-ideal.*

Proof. Recall that a primary ideal Q is a proper ideal where for any element xy of Q, then x or y^n is also an element of Q, for some $n > 0$. Taking a primary ideal Q and a z-ideal I which are not in a chain, we let P be a prime ideal, where $\sqrt{Q} = P$. We use contradiction to show that I and P are not in a chain. Let $I \subseteq P$, then by Corollary 3.2.5(1), we have $I \subseteq Q$ hence contradiction. If $P \subseteq I$, then $Q \subseteq P \subseteq I$ and so I and P are not in a chain. Thereby, $I + P$ is a prime z-ideal. Now $\sqrt{I+Q} = \sqrt{\sqrt{I} + \sqrt{Q}} = \sqrt{I+P} = I + P$ implies that $\sqrt{I+Q}$ is a z-ideal. However, by Corollary 3.2.5(2), $I + Q$ is also a z-ideal and thus $I + Q = \sqrt{I+Q} = I + P$ is a prime z-ideal. □

Theorem 3.2.1. *If the sum of any two minimal prime ℓ-ideals in an f-ring with bounded inversion R is a prime z-ideal. Then for any two prime ℓ-ideals not in a chain, their sum is a z-ideal.*

Proof. Assume that R_1 and R_2 as minimal prime ℓ-ideals to have $R_1 \subseteq S$ and $R_2 \subseteq T$ with prime ℓ-ideals of R, S and T. From our assumption, $R_1 + R_2$ is a prime z-ideal. Since $R_1 \subseteq S$ and $R_2 \subseteq T$, then $R_1 + R_2 \subseteq S + T$. Conversely, $R_1 + R_2$ is a prime ℓ-ideal with $R_1, R_2 \subseteq R_1 + R_2$ and is in a chain with both S and T. Because S and T are not in a chain, both S and T have to sit in $R_1 + R_2$ where $R_1 + R_2 = S + T$ is a prime z-ideal. □

We recall the definition of a square dominated ideal from 2.1 as an ideal I of an f-ring R where

$$I = \{r \in R \mid |r| \leq a^2 \text{ for some } a \in R \text{ with } a^2 \in I\}$$

and get the following theorem.

Theorem 3.2.2. *Taking square dominated minimal prime ℓ-ideals of an f-ring R, assume the sum of any two minimal prime ℓ-ideals of R is a z-ideal. Then for any two z-ideals of R that are ℓ-ideals, their sum is a z-ideal.*

Proof. Assuming X and Y are z-ideals that are ℓ-ideals, because every z-ideal is a radical ℓ-ideal, then X, Y are radical ℓ-ideals and $X + Y$ is a radical ℓ-ideal. By letting $s \in R$ such that $z \notin X + Y$ then we see that $X + Y$ is the intersection of z-ideals. Because $X + Y$ is a radical ℓ-ideal, there exists a prime ℓ-ideal P that contains $X + Y$ but not s. Taking two prime ℓ-ideals $P_1, P_2 \subseteq P$ as minimal with regards to containing X and Y respectively, then P_1, P_2 are prime z-ideals and $P_1 + P_2$ is a z-ideal. Moreover, $X + Y \subseteq P_1 + P_2$ and $s \notin (P_1 + P_2)$ because $P_1 + P_2 \subseteq P$. □

Corollary 3.2.7. *Taking square dominated minimal prime ℓ-ideals of a reduced f-ring R, assume that for each $r, s \in R^+$,* $\mathrm{Ann}(r) + \mathrm{Ann}(s)$ *is a z-ideal. Then for any two z-ideals that are ℓ-ideals of R, their sum is a z-ideal.*

Proof. Given some u, v in the same set of maximal ideals and $u \in S + T$ where S and T are minimal prime ℓ-ideals, then for some $s \in S$ and $t \in T$, $v = s + t$. Moreover, there exists

$s_1, t_1 \in R^+$ such that $s_1 \notin S$, $t_1 \notin T$, and $ss_1 = 0$, $tt_1 = 0$. So $v = s + t \in \text{Ann}(s_1) + \text{Ann}(t_1)$. By hypothesis, $\text{Ann}(s_1) + \text{Ann}(t_1)$ is a z-ideal. So $u \in \text{Ann}(s_1) + \text{Ann}(t_1) \subseteq S + T$. $\square$

Following [45], an f-ring R is said to be *normal* if $R = \text{Ann}(r^+) + \text{Ann}(r^-)$ for all $r \in R$. This is equivalent to saying if $r \wedge s = 0$ then $R = \text{Ann}(r) + \text{Ann}(s)$.

Corollary 3.2.8. *Given a reduced normal f-ring R, for any two z-ideals that are ℓ-ideals, their sum is a z-ideal.*

Proof. Minimal prime ℓ-ideals of R are square dominated. Let S and T be minimal prime ℓ-ideals. If $S \neq T$, then there exists elements $s \in S \setminus T$ and $t \notin S$ such that $st = 0$ because S is a minimal prime ℓ-ideal. As such, $s \wedge t = 0$, and $\text{Ann}(s) + \text{Ann}(t) = r$. However $\text{Ann}(s) \subseteq T$, $\text{Ann}(t) \subseteq S$ and so $R = S + T$. $\square$

3.3 Characterizations of d-ideals

The concept of d-ideals has been studied by Picado and Pultr [60] who extended concepts from classical topology and ring theory into the point-free context. Dube and Ighedo also worked on the same in [29] where they worked on the lattices of z-ideals and d-ideals in the ring of continuous real-valued functions on completely regular frames. Some authors refer to d-ideals as z°-ideals.

Definition 3.3.1. An ideal I of a ring R is a *d-ideal* if for any $a, b \in R$, $\text{Ann}(a) = \text{Ann}(b)$ and $a \in I$ imply $b \in I$. This is equivalent to saying I is a d-ideal iff for any $a, b \in R$, $\text{Ann}^2(a) = \text{Ann}^2(b)$ and $a \in I$ then $b \in I$.

Examples of d-ideals of a commutative ring are given in [5], such examples are as follows:

i. The zero ideal.

ii. Every minimal prime ideal is a d-ideal.

iii. The $\text{Ann}(I)$ for a nonzero ideal I in a reduced ring R is a d-ideal since

$$\text{Ann}(I) = \bigcap \{P \in \text{Min}(A) : P \in D(I)\}.$$

iv. Expansively, if $S \subseteq R$ is not contained in a d-ideal I of R, then

$$(I : S) = \{r \in R : rS \subseteq I)\}$$

is a d-ideal[5].

Here is a less obvious example of a d-ideal.

Proposition 3.3.2. *The ideal* $\boldsymbol{O}^J$ *is a d-ideal.*

Proof. If $\delta, \gamma \in \boldsymbol{O}^h$, then

$$(\operatorname{coz}\gamma \vee \operatorname{coz}\delta)^* \leq (\operatorname{coz}(\gamma+\delta))^*,$$

that is, $(\operatorname{coz}\gamma)^* \wedge (\operatorname{coz}\delta)^* \leq (\operatorname{coz}(\gamma+\delta))^*$, and therefore $h(r_L(\operatorname{coz}(\gamma+\delta)^*)) = 1$, showing that $\gamma+\delta \in \boldsymbol{O}^h$. Similarly, if $\varphi \in \boldsymbol{O}^h$ and $\alpha \in \mathcal{R}L$, then $h(r_L((\operatorname{coz}\alpha\varphi)^*)) = 1$ because $(\operatorname{coz}\alpha\varphi)^* = (\operatorname{coz}\alpha \wedge \operatorname{coz}\varphi)^* \geq (\operatorname{coz}\varphi)^*$. So $\boldsymbol{O}^h$ is an ideal. Furthermore, if $\gamma \in \boldsymbol{O}^h$ and $\delta \in \mathcal{R}L$ such that $\operatorname{coz}\delta \leq (\operatorname{coz}\gamma)^{**}$, then $1 = h(r_L((\operatorname{coz}\gamma)^*)) \leq h(r_L((\operatorname{coz}\delta)^*))$, showing that $\boldsymbol{O}^h$ is a d-ideal. □

Remark 3.3.3. If $h \colon \beta L \to K$ is dense, then $\boldsymbol{M}^h = \boldsymbol{O}^h = \{\boldsymbol{0}\}$. Conversely, if $\boldsymbol{O}^h$ is the zero ideal, then h is dense. Suppose $h(I) = 0$ for any $I \in \beta L$. Let $\delta \in \mathcal{R}L$ such that $\operatorname{coz}\delta \in I$. Then $r_L(\ \delta) \prec\!\!\prec I$, and therefore $h(r_L((\operatorname{coz}\delta)^*)) = 1$, implying that $\delta \in \boldsymbol{O}^h$. So the only cozero element contained in I is 0. Therefore $I = 0$.

For any $a \in R$, let

$$P(a) = \bigcap\{Q \in \operatorname{Min}(R) \mid a \in Q\}.$$

This is referred as a *basic d-ideal*. It has been shown in [54] that $P(a) = \operatorname{Ann}^2(a)$.

We show that these ideals are characterizable in terms of the cozero map. From [17], we have the lemma below.

Lemma 3.3.4. *Let* $\delta, \gamma \in \mathcal{R}L$. *Then* $(\operatorname{coz}\gamma)^* \leq (\operatorname{coz}\delta)^*$ *if and only if* $\operatorname{Ann}(\gamma) \subseteq \operatorname{Ann}(\delta)$. *Hence,*

$$\operatorname{Ann}(\gamma) = \operatorname{Ann}(\delta) \text{ iff } (\operatorname{coz}\gamma)^* = (\operatorname{coz}\delta)^*.$$

Theorem 3.3.1. *The statements below are equivalent for a singular ideal* I *of* $\mathcal{R}L$.

1. *I is a d-ideal.*

2. *For any $\delta, \gamma \in \mathcal{R}L$, if $\gamma \in I$ and $(\operatorname{coz}\gamma)^* = (\operatorname{coz}\delta)^*$, then $\delta \in I$.*

3. *For any $\delta, \gamma \in \mathcal{R}L$, if $\gamma \in I$ and $(\operatorname{coz}\gamma)^* \leq (\operatorname{coz}\delta)^*$, then $\delta \in I$.*

4. *For any $\delta, \gamma \in \mathcal{R}L$, if $\gamma \in I$ and $\operatorname{coz}\delta \leq (\operatorname{coz}\gamma)^{**}$, then $\delta \in I$.*

Proof. (1) $\Rightarrow$ (2): Suppose I is a d-ideal and consider $\delta, \gamma \in \mathcal{R}L$ and have $\gamma \in I$ and $(\operatorname{coz}\gamma)^* = (\operatorname{coz}\delta)^*$. We claim that $\delta \in Q$. By Lemma 3.3.4, we know that $\operatorname{Ann}(\gamma) = \operatorname{Ann}(\delta)$ if and only if $(\operatorname{coz}\gamma)^* = (\operatorname{coz}\delta)^*$. By hypothesis, $\delta \in I$.
(2) $\Rightarrow$ (1): Assume (2) holds and let $\gamma \in I$ and $\operatorname{Ann}(\gamma) = \operatorname{Ann}(\delta)$ and by the definition then I is a d-ideal.
(2) $\Rightarrow$ (3): Notice that if $s, t \in L$ with $s^* \leq t^*$, then $(s \wedge t)^* \leq t^*$ and $t^{**} \leq s^{**}$, such that

$$(s \wedge t)^* \wedge t \leq (s \wedge t)^* \wedge s^{**}.$$

Additionally, $(s \wedge t)^* \wedge t \leq (s \wedge t)^* \wedge t^{**}$, and consequently

$$(s \wedge t)^* \wedge t \leq (s \wedge t)^* \wedge (s^{**} \wedge t^{**}) = (s \wedge t)^* \wedge (s \wedge t)^{**} = 0,$$

from which assertion follows.

Next, supposing that (2) holds and letting $\gamma, \delta \in \mathcal{R}L$ be such that $\gamma \in I$ and $(\operatorname{coz}\gamma)^* \leq (\operatorname{coz}\delta)^*$, then $\gamma\delta \in I$ such that

$$\begin{aligned}(\operatorname{coz}\gamma\delta)^* =& (\operatorname{coz}\gamma \wedge \operatorname{coz}\delta)^* \\ \leq& (\operatorname{coz}\delta)^* \\ \leq& (\operatorname{coz}\gamma\delta)^*.\end{aligned}$$

As such, $(\operatorname{coz}\gamma\delta)^* = (\operatorname{coz}\delta)^*$, and therefore by (2), $\delta \in I$ and (3) holds.
(3) $\Rightarrow$ (4): Assume (3) holds and $\operatorname{coz}\delta \leq (\operatorname{coz}\gamma)^{**}$ for $\gamma \in I$ and $\delta \in \mathcal{R}L$. Then $(\operatorname{coz}\gamma)^* \leq (\operatorname{coz}\delta)^*$, such that, by (3), $\delta \in I$. Therefore Q satisfies (4).
(4) $\Rightarrow$ (1): Let $\gamma \in I$ and $\tau \in P_\gamma = \mathbf{M}_{(\operatorname{coz}\gamma)^{**}}$ where for any $\gamma \in \mathcal{R}L, \operatorname{Ann}(\gamma) = \mathbf{M}_{(\operatorname{coz}\gamma)^*}$. Then $\operatorname{coz}\tau \leq (\operatorname{coz}\gamma)^{**}$. So, by (4), $\tau \in I$, and hence $P_\gamma \subseteq I$ and so I is a d-ideal. $\square$

Recall that R is a *von Neumann regular ring* if for every $a \in R$, there is $x \in R$ such that $a^2 x = a$.

From [5], we get these results in terms of d-ideals.

Proposition 3.3.5. *1. Every ideal in a von Neumann regular ring is an intersection of minimal prime ideals so it is a d-ideal.*

*2. The ring $\mathcal{R}L$ is von Neumann regular iff L is a P-*frame*, that is, every cozero element is complemented (see Chapter 4 for the proof)*

3. Every d-ideal I in a reduced ring is a radical ideal as it is an intersection of prime ideals. The ring $\mathcal{R}L$ is a reduced ring with bounded inversion, hence all d-ideals in $\mathcal{R}L$ are radical ideals.

4. The principal ideal is a d-ideal iff it is a basic d-ideal.

3.4 Sums of d-ideals

In this section, we will give sufficient conditions when the sums of two d-ideals is a d-ideal. Rings in which d-ideals coincide with z-ideals will be characterized.

Proposition 3.4.1. *The sum of d-ideals in $\mathcal{R}L$ is a d-ideal.*

The *Jacobson radical* of A is the intersection of all maximal ideals of A and is denoted $\mathrm{Jac}(A)$.

It has been shown in [5] and [50] that every d-ideal is a z-ideal in a large class of rings. In fact, the Jacobson radical $\mathrm{Jac}(A)$ of a ring A is zero if and only if every d-ideal is a z-ideal. The proofs for this can be found in Mason's paper [54, Proposition 2.12], and [3, Theorem 2.3].

Dube and Ighedo in [30] show the following. We shall include the proofs for convenience.

Lemma 3.4.2. *1. Let $\{I_\lambda \mid \lambda \in \Lambda\}$ be a subset of βL. Then*

$$\bigcap_\lambda \boldsymbol{M}^{I_\lambda} = \boldsymbol{M}^I$$

where $I = \bigwedge_\lambda I_\lambda$.

2. An ideal of $\mathcal{R}L$ is an intersection of maximal ideals if it is of the form $\boldsymbol{M}^I$, for some $I \in \beta L$.

3. The Jacobson radical of a ring $\mathcal{R}L$ is zero.

Proof. (1) For any $\alpha \in \mathcal{R}L$, we have

$$\begin{aligned} \alpha \in \bigcap_{\lambda \in \Lambda} \mathbf{M}^{I_\lambda} &\iff \alpha \in \mathbf{M}^{I_\lambda} \text{ for every } \lambda \\ &\iff r_L(\operatorname{coz} \alpha) \leq I_\lambda \text{ for every } \lambda \\ &\iff r_L(\operatorname{coz} \alpha) \leq \bigwedge_{\lambda \in \Lambda} I_\lambda \\ &\iff \alpha \in \mathbf{M}^I. \end{aligned}$$

(2) An intersection of maximal ideals is of the form $\boldsymbol{M}^I$ for some $I \in \beta$ follows from (1). In the same way, let $I \in \beta L$. We see that it is the empty meet of maximal ideals because if $I = 1_{\beta L}$, then $\boldsymbol{M}^I = \mathcal{R}L$. Now suppose that $I < 1_{\beta L}$. Note that $I = \bigwedge\{J \in \operatorname{Pt}(\beta L) \mid I \leq J\}$ because βL is spatial. Therefore, by (1),

$$\boldsymbol{M}^I = \bigcap\{\boldsymbol{M}^J \mid J \in \operatorname{Pt}(\beta L) \text{ and } J \geq I\}$$

is an intersection of maximal ideals.

(3) We use part (1) to see that $\operatorname{Jac}(\mathcal{R}L)$ is the zero ideal. Given that βL is spatial, $\bigwedge \operatorname{Pt}(\beta L) = 0_{\beta L}$, and so

$$\operatorname{Jac}(\mathcal{R}L) = \bigcap\{\boldsymbol{M}^I \mid I \in \operatorname{Pt}(\beta L)\} = \boldsymbol{M}^{0_{\beta L}} = \{\mathbf{0}\}.$$

This concludes the proof. □

By the argument above and Lemma 3.4.2(3), the following proposition is immediate.

Proposition 3.4.3. *In an f-ring $\mathcal{R}L$, every d-ideal is a z-ideal.*

A ring A is a *Baer ring* if, for every $S \subseteq A$, $\operatorname{Ann}(S)$ is principal generated by an idempotent. Following the definition of a Baer ring, then we have the following proposition as shown in [23].

Proposition 3.4.4. *In a Baer ring, the sum of two d-ideals is a d-ideal.*

Proof. Given a Baer ring R, take the d-ideals D_1 and D_2. Select $a \in D_1$ and $b \in D_2$ such that $d = a + b$ such that $d \in D_1 + D_2$. Then, $\operatorname{Ann}(a) \cap \operatorname{Ann}(b) \subseteq \operatorname{Ann}(d)$. Taking idempotents g and h where $\operatorname{Ann}(a) = \langle g \rangle$ and $\operatorname{Ann}(b) = \langle h \rangle$, we then have $\operatorname{Ann}(a) = \operatorname{Ann}^2(g)$ and $\operatorname{Ann}(b) = \operatorname{Ann}^2(h)$, and consequently

$$\operatorname{Ann}^2(gh) = \operatorname{Ann}^2(g) \cap \operatorname{Ann}^2(h) \subseteq \operatorname{Ann}(d),$$

when $\mathrm{Ann}^2(d) \subseteq \mathrm{Ann}(gh) = \langle 1 - gh \rangle$. But $\mathrm{Ann}(a) = \langle g \rangle$ implies $1 - g \in \mathrm{Ann}^2(a) \subseteq D_1$, and $\mathrm{Ann}(b) = \langle h \rangle$ implies $1 - h \in \mathrm{Ann}^2(b) \subseteq D_2$, such that $1 - gh = h(1 - g) + (1 - h) \in D_1 + D_2$, and so $\mathrm{Ann}^2(d) \subseteq D_1 + D_2$. Therefore $D_1 + D_2$ is a d-ideal. □

We define a frame L as *compact* if every cover has a finite subcover and if a frame L is compact, then L is regular iff L is completely regular. We say that a compact regular frame L is *extremely disconnected* if $a^* \vee a^{**} = 1$ for all $a \in L$. We have the following proposition from [17, Proposition 2.4].

Proposition 3.4.5. *A frame L is extremely disconnected if and only if $\mathcal{R}L$ is Baer.*

Recall that the set of all basic d-ideals of the ring A is $\{P_a \mid a \in A\}$ and in the lattice of d-ideals with $I \vee J = (I + J)_\circ$ and $I \wedge J = I \cap J$ where for an ideal K of R, $K_\circ$ denotes the smallest $z_\circ$-ideal containing K. Ahmadi and Taherifar in [2, Proposition 3.1] characterized rings in which the sum of two d-ideals is a d-ideal as in the proposition below but first we use this lemma from [2, Lemma 2.2].

Lemma 3.4.6. *Given a reduced ring R, then:*

1. *If I is a d-ideal in R, then every prime ideal minimal over I is a prime d-ideal.*
2. *A proper ideal I of R is a d-ideal if and only if it is an intersection of prime d-ideals.*
3. *If I, J are two ideals of R, then $(I \cap J)_\circ = I_\circ \cap J_\circ$.*

Proposition 3.4.7. *The statements below are equivalent.*

1. *The sum of two d-ideals in A is a d-ideal.*
2. *Given any two ideals I, J of A, $(I_\circ + J_\circ)_\circ = I_\circ + J_\circ$.*
3. *The lattice $\mathrm{DId}(A)$ is a sublattice of the lattice of ideals of A.*
4. *For every two families $\{P_a \mid a \in S\}$ and $\{P_b \mid b \in T\}$ of basic d-ideals,*

$$\left(\sum_{a \in S} P_a + \sum_{b \in T} P_b\right)_\circ = \left(\sum_{a \in S} P_a\right)_\circ + \left(\sum_{b \in T} P_b\right)_\circ.$$

Proof. (1) $\Rightarrow$ (2): Consider two ideals I, J of the ring A and suppose that their sum $I_\circ + J_\circ$ is a d-ideal of A such that (1) holds. Then, $(I_\circ + J_\circ)_\circ = I_\circ + J_\circ$.

(2) $\Rightarrow$ (3): Suppose that (2) holds. Then,

$$\begin{aligned} I + J &= I_\circ + J_\circ \\ &= (I_\circ + J_\circ)_\circ \end{aligned}$$

thereby having $I + J$ as a d-ideal for two d-ideals I and J of A.

(3) $\Rightarrow$ (4): Suppose that $(\Sigma_{a\in S} P_a)_\circ + (\Sigma_{b\in T} P_b)_\circ \in \mathrm{DId}(A)$. Then,

$$(\Sigma_{a\in S} P_a) + (\Sigma_{b\in T} P_b) \subseteq (\Sigma_{a\in S} P_a)_\circ + (\Sigma_{b\in T} P_b)_\circ \in \mathrm{DId}(A).$$

Also, $(\Sigma_{a\in S} P_a)_\circ + (\Sigma_{b\in T} P_b)_\circ \in \mathrm{DId}(A)$ and

$$(\Sigma_{a\in S} P_a)_\circ + (\Sigma_{b\in T} P_b)_\circ \subseteq (\Sigma_{a\in S} P_a + \Sigma_{b\in T} P_b)_\circ.$$

(4) $\Rightarrow$ (1): Now consider two d-ideals I, J of A such that $I = \Sigma_{a\in I} P_a$ and $J = \Sigma_{b\in J} P_b$.

$$\begin{aligned} I + J &= \left(\sum_{a\in I} P_a\right) + \left(\sum_{b\in J} P_b\right) \\ &= \left(\sum_{a\in I} P_a\right)_\circ + \left(\sum_{b\in J} P_b\right)_\circ \\ &= \left(\sum_{a\in I} P_a + \sum_{b\in J} P_b\right)_\circ. \end{aligned}$$

□

Proposition 3.4.8. *Assume for each minimal prime ℓ-ideal of an f-ring with bounded inversion R, their sum is a prime d-ideal. Also, for each prime ℓ-ideal not in a chain, their sum is a d-ideal.*

Proof. Taking minimal prime ℓ-ideals Q_1 and Q_2 where $Q_1 \subseteq X$, $Q_2 \subseteq Y$ and X, Y are prime ℓ-ideals of R, then $Q_1 + Q_2 \subseteq X + Y$ and by assumption $Q_1 + Q_2$ is a prime d-ideal. Also, because $Q_1, Q_2 \subseteq Q_1 + Q_2$, $Q_1 + Q_2$ is in chain with X and Y. However, X and Y are not in chain and so X, Y must be contained in $Q_1 + Q_2 = X + Y$ which is a d-ideal. □

Lemma 3.4.9. *Given the class of prime ideals, we can take a minimal prime ideal P which is contained in an f-ring R. Then taking a d-ideal I that is an ℓ-ideal, P is a d-ideal.*

Proposition 3.4.10. *Taking square dominated minimal prime ℓ-ideals of an f-ring R, for each d-ideal which is an ℓ-ideal in R, their sum is a d-ideal.*

Proof. Taking two d-ideals S and T, we say they are radical ℓ-ideals because all d-ideals are radical ℓ-ideals, and so $S + T$ is a radical ℓ-ideal. It follows that $S + T$ is the intersection of prime ℓ-ideals meaning that there exists a prime d-ideal P containing $S + T$ but not $z \in R$. Let $P_1, P_2 \subseteq P$ be minimal with respect to containing S and T respectively. We have that $P_1 + P_2$ is a d-ideal and $S + T \subseteq P_1 + P_2$ and $z \notin (P_1 + P_2)$ because $P_1 + P_2 \subseteq P$ and $z \in P$. □

For square dominated minimal prime ℓ-ideals in a commutative ℓ-ring with identity, the sum of two minimal prime ℓ-ideals is not necessarily a d-ideal or z-ideal (see [52]).

Example 3.4.11. In $C([0, 1])$, take that

$$I = \{f \in C([0, 1]) : |f^n| \leq mi \text{ for some } n, m \in \mathbb{N}\}$$

where $i(x) = x$. Then I is a semiprime ℓ-ideal of $C([0, 1])$. There is a commutative semiprime f-ring with identity element $R = \{(f, g) \in C([0, l]) \times C([0, l]) : f - g \in 1\}$ (see [44]) whose minimal prime ℓ-ideals have the form $\{(f + g, f) : f \in S, g \in I\}$ or $\{(f, f + g) : f \in S, g \in I\}$ for some minimal prime ℓ-ideal S of $C([0, 1])$. Define a function h by

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{2}^n x^{\frac{1}{n}}.$$

Then $h \in C([0, 1])$ and $h \notin I$. Let $I \subseteq S$, $h \notin S$, and S_1 be a minimal prime ℓ-ideal that sits in S. In R, let $Q_1 = \{(f + g, f) : f \in S_1, g \in I\}$ and $Q_2 = \{(f, f + g) : f \in S_1, g \in I\}$. Then Q_1 and Q_2 are minimal prime ℓ-ideals of R and therefore are z-ideals and d-ideals. However, because the only maximal ideal (h, h) or (i, i) is contained in $M = \{(f, g) : f(0) = g(0) = 0\}$ and $(i, i) \in Q_1 + Q_2$ then $(h, h) \notin Q_1 + Q_2$ so $Q_1 + Q_2$ is not a z-ideal. Also because $\{(i, i)\}^{dd} = R$ and $R \nsubseteq Q_1 + Q_2$ then $Q_1 + Q_2$ is not a d-ideal.

Chapter 4

P-frames and F-frames

In [8], Ball and Walters-Wayland coined the term P-frames from the topological definition of P-spaces. They also defined F-frames using a condition based on frames which is the analogue of every cozero set is C^*-embedded in topology. We use [14], [24] and [25] as our references for this chapter.

4.1 Characterizations of P-frames and almost P-frames

4.1.1 Characterizations of P-frames

Definition 4.1.1. For a given frame L, if $a \vee a^* = 1$ for each $a \in \operatorname{Coz} L$, then L is called a *P-frame*.

We recall the following given a ring A.

1. The ring A is regular if for each $a \in A$ there exists $b \in A$ with $a = aba$.
2. The pure part of an ideal I of the ring A is the ideal $mI = \{a \in A \mid a = ab \text{ for some } b \in I\}$.
3. The *Von Neumann inverse* (abbreviated VN-inverse) of $a \in A$ is an element b such that $a = a^2b$. We then characterize the elements of A that have VN-inverses as follows.

   (a) x has a VN-inverse iff for each maximal ideal M, $x \in M$ implies $x \in mM$.

(b) x has a VN-inverse iff xu is idempotent for some invertible u.

We are now going to give the ring-theoretic characterization of P-frames as in [14].

Proposition 4.1.2. *The following are equivalent for a frame L.*

1. *L is P-frame.*
2. *$\mathcal{R}L$ is a regular ring.*
3. *Every ideal of $\mathcal{R}L$ is a z-ideal.*
4. *Every ideal of $\mathcal{R}L$ is an intersection of prime ideals.*
5. *Every ideal of $\mathcal{R}L$ is an intersection of maximal ideals.*
6. *Every prime ideal of $\mathcal{R}L$ is an intersection of maximal ideals.*
7. *Every z-ideal of $\mathcal{R}L$ is an intersection of maximal ideals.*
8. *For every $\alpha, \beta \in \mathcal{R}L, \langle \alpha, \beta \rangle = \langle \alpha^2 + \beta^2 \rangle$.*
9. *Every principal ideal of $\mathcal{R}L$ is generated by an idempotent.*
10. *$\boldsymbol{O}^J = \boldsymbol{M}^J$ for each $J \in \mathrm{Pt}(\beta L)$.*

Proof. (1) $\Leftrightarrow$ (2): Suppose L is a P-frame and let $\varphi \in \mathcal{R}L$. Then $\operatorname{coz} \varphi$ is complemented. Therefore, if $\varphi \in \boldsymbol{M}^I$ for some $I \in \mathrm{Pt}(\beta L)$, then $r(\operatorname{coz} \varphi) \prec\!\!\prec r(\operatorname{coz} \varphi) \leq I$ since r preserves the completely below relation, meaning that $\varphi \in \boldsymbol{O}^I$. Consequently, φ has a VN-inverse from the characterization above and as such $\mathcal{R}L$ is regular.
Conversely, suppose $\mathcal{R}L$ is regular and let $\gamma \in \mathcal{R}L$. Then γ has a VN-inverse, and therefore, there is an invertible $\rho \in \mathcal{R}L$ such that $\gamma\rho$ is idempotent. Therefore $\operatorname{coz} \gamma\rho$ is complemented. However, $\operatorname{coz} \gamma\rho = \operatorname{coz} \gamma \wedge \operatorname{coz} \rho = \operatorname{coz} \gamma \wedge 1 = \operatorname{coz} \gamma$ such that $\operatorname{coz} \gamma$ is complemented thereby showing that L is a P-frame.
(3) $\Rightarrow$ (4): Let Q be an ideal of $\mathcal{R}L$ and $\varphi \in \sqrt{Q}$. Then there is a natural number n such that $\varphi^n \in Q$. But $\operatorname{coz} \varphi = \operatorname{coz} \varphi^n$, and so $\varphi \in Q$, since Q is a z-ideal by hypothesis. This shows that $Q = \sqrt{Q}$, an intersection of prime ideals.
(4) $\Rightarrow$ (2): Let $\varphi \in \mathcal{R}L$. Any prime ideal of $\mathcal{R}L$ contains φ if and only if it contains φ^2. By

hypothesis, $\langle \varphi \rangle = \langle \varphi^2 \rangle$, and therefore $\varphi = \rho\varphi^2$, for some $\rho \in \mathcal{R}L$, implying that $\mathcal{R}L$ is regular. The implications (5) $\Rightarrow$ (6) and (5) $\Rightarrow$ (7) are trivial whereas the equivalences of (1), (3), (5), (9) and (10) follow from [14, Proposition 3.2].

(6) $\Rightarrow$ (2): Since every prime ideal is contained in a unique maximal ideal, (6) implies that every prime ideal is maximal, such that $\mathcal{R}L$ is regular.

(7) $\Rightarrow$ (10): For any $J \in \mathrm{Pt}(\beta L)$, $\boldsymbol{O}^J$ is a z-ideal, and $\boldsymbol{M}^J$ is the unique maximal ideal containing it. Then by hypothesis $\boldsymbol{O}^J = \boldsymbol{M}^J$.

(3) $\Rightarrow$ (8): Check that, $\langle \alpha^2+\beta^2 \rangle \subseteq \langle \alpha, \beta \rangle$. Now let $\varphi \in \langle \alpha, \beta \rangle$. Then $\varphi = \tau_1\alpha + \tau_2\beta$ for some $\tau_1, \tau_2 \in \mathcal{R}L$. Therefore $\operatorname{coz}\varphi \leq \operatorname{coz}\alpha \vee \operatorname{coz}\beta = \operatorname{coz}\alpha^2 \vee \operatorname{coz}\beta^2 = \operatorname{coz}(\alpha^2+\beta^2)$. Since, by hypothesis, $\langle \alpha^2+\beta^2 \rangle$ is a z-ideal, then $\varphi \in \langle \alpha^2+\beta^2 \rangle$. Therefore $\langle \alpha, \beta \rangle = \langle \alpha^2+\beta^2 \rangle$.

(8) $\Rightarrow$ (2): Let $\alpha \in \mathcal{R}L$, and consider the principal ideal $\langle \alpha \rangle \in \mathcal{R}L$ then by hypothesis, $\langle \alpha \rangle = \langle \alpha, \mathbf{0} \rangle = \langle \alpha^2 \rangle$. Therefore $\alpha = \rho\alpha^2$ for some $\rho \in \mathcal{R}L$ implying $\mathcal{R}L$ is regular. $\square$

Corollary 4.1.3. *A frame L is a P-frame if and only if every ideal of $\mathcal{R}L$ is pure.*

Proof. For

$$I = \bigvee \{ r(\operatorname{coz}\alpha) \mid \alpha \in A \},$$

an ideal $A \in \mathcal{R}L$ is precisely the ideal $\boldsymbol{O}^I$ for $I \in \beta L$. This can be shown by letting $\varphi \in \boldsymbol{O}^I$. Then $r(\operatorname{coz}\varphi) \prec\!\!\prec I$, which implies that $\varphi \in A$ and therefore $\boldsymbol{O}^I \subseteq A$. To show containment the other way round, we let $\varphi \in A$ and since we know that L is a P-frame then $\operatorname{coz}\varphi \prec\!\!\prec \operatorname{coz}\varphi$, meaning $r(\operatorname{coz}\varphi) \prec\!\!\prec r(\operatorname{coz}\varphi) \leq I$. As such $\varphi \in \boldsymbol{O}^I$ and so $A \subseteq \boldsymbol{O}^I$. As a consequence, all ideals of $\mathcal{R}L$ are pure. Note that the supposition means that $\boldsymbol{M}^I = m\boldsymbol{M}^I = \boldsymbol{O}^I$ for each $I \in \mathrm{Pt}(\beta L)$ and therefore the converse holds. $\square$

The following proposition also follows from [25].

Proposition 4.1.4. *A frame L is a P-frame iff every prime ideal of $\mathcal{R}L$ is minimally prime.*

Proof. Given that L is a P-frame, then $\operatorname{Min}\mathcal{R}L$ is compact and because $\mathcal{R}L$ satisfies the annihilator condition, then each prime ideal of $\mathcal{R}L$ is a minimal prime.

Conversely, let P be a prime ideal of $\operatorname{Coz}L$. Since primality is implied by maximality, a prime ideal I in $\mathcal{R}L$ defined as $I = \{\varphi \in \mathcal{R}L \mid \operatorname{coz}\varphi \in P\}$ is maximal. Taking $a \in \operatorname{Coz}L$ then for a given element $\alpha \in \mathcal{R}L$, $a = \operatorname{coz} \alpha$ and for every $b \in P$, $a \vee b \neq 1$. Assume that $a \notin P$ and so by the maximality of I, for some given $\delta \in I$ and $\tau \in \mathcal{R}L$, $\mathbf{1} = \delta + \tau\alpha$ which shows that

$$\begin{aligned} 1 &= \operatorname{coz}(\delta + \tau\alpha) \\ &\leq \operatorname{coz} \delta \vee (\operatorname{coz} \tau \wedge \operatorname{coz} \alpha) \\ &= (\operatorname{coz} \delta \vee \operatorname{coz} \tau) \wedge (\operatorname{coz} \delta \vee \operatorname{coz} \alpha), \end{aligned}$$

when $a \vee \operatorname{coz} \delta = 1$, contradicting the nature of a as $\operatorname{coz} \delta \in P$ so $a \in P$. □

The following proposition follows from [14, Proposition 3.13].

Proposition 4.1.5. *A frame L is a P-frame if and only if every maximal ideal of $\mathcal{R}L$ is pure.*

Proof. Suppose L is a P-frame and Q is an ideal of $\mathcal{R}L$. We need to show that $Q = \boldsymbol{O}^I$ for $I = \bigvee\{r(\operatorname{coz} \alpha) \mid \alpha \in Q\}$. For the first containment, let $\varphi \in \boldsymbol{O}^I$. Then, $r(\operatorname{coz} \varphi) \prec\!\!\prec I$ implying that $\varphi \in Q$. As such, $\boldsymbol{O}^I \subseteq Q$. For the other containment, let $\varphi \in Q$. Then, since L is a P-frame, $\operatorname{coz} \varphi \prec\!\!\prec \operatorname{coz} \varphi$, implying that $r(\operatorname{coz} \varphi) \prec\!\!\prec r(\operatorname{coz} \varphi) \leq I$. Hence $\varphi \in \boldsymbol{O}^I$, and therefore $Q \subseteq \boldsymbol{O}^I$. Consequently, every ideal of $\mathcal{R}L$ is pure. The converse holds because the hypothesis implies that $\boldsymbol{M}^I = m\boldsymbol{M}^I = \boldsymbol{O}^I$ for every $I \in \operatorname{Pt}(\beta L)$. □

Proposition 4.1.6. *A frame L is a P-frame if and only if the countably generated ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{O}^I$ for $I \in \operatorname{Coz} \beta L$.*

Proof. ($\Leftarrow$): Let $\varphi \in \mathcal{R}L$ and by hypothesis, $\langle\varphi\rangle = \boldsymbol{O}^I$ for some $I \in \operatorname{Coz} \beta L$. There exists $\gamma \in \mathcal{R}L$ such that $\operatorname{coz} \varphi \prec\!\!\prec \operatorname{coz} \gamma \in I$ since I is a completely regular ideal and $\operatorname{coz} \varphi \in I$. Then $\gamma \in \boldsymbol{O}^I = \langle\varphi\rangle$, and so $\gamma = \alpha\varphi$ for some $\alpha \in \mathcal{R}L$. As such, $\operatorname{coz} \varphi \prec\!\!\prec \operatorname{coz} \alpha\varphi \leq \operatorname{coz} \varphi$ meaning that $\operatorname{coz} \varphi$ is complemented and therefore L is a P-frame.
($\Rightarrow$): Taking a sequence (I_n) in βL such that

$$I_n \prec\!\!\prec I_{n+1} \text{ for each } n, \text{ and } I = \bigvee_{\beta L} I_n = \bigcup_n I_n,$$

then taking $\varphi_n \in \mathcal{R}L$ for every n, we have

$$\bigvee I_n \prec\!\!\prec \operatorname{coz} \varphi_n \prec\!\!\prec \bigvee I_{n+1}.$$

Then, for each n, $\bigvee I_n \in I_{n+1}$, since $I_n \prec\!\!\prec I_{n+1}$. Consequently, $\operatorname{coz} \varphi_n \in I_{n+2} \subseteq I$, implying that $\varphi_n \in \boldsymbol{O}^I$, and hence $\langle\varphi_1, \varphi_2, \dots\rangle \subseteq I$. Now, suppose $\varphi \in \boldsymbol{O}^I$. Then $\operatorname{coz} \varphi \in I$, and therefore

$\operatorname{coz}\varphi \in I_n$ for some index n. However, this means that $\operatorname{coz}\varphi \leq \bigvee I_n \prec\!\!\prec \operatorname{coz}\varphi_n$, such that φ is a multiple of φ_n. This implies that $\boldsymbol{O}^{\boldsymbol{I}} \subseteq \langle \varphi_1, \varphi_2, \dots \rangle$, and therefore $\boldsymbol{O}^{\boldsymbol{I}} = \langle \varphi_1, \varphi_2, \dots \rangle$ showing that for any frame L, $\boldsymbol{O}^{\boldsymbol{I}}$ is countably generated if $I \in \operatorname{Coz}\beta L$.
Next, take that $\boldsymbol{O}^{\boldsymbol{I}} = \langle \varphi_1, \varphi_2, \dots \rangle$. We need to show that $I = \bigvee_n r(\operatorname{coz}\varphi_n)$. For the first inequality, $r(\operatorname{coz}\varphi_n) \prec\!\!\prec I$ since $\varphi_n \in \boldsymbol{O}^{\boldsymbol{I}}$ for each n, and therefore $\bigvee_n r(\operatorname{coz}\varphi_n) \leq I$. For the other inequality, let $\varphi \in \mathcal{R}L$ such that $\operatorname{coz}\varphi \in I$. Then $\varphi \in \boldsymbol{O}^{\boldsymbol{I}}$, and as such there are finitely many indices $m_1, \dots, m_k$ and elements $\alpha_{m_1}, \dots, \alpha_{m_k}$ of $\mathcal{R}L$ such that

$$\varphi = \alpha_{m_1}\varphi_{m_1} + \cdots + \alpha_{m_k}\varphi_{m_k}.$$

This implies that

$$\operatorname{coz}\varphi \leq \operatorname{coz}\varphi_{m_1} \vee \cdots \vee \operatorname{coz}\varphi_{m_k}.$$

Then

$$\operatorname{coz}\varphi_{m_i} \prec\!\!\prec \operatorname{coz}\varphi_{m_i} \text{ for each } i \in \{1, \dots, k\}$$

since L is a P-frame, and therefore $\operatorname{coz}\varphi_{m_i} \in r(\operatorname{coz}\varphi_{m_i})$, showing that

$$\operatorname{coz}\varphi \in r(\operatorname{coz}\varphi_{m_1}) \vee \cdots \vee r(\operatorname{coz}\varphi_{m_k}).$$

This implies that all cozero elements that belongs to I also belongs to $\bigvee_n r(\operatorname{coz}\varphi_n)$, and therefore $I \leq \bigvee_n r(\operatorname{coz}\varphi_n)$. Consequently, $I = \bigvee_n r(\operatorname{coz}\varphi_n)$, showing that $I \in \operatorname{Coz}\beta L$ since $r(\operatorname{coz}\varphi_n) \prec\!\!\prec I$ for each n.
Now considering a countably generated ideal Q of $\mathcal{R}L$, let us take $Q = \langle \varphi_1, \varphi_2, \dots \rangle$ and have $I = \bigvee_n r(\operatorname{coz}\varphi_n)$. Check that $I = \bigvee_{\alpha\in Q} r(\operatorname{coz}\alpha)$ and by hypothesis, for the first containment we have that for any n, $\operatorname{coz}\varphi_n \prec\!\!\prec \operatorname{coz}\varphi_n$ since L is a P-frame, and therefore $\operatorname{coz}\varphi_n \in r(\operatorname{coz}\varphi_n) \subseteq I$, meaning that $\varphi_n \in \boldsymbol{O}^{\boldsymbol{I}}$. As such, $Q \subseteq \boldsymbol{O}^{\boldsymbol{I}}$. For the other containment, let $\varphi \in \boldsymbol{O}^{\boldsymbol{I}}$. Then $r(\operatorname{coz}\varphi) \prec\!\!\prec I = \bigvee_{\alpha\in Q} r(\operatorname{coz}\alpha)$, which implies that $\varphi \in Q$. Therefore $\boldsymbol{O}^{\boldsymbol{I}} \subseteq Q$, and hence $Q = \boldsymbol{O}^{\boldsymbol{I}}$ finally having $I \in \operatorname{Coz}\beta L$ by what we have above. □

We then show that the lattice $\operatorname{PId}(\mathcal{R}L)$ of pure ideals of $\mathcal{R}L$ is isomorphic to βL for any completely regular frame L. The proposition below follows from [14, Proposition 3.11].

Proposition 4.1.7. *Given a P-frame L, then $\beta L \cong \operatorname{Idl}(\mathcal{R}L)$.*

Proof. Taking the ideal $\sum_{\gamma\in\Gamma} I_\gamma$ which is the join of a set $\{I_\gamma \mid \gamma\in\Gamma\}$ in $\mathrm{PId}(A)$, then the join is described in $\mathcal{R}L$ as follows:

$$\bigvee_{\mathrm{PId}(\mathcal{R}L)} \boldsymbol{O}^{I_\gamma} = \boldsymbol{O}^I,$$

where $I = \bigvee\{I_\gamma \mid \gamma\in\Gamma\}$. We see that $\subseteq$ is immediate.
Conversely, take $\varphi \in \boldsymbol{O}^I$ such that $r(\operatorname{coz}\varphi) \prec\!\!\prec I$, then

$$r(\operatorname{coz}\varphi)^* \vee \bigvee_{\beta L} I_\gamma = 1,$$

and so, by compactness

$$r(\operatorname{coz}\varphi) \prec\!\!\prec I_{\gamma_1}\vee\cdots\vee I_{\gamma_m} \text{ for some finitely many indices } \gamma_1,\ldots,\gamma_m.$$

This means $\operatorname{coz}\varphi \in I_{\gamma_1}\vee\cdots\vee I_{\gamma_m}$.
Now let $a_i \in I_{\gamma_i}$ to have $\operatorname{coz}\varphi = a_1\vee\cdots\vee a_m$ for every $i\in 1,\ldots,m$. There exists cozero elements $c_i\in I_{\gamma_i}$ where $a_i \prec\!\!\prec c_i$ because the I_{γ_i} are completely regular ideals of L. We then take $\delta_i\in\mathcal{R}L$ for each i such that $c_i = \operatorname{coz}\delta_i$. Then $\delta_i \in \boldsymbol{O}^{I_{\gamma_i}}$. Consequently,

$$\delta_1^2+\cdots+\delta_m^2 \text{ is in } \Sigma_\gamma \boldsymbol{O}^{I_{\gamma_i}}$$

and

$$\operatorname{coz}\varphi \prec\!\!\prec \operatorname{coz}(\delta_1^2+\cdots+\delta_1^2).$$

Therefore, φ is a multiple of $\delta_1^2+\cdots+\delta_m^2$, showing that $\varphi\in\Sigma_\gamma\boldsymbol{O}^{I_{\gamma_i}}$ for the other inclusion.
Consequently, given that $\boldsymbol{O}^I\wedge\boldsymbol{O}^J = \boldsymbol{O}^{I\wedge J}$ for all $I,J\in\beta L$, let $s\in I$ and c be a cozero element of L where $s\le c\in I$. Take $\delta\in\mathcal{R}L$ such that $c=\operatorname{coz}\delta$. Then $\delta\in\boldsymbol{O}^I=\boldsymbol{O}^J$ meaning $\operatorname{coz}\delta\in J$, and therefore $s\in J$. As such $I\subseteq J$, and by symmetry $I=J$. □

4.1.2 Characterizations of almost P-frames

Recall that the regular element a is regular if and only if $a=a^{**}$. From [25], we say L that is an *almost P-frame* if every cozero element in a frame L is regular. To start this section, we have the following propositions.

Proposition 4.1.8. *A frame L is almost-P if and only if*

$$\bigcup\{\boldsymbol{O}^J \mid J\in\mathrm{Pt}(\beta L)\} = \bigcup\{\boldsymbol{M}^J \mid J\in\mathrm{Pt}(\beta L)\}.$$

Proof. ($\Leftarrow$): For a given dense element $r \in \operatorname{Coz} L$, then $r = \operatorname{coz} \gamma$ where $\gamma \in \mathcal{R}L$. Using contradiction, suppose $r \neq 1$ and pick $J \in \operatorname{Pt}(\beta L)$ such that $r_L(\operatorname{coz} \gamma) \subseteq J$. Hypothetically, there is an $I \in \operatorname{Pt}(\beta L)$ such that $\gamma \in \boldsymbol{O}^I$ and $\gamma \in \boldsymbol{M}^J$ which means $\operatorname{coz} \gamma \in I$ which does not hold since $I \neq 1$.

($\Rightarrow$): We check both inclusions to prove equality. We know that $\boldsymbol{O}^J \subseteq \boldsymbol{M}^J$ for each J from Proposition 2.1.1. To show $\boldsymbol{O}^J \supseteq \boldsymbol{M}^J$, let $\alpha \in \boldsymbol{M}^J$ and since $\boldsymbol{M}^J$ is a proper ideal, then α is a zero divisor as seen in [7, Proposition 4.5]. We then pick a nonzero $\delta \in \mathcal{R}L$ to have $\alpha\delta = \mathbf{0}$ and because $\delta \neq \mathbf{0}$, $r_L(\operatorname{coz} \delta) \neq 0$. By spatiality of βL then there exists $I \in \operatorname{Pt}(\beta L)$ such that $r_L(\operatorname{coz} \delta) \nsubseteq I$. As such, by maximality of I, $r_L(\operatorname{coz} \delta) \vee I = 1$. However, $r_L(\operatorname{coz} \delta) \wedge r_L(\operatorname{coz} \alpha) = 0$, and so $r_L(\operatorname{coz} \alpha) \prec\!\!\prec I$, which means that $\alpha \in \boldsymbol{O}^I$. ☐

We now present the frame-theoretic characterizations of almost P-frames.

Proposition 4.1.9. *Let $h : L \to M$ be a dense onto frame homomorphism.*

1. *If h is coz-onto and L is almost-P, then M is also an almost P-frame.*
2. *If h is coz-codense and M is almost-P, then L is also an almost P-frame.*

Proof. (1): Let v and u be elements of $\operatorname{Coz} M$ and $\operatorname{Coz} L$ respectively such that $h(u) = v$. Since dense onto frame homomorphism preserves pseudocomplements, we then have

$$v = h(u) = h(u^{**}) = h(u)^{**} = v^{**},$$

and so M is an almost-P frame.

(2): Taking a dense cozero element u of L to have a dense cozero element $h(u)$ of M then $h(u) = 1$ because M is almost-P. Also, $u = 1$ because h is coz-codense, and therefore L is almost-P. ☐

Proposition 4.1.10. *A frame L is almost-P if and only if the realcompactification υL of L is almost-P.*

Proof. This follows from Proposition 4.1.9, because the join map $\upsilon L \to L$ has the properties; dense ontoness, coz-ontoness and coz-codensity. ☐

Next, we present the ring-theoretic characterizations of almost P-frames.

Proposition 4.1.11. *The following are equivalent for a frame L.*

1. *L is almost-P.*

2. *Every nonzero divisor in $\mathcal{R}L$ is invertible.*

3. *Every proper ideal of $\mathcal{R}L$ contains only zero divisors.*

4. *Every proper principal ideal of $\mathcal{R}L$ is non-essential.*

5. *Every fixed countably generated ideal of $\mathcal{R}L$ is contained in non-essential principal ideals.*

For our next characterization, we use $\operatorname{Min}\mathcal{R}L$ to denote the space of minimal prime ideals of $\mathcal{R}L$. Recall that for any $\varphi \in \mathcal{R}L$, $\operatorname{Ann}(\varphi) = \bigcap\{P \in \operatorname{Min}\mathcal{R}L \mid \varphi \notin P\}$.

Proposition 4.1.12. *A frame L is almost-P iff for every non-invertible $\delta \in \mathcal{R}L$, there exists a nonzero $\gamma \in \mathcal{R}L$ such that*

$$\bigcap\{I \in \operatorname{Min}\mathcal{R}L \mid \gamma \in I\} \subseteq \operatorname{Ann}(\delta).$$

Proof. ($\Rightarrow$): Because L is almost-P, from Proposition 4.1.11 (1) implies (2) meaning that δ is a zero divisor because it is non-invertible and so there is a nonzero $\gamma \in \mathcal{R}L$ which has it that $\gamma\delta = \mathbf{0}$ and so $\gamma \in \operatorname{Ann}(\delta)$. If P is a minimal prime ideal of $\mathcal{R}L$, then by primeness if $\delta \notin P$, then $\gamma \in P$. This shows that

$$\{A \in \operatorname{Min}\mathcal{R}L \mid \delta \notin A\} \subseteq \{B \in \operatorname{Min}\mathcal{R}L \mid \gamma \in B\}.$$

Taking intersections, we get

$$\begin{aligned}\bigcap\{B \in \operatorname{Min}\mathcal{R}L \mid \gamma \in B\} &\subseteq \bigcap\{A \in \operatorname{Min}\mathcal{R}L \mid \delta \notin A\}\\ &= \operatorname{Ann}(\delta),\end{aligned}$$

as needed.

($\Leftarrow$): Let $1 \neq c \in \operatorname{Coz} L$ and have that $\delta \in \mathcal{R}L$ such that $\operatorname{coz}\delta = c$. Then δ is non-invertible. We then have that $\gamma \neq \mathbf{0}$ such that

$$\bigcap\{P \in \operatorname{Min}\mathcal{R}L \mid \gamma \in P\} \subseteq \operatorname{Ann}(\gamma).$$

Therefore $\delta \in \operatorname{Ann}(\gamma)$ and we see that L is almost-P because c is not dense as $\operatorname{coz}\gamma$ is a nonzero element of L that misses c. $\square$

4.2 Characterizations of F-frames and F'-frames

4.2.1 Characterizations of F-frames

F-frames were defined in [8] by Ball and Walters-Wayland and Dube in [24] characterized F-frames L in terms of the ring $\mathcal{R}L$. We show that given a frame L, when $\mathcal{R}L$ is a Bézout ring, then L is an F-frame.

Definition 4.2.1. If the open quotient map $L \to {\downarrow}c$ for a frame L is a C^*-quotient map for each $c \in \operatorname{Coz} L$, then L is referred to as an *F-frame.*

Note from [8] that L is an F-frame iff whenever $u \wedge v = 0$ in $\operatorname{Coz} L$, then there exist $a, b \in \operatorname{Coz} L$ such that $a \vee b = 1$, and $a \wedge u = 0 = b \wedge u$.

We first show that indeed $\mathcal{R}L$ is indeed a Bézout ring. Recall that a Bézout ring is a ring A such that every finitely generated ideal is principal.

Lemma 4.2.2. *Given a reduced commutative f-ring with unit R, with the bounded part R^*, then these statements are true.*

1. *If a is a nonnegative invertible element of R, then $\frac{1}{a}$ is nonnegative.*

2. *Assuming R has bounded inversion property, then it is Bézout iff R^* is Bézout.*

Proof. 1. $a(\frac{1}{a} \vee 0) = 1 \vee 0 = 1$ therefore $\frac{1}{a} = \frac{1}{a} \vee 0 \geq 0$.

2. Suppose R is Bézout and let $0 \leq r \leq s$ in R^*. Since $R^* \subseteq R$ then $0 \leq r \leq s$ in R, and so there exists $a \in R$ such that $r = sa$. We then have that $a \wedge 1$ is an element of R^* with $(a \wedge 1)s = as \wedge s = r \wedge s = r$ and so R^* is Bézout.

Conversely, let $0 \leq r \leq s$ in R. We then have $r + rs \leq s + rs$, which implies

$$r(1+s) \leq s(1+r). \tag{4.1}$$

From above, R has bounded inversion and $1+r$ and $1+s$ are nonnegative, they are invertible. Therefore $\frac{1}{1+r} \cdot \frac{1}{1+s}$ is a nonnegative element of R by (1) above.

Multiplying it by both sides of (Equation 4.1) gives $0 \leq \frac{r}{1+r} \leq \frac{s}{1+s}$. Then, for any $a \geq 0$ in R, we can do the same calculation beginning with $0 \leq a \leq 1 + a$ which then gives $0 \leq \frac{a}{1+a} \leq 1$, such that $\frac{u}{1+u} \in R^*$. Therefore, $0 \leq \frac{r}{1+r} \leq \frac{s}{1+s}$ implies there exists some $v \in R^*$ such that $\frac{r}{1+r} \leq v\frac{s}{1+s}$ since R^* is Bézout from our hypothesis. Moreover, $r = es$ for some $e \in R$, and therefore R is Bézout. $\square$

We now look into characterizations in terms of ideals.

Proposition 4.2.3. *For a completely regular frame L, the following are equivalent.*

1. *L is an F-frame.*
2. *$\boldsymbol{O}^I$ is prime for every $I \in \mathrm{Pt}(\beta L)$.*
3. *Minimal prime ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{O}^I$ for $I \in \mathrm{Pt}(\beta L)$.*
4. *Every maximal ideal of $\mathcal{R}L$ contains a unique minimal prime ideal.*
5. *Every prime ideal of $\mathcal{R}L$ contains a unique minimal prime ideal.*

Proposition 4.2.4. *Prime ideals of $\mathcal{R}L$ that have a given prime ideal form a chain.*

Proof. Given a minimal prime ideal P of $\mathcal{R}L$, suppose $\operatorname{coz}\varphi = \operatorname{coz}\delta$ with $\varphi \in P$. Taking $\gamma \notin P$ such that $\varphi\gamma = \mathbf{0}$, then

$$\operatorname{coz}\delta\gamma = \operatorname{coz}\delta \wedge \operatorname{coz}\gamma = \operatorname{coz}\varphi \wedge \operatorname{coz}\gamma = \mathbf{0},$$

implying $\delta\gamma = \mathbf{0} \in P$, and by primeness $\delta \in P$. $\square$

Corollary 4.2.5. *The statements below are equivalent for a completely regular frame L.*

1. *L is an F-frame.*
2. *Prime ideals of $\mathcal{R}L$ contained in any maximal ideal form a chain.*
3. *Prime z-ideals of $\mathcal{R}L$ contained in any maximal ideal form a chain.*

Proof. (1) $\Rightarrow$ (2): Let L is an F-frame and $I \in \mathrm{Pt}(\beta L)$. Given the prime ideals P and Q contained in $\boldsymbol{M}^I$ then $\boldsymbol{O}^I$ is contained in both P and Q. Hypothetically, $\boldsymbol{O}^I$ is a prime ideal because L is an F-frame, and therefore by Proposition 4.2.4, $P \subseteq Q$ or $Q \subseteq P$.

The implication (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1): Let P, Q be minimal prime ideals contained in $\boldsymbol{M}^I$ where $I \in \mathrm{Pt}(\beta L)$. Then P and Q are prime z-ideals. Now suppose $P \subseteq Q$ which by minimality implies $P = Q$, then each maximal ideal of $\mathcal{R}L$ has a unique minimal prime ideal. $\square$

4.2.2 Characterizations of F'-frames

Balls and Walters in [8] defined an *F'-frame* L such that if $u \wedge v = 0$ in $\mathrm{Coz}\, L$ then $u^* \vee v^* = 1$. In [36, Lemma 4.4], it is stated that if $h : L \to M$ is coz-onto whenever L is an F'-frame, then M is also an F'-frame. We show that every open quotient of an F'-frame is F'. Our main reference for this section will be [24].

Proposition 4.2.6. *For every a in an F'-frame L, $\downarrow a$ is an F'-frame.*

Proof. Take some $p, q \in \mathrm{Coz}(\downarrow a)$ such that $p \wedge q = 0$ and have a cozero element c of L such that $c \leq a$. From the mapping $x \mapsto c \wedge x$ where $x \in \downarrow a$, we get $\downarrow a \to \downarrow c$ is a frame homomorphism. Then, we have cozero elements of $\downarrow c$; $c \wedge p$ and $c \wedge q$ with $(c \wedge p) \wedge (c \wedge q) = 0$. It can then be seen that $\downarrow c$ is an F'-frame and therefore:

$$(c \wedge (c \wedge p)^*) \vee (c \wedge (c \wedge q)^*) = c. \tag{4.2}$$

Then

$$(c \wedge (c \wedge p)^*) \wedge p = (c \wedge p) \wedge (c \wedge p)^* = 0,$$

which implies $(c \wedge (c \wedge p)^*) \leq p^*$. Moreover, $(c \wedge (c \wedge q)^*) \leq q^*$ and it follows from (Eqn 4.2) that $c \leq p^* \vee q^*$. Also, when $a = a \wedge (p^* \vee q^*)$ then $a \leq p^* \vee q^*$. $\square$

Recall that a frame L is *locally F'* if for every $s \in L$ there exists $S \subseteq L$ whenever $s = \bigvee S$, and $\downarrow r$ is an F'-frame for every $r \in S$.

Corollary 4.2.7. *A completely regular frame is an F'-frame iff it is locally F'.*

Proof. The proof for ($\Rightarrow$) follows from Proposition 4.2.6.

($\Leftarrow$): Let $a \wedge b = 0$ for some $a, b \in \mathrm{Coz}\, L$. Suppose $S \subseteq L$ such that $\bigvee S = 1$ and $\downarrow x$ is F' for every $x \in S$. For any $s \in S$, we have cozero elements $a \wedge s$ and $b \wedge s$ of $\downarrow s$ which meet at the bottom and thereby, $(s \wedge a^*) \vee (s \wedge b^*) = s$. This implies that $s \wedge (a^* \vee b^*) = s$ and so, $s \leq (a^* \vee b^*)$. As such, $1 \leq a^* \vee b^*$, and therefore L is an F'-frame. $\square$

We recall the definition of a fixed ideal from Chapter 2 and proceed with the lemma below.

Lemma 4.2.8. *A z-ideal is prime if and only if it contains a prime ideal. Essentially, a z-ideal I is prime iff when $\alpha\beta = \mathbf{0}$, then $\alpha \in I$ or $\beta \in I$.*

Proof. ($\Rightarrow$): This is trivial from the definition.

($\Leftarrow$): Let I be a z-ideal with $\alpha \in I$ or $\beta \in I$. Then $\operatorname{coz}[I]$ is an ideal of $\operatorname{Coz} L$ such that whenever $a \wedge b = 0$ in $\operatorname{Coz} L$, then $a \in \operatorname{coz}[I]$ or $b \in \operatorname{coz}[I]$. Thus $\operatorname{coz}[I]$ is a prime ideal of $\operatorname{Coz} L$ by [36, Lemma 3.8]. Now suppose $\alpha\beta \in I$. Then $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta \in \operatorname{coz}[I]$. Assuming $\operatorname{coz}\alpha \in \operatorname{coz}[I]$, because I is a z-ideal then $\alpha \in I$ and so I is prime. $\square$

We then have the following lemma which shows that if a prime ideal P of $\mathcal{R}L$ with $\boldsymbol{O}^I \subseteq P \subseteq \boldsymbol{M}^I$ is fixed, then $\boldsymbol{M}^I$ is also fixed.

Lemma 4.2.9. *For any $I \in \beta L$, $\operatorname{coz}[\boldsymbol{O}^I] = \operatorname{coz}[\boldsymbol{M}^I] = I$.*

Proof. If $\delta \in \boldsymbol{M}^I$, then $r_L(\operatorname{coz}\delta) \subseteq I$. When we take the joins we get $\operatorname{coz}\delta \leq \bigvee I$. Therefore,

$$\bigvee \operatorname{coz}[\boldsymbol{O}^I] \leq \operatorname{coz}[\boldsymbol{M}^I] \leq \bigvee I.$$

Now, take a cozero element r of L in I and have $r = \operatorname{coz}\delta$ for some $\delta \in \mathcal{R}L$. Then $\delta \in \boldsymbol{O}^I$, and $r \leq \bigvee \operatorname{coz}[\boldsymbol{O}^I]$. As such, $\bigvee I \leq \bigvee \operatorname{coz}[\boldsymbol{O}^I]$ because all elements of I are below some cozero element in I. $\square$

Proposition 4.2.10. *Consider the following for a completely regular frame L.*

1. *L is an F'-frame.*
2. *$\boldsymbol{O}^I$ is a prime ideal for every point I of βL with $\bigvee I = 1$.*
3. *The prime ideals of $\mathcal{R}L$ contained in any fixed maximal ideal form a chain.*
4. *For any point I of βL with $\bigvee I = 1$, the prime ideals of $\mathcal{R}L$ that contain $\boldsymbol{O}^I$ form a chain.*

The proof is documented in [24] where we see that (1) implies (2);(4) implies (1) if L has enough points and (2), (3) and (4) are equivalent.

We conclude by showing that if L is a strongly zero-dimensional F'-frame, then $\mathcal{R}L$ is almost weak Baer, and that the converse holds.

Definition 4.2.11. If the Stone-Čech compactification of a frame is generated by complemented elements, then we say the frame is *strongly zero-dimensional.*

From [11], we have that a frame L is strongly zero-dimensional iff when $a \prec\!\!\prec b$ in L, then $a \leq c \leq b$ for some complemented c.

Definition 4.2.12. If $\mathrm{Ann}(a)$ is generated by idempotents for each $a \in R$, then we say the ring R is *almost weak Baer.*

From [57, Lemma 1.2], an ideal I is generated by idempotents iff for every $r \in I$ there is an idempotent $e \in I$ such that $r = er$.

Lemma 4.2.13. *A frame L is an F-frame iff $\boldsymbol{M}^{r(c^*)} = \boldsymbol{O}^{r(c^*)}$ for every $c \in \operatorname{Coz} L$.*

Proof. Let L be an F-frame with $c \in \operatorname{Coz} L$ and $\alpha \in \boldsymbol{M}^{r(c^*)}$, to have $d = \operatorname{coz} \alpha$ as shorthand. Then $r(\operatorname{coz} \alpha) \leq r(c^*)$, to get $d \leq c^*$, and so $c \wedge d = 0$. Also, because L is an F-frame then there exist cozero elements a, b such that $a \vee b = 1$ and $a \wedge c = 0 = b \wedge d$. We can find $m \prec\!\!\prec a$ such that $m \vee b = 1$ by normality of $\operatorname{Coz} L$. As such, $d \wedge b = 0$ meaning $d \leq u \prec\!\!\prec a \leq c^*$ and so it follows $\operatorname{coz} \alpha \in r(c^*)$ meaning $\alpha \in \boldsymbol{O}^{r(c^*)}$. Therefore, $\boldsymbol{M}^{r(c^*)} \subseteq \boldsymbol{O}^{r(c^*)}$. As the other inclusion holds, then we have equality.

Conversely, assume $c \wedge d = 0$ in $\operatorname{Coz} L$. Take $\delta \in \mathcal{R}L$ such that $\operatorname{coz} \delta = c$. We have $r(\operatorname{coz} \delta) \leq r(d^*)$ because $c \leq d^*$, and therefore $\delta \in \boldsymbol{M}^{r(d^*)}$; and so $\delta \in \boldsymbol{O}^{r(d^*)}$ by supposition. We then pick cozero elements p and q where $c \prec\!\!\prec p \prec\!\!\prec q \prec\!\!\prec d^*$ and pick another a cozero element s to have $p \wedge s = 0$ and $s \vee q = 0$. Because $p \wedge s = 0$ and $c \leq p, c \wedge s = 0 = q \wedge d$, with the latter because $q \leq d^*$. As such, L is an F-frame as seen from [8]. □

Before we conclude this section, we have the following lemma.

Lemma 4.2.14. *Suppose* $\operatorname{coz} \gamma \prec\!\!\prec \operatorname{coz} \delta$ for some $\gamma, \varphi \in \mathcal{R}L$. *Then there exists an invertible $\tau \in \mathcal{R}L$ such that $\gamma = \gamma\tau\varphi^2$. Therefore, γ is a multiple of φ.*

The proof of this lemma follows from Lemma 2.1.2.

If we let c be a complemented element of L and take $\gamma \in \mathcal{R}L$ such that $\operatorname{coz} \gamma = c$, then $\operatorname{coz} \gamma \prec\!\!\prec \operatorname{coz} \gamma$, and hence, by Lemma 4.2.14, there is an invertible $\tau \in \mathcal{R}L$ such that $\gamma = \gamma^3\tau$.

This implies $\gamma^2\tau = (\gamma^2\tau)^2$. Since τ is invertible, $\operatorname{coz}\gamma^2\tau = \operatorname{coz}\gamma^2 \wedge \operatorname{coz}\tau = \operatorname{coz}\gamma = c$. Therefore, $\gamma^2\tau$ is an idempotent with $\operatorname{coz}\gamma^2\tau = c$. The forward implication is shown in Lemma 2.2.1 which then gives us that $\operatorname{coz}\delta \vee \operatorname{coz}(\mathbf{1}-\delta) = 1$ since $\delta = \delta^2$ and $\mathbf{1}-\delta = (\mathbf{1}-\delta)^2$ and therefore, $\operatorname{coz}\delta$ is complemented. We then have the following proposition from [24].

Proposition 4.2.15. *$\mathcal{R}L$ is almost weak Baer if and only if L is a strongly zero-dimensional F-frame.*

Proof. ($\Rightarrow$): Take $\alpha, \beta \in \mathcal{R}L$ such that $a \prec\!\!\prec \operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\beta \prec\!\!\prec b$ and have $a \prec\!\!\prec b$ in L for soe $a, b \in L$. We then pick $\gamma \in \mathcal{R}L$ such that $\operatorname{coz}\alpha \wedge \operatorname{coz}\gamma = 0$ and $\operatorname{coz}\gamma \vee \operatorname{coz}\beta = 1$. Then $\alpha \in \operatorname{Ann}(\gamma)$, and by supposition, there is an idempotent $\delta \in \operatorname{Ann}(\gamma)$ to have $\alpha = \alpha\delta$. As such, $\operatorname{coz}\alpha \leq \operatorname{coz}\delta$. Since $\delta \in \operatorname{Ann}(\gamma)$, we have $\operatorname{coz}\delta \wedge \operatorname{coz}\gamma = 0$, and so $\operatorname{coz}\delta \leq \operatorname{coz}\beta$. Therefore, $\operatorname{coz}\delta$ is a complemented element with $a \leq \operatorname{coz}\delta \leq b$ and therefore L is strongly zero-dimensional. To show L is an F-frame, we let $\operatorname{coz}\gamma \in \operatorname{Coz}L$ and take $\gamma \in \mathcal{R}L$. Let $\varphi \in \boldsymbol{M}^{r(c^*)}$. Then $r(\operatorname{coz}\varphi) \leq r(\operatorname{coz}\gamma^*)$, which implies $\operatorname{coz}\varphi \wedge \operatorname{coz}\gamma = \mathbf{0}$, such that $\varphi \in \operatorname{Ann}(\gamma)$. There is an idempotent $\delta \in \operatorname{Ann}(\gamma)$ such that $\varphi = \varphi\delta$ by hypothesis. Now, $\delta\gamma = \mathbf{0}$ implies $\operatorname{coz}\delta \wedge \operatorname{coz}\gamma = 0$, and consequently, $\operatorname{coz}\varphi \leq \operatorname{coz}\delta \prec\!\!\prec \operatorname{coz}\delta \leq \operatorname{coz}\gamma^*$. Therefore $\operatorname{coz}\varphi \in r(\operatorname{coz}\gamma^*)$, showing that $\varphi \in \boldsymbol{O}^{r(\operatorname{coz}\gamma^*)}$. Thus, $\boldsymbol{M}^{r(\operatorname{coz}\gamma^*)} \subseteq \boldsymbol{O}^{r(\operatorname{coz}\gamma^*)}$, and hence, $\boldsymbol{M}^{r(\operatorname{coz}\gamma^*)} = \boldsymbol{O}^{r(\operatorname{coz}\gamma^*)}$ and so L is an F-frame.

($\Leftarrow$): Let $\alpha \in \mathcal{R}L$ and $\operatorname{coz}\alpha$. For any $\varphi \in \operatorname{Ann}(\alpha)$, we have $\operatorname{coz}\varphi \leq \operatorname{coz}\alpha^*$, such that $\varphi \in \boldsymbol{M}^{r(\operatorname{coz}\alpha^*)}$. Since L is an F-frame by hypothesis, $\varphi \in \boldsymbol{O}^{r(\operatorname{coz}\alpha^*)}$, and so $\operatorname{coz}\varphi \prec\!\!\prec \operatorname{coz}\alpha^*$. Because hypothetically L is strongly zero-dimensional, there is a complemented element $\operatorname{coz}\delta$ with $\operatorname{coz}\varphi \prec\!\!\prec \operatorname{coz}\delta \prec\!\!\prec \operatorname{coz}\alpha^*$. Take an idempotent $\delta \in \mathcal{R}L$. Now, $\operatorname{coz}\varphi \prec\!\!\prec \operatorname{coz}\delta$ implies $\varphi \in \langle\delta\rangle$, and $\operatorname{coz}\delta \leq (\operatorname{coz}\alpha)^*$ implies $\delta \in \operatorname{Ann}(\alpha)$. As such, φ is a multiple of an idempotent belonging to $\operatorname{Ann}(\alpha)$, and therefore $\operatorname{Ann}(\alpha)$ is generated by idempotents proving $\mathcal{R}L$ is an almost weak Baer ring. $\square$

4.3 Characterizations of quasi P-frames and quasi F-frames

In this section, we will introduce the idea of quasi P-frames and quasi F-frames. We define quasi P-frames based on the definition of quasi P-spaces. A Tychonoff space is called a P-space if the collection of its open sets forms a P-frame. However, Ball, Walters-Wayland [8], and Zenk[53] show that while P-spaces behave nicely under certain operations, P-frames do not always do the same. In fact, there are cases where a P-frame can be broken down into quotients that are no longer P-frames. In this section, we will study how much the theory of quasi P-frames is similar to the theory of quasi P-spaces, which was developed by Henriksen, Martínez, and Woods in [43].

From [8], we see that every F-frame is also an F'-frame, and every F'-frame is a quasi F-frame though the reverse need not be true. A completely regular topological space X is called a quasi F-space if and only if the frame formed by its open sets is a quasi F-frame. This is extended to frames analogously.

4.3.1 Characterizations of Quasi P-frames

For this section we will be considering frames L whose all prime z-ideals of $\mathcal{R}L$ are either minimal or maximal. Our main references will be [33] , [43] and [53].

Definition 4.3.1. Given a prime z-ideal Q of $\mathcal{R}L$ where $Q \subseteq \boldsymbol{M}^I$, then $Q = \boldsymbol{M}^I$ or Q is a minimal prime ideal, then I which is a point of βL is a *quasi P-point.*

If every point of βL is a quasi P-point then the frame L is *quasi P-frame.* L is a quasi P-frame precisely if every prime z-ideal of $\mathcal{R}L$ is minimal or maximal. Moreover, L is a quasi P-frame iff $\dim(\mathrm{ZId}(\mathcal{R}L)) \leq 1$. We then have the following characterization.

Proposition 4.3.2. *A frame L is said to be a quasi P-frame if and only if for every pair of cozero elements m, n of L, there is a pair of cozero elements u, v of L such that*

$$m \wedge u = 0, n \vee v = 1, n \wedge v \leq m \vee u.$$

This can also be shown directly from the definition. We then use Proposition 4.3.2 to show that if L is a quasi P-frame and $L \to M$ is a coz-onto homomorphism, then M is also a quasi P-frame in the lemma below.

Lemma 4.3.3. *If L is a quasi P-frame and $h\colon L \to M$ is a coz-onto homomorphism, then M is a quasi P-frame.*

Proof. Let $c, d \in \operatorname{Coz} M$. Because h is coz-onto, there exist $a, b \in \operatorname{Coz} L$ such that $h(a) = c$ and $h(b) = d$. Since L is quasi P-frame, there exist $a', b' \in \operatorname{Coz} L$ such that

$$a \wedge a' = 0, b \vee b' = 1 \text{ and } b \wedge b' \leq a \vee a'.$$

Also,

$$h(a) \wedge h(a') = h(0), h(b) \vee h(b') = h(1)$$

because frame homomorphisms preserve cozero elements and

$$h(b) \wedge h(b') \leq h(a) \vee h(a').$$

Moreover,

$$c \wedge h(a') = 0, d \vee h(b') = 1 \text{ and } d \wedge h(b') \leq c \vee h(a').$$

As such $h(a')$ and $h(b')$ are cozero elements in M, and therefore M is quasi P-frame. $\square$

From [36], we see that given a frame surjection $h : L \to M$, then if M is Lindelöf and L is completely regular, then h is coz-onto which gives the following.

Proposition 4.3.4. *Given a quasi P-frame L and $h : L \to M$ be a quotient of L with an M Lindelöf frame, then M is a quasi P-frame.*

Corollary 4.3.5. *Given a quasi P-frame L, $\downarrow c$ is a quasi P-frame for all $c \in \operatorname{Coz} L$.*

Therefore, from Lemma 4.3.3, we have that if βL is a quasi P-frame, then L is a quasi P-frame. Moreover, we can say that a pseudocompact frame is quasi P-frame iff its Stone-Čech compactification is a quasi P-frame because a frame L is pseudocompact exactly when βL is isomorphic to υL and $\mathcal{R}L$ is isomorphic to $\mathcal{R}(\upsilon L)$.

Lemma 4.3.6. *Given a cozero complemented frame L, if I is a prime ideal (not minimally prime) of $\mathcal{R}L$, then I contains a nonzero divisor.*

Proof. Let $\gamma \in I$ such that $\mathrm{Ann}(\gamma) \subseteq I$. Because L is a cozero complemented frame, there exists an element $\delta \in \mathcal{R}L$ such that $\operatorname{coz}\gamma \wedge \operatorname{coz}\delta = 0$ and $\operatorname{coz}\gamma \vee \operatorname{coz}\delta$ is dense which implies that $\operatorname{coz}(\gamma^2 + \delta^2)$ is dense. Therefore, $\gamma^2 + \delta^2$ is a nonzero divisor and since $\gamma\delta = \mathbf{0}$, then $\delta \in I$, when $\gamma^2 + \delta^2 \in I$. □

The following propositions follow from [59, Proposition 3.2.2].

Proposition 4.3.7. *Given a completely regular quasi P-frame L and $I \subseteq \operatorname{Coz} L$ a prime ideal in $\operatorname{Coz} L$ containing a dense cozero element, then I is a maximal ideal in $\operatorname{Coz} L$.*

Proof. Let $Q = \{\gamma \in \mathcal{R}L \mid \operatorname{coz}\gamma \in I\}$. We claim that Q is a prime z-ideal. We first show that Q is an ideal of $\mathcal{R}L$. Consider any $\alpha, \beta \in Q$, and any $\gamma \in \mathcal{R}L$. Then $\operatorname{coz}\alpha$ and $\operatorname{coz}\beta$ are elements of I, which implies $\operatorname{coz}\alpha \vee \operatorname{coz}\beta \in I$ since I is an ideal in $\operatorname{Coz} L$. Since $\operatorname{coz}(\alpha+\beta) \leq \operatorname{coz}\alpha \vee \operatorname{coz}\beta$, it follows that $\operatorname{coz}(\alpha + \beta) \in I$ since I is a downset. Thus, $\alpha + \beta \in Q$. Similarly, from the relations $\operatorname{coz}(\alpha\gamma) = \operatorname{coz}\alpha \wedge \operatorname{coz}\gamma \leq \operatorname{coz}\alpha$, we deduce that $\alpha\gamma \in Q$. Therefore Q is an ideal in $\mathcal{R}L$. It is easy to see that Q is a z-ideal since $\operatorname{coz}\alpha = \operatorname{coz}\beta$ and $\beta \in Q$ implies $\operatorname{coz}\alpha \in I$, when $\alpha \in Q$. To see that Q is prime, consider any $\alpha, \beta \in \mathcal{R}L$ such that $\alpha\beta \in Q$. Then $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \operatorname{coz}(\alpha\beta) \in I$, implying $\operatorname{coz}\alpha \in I$ or $\operatorname{coz}\beta \in I$ since I is a prime ideal in $\operatorname{Coz} L$. It follows therefore that $\alpha \in Q$ or $\beta \in Q$. Hence Q is prime. Now, being a prime ideal in $\mathcal{R}L$, Q is contained in a maximal ideal say M, and contains a minimal prime ideal say P. By hypothesis, there is a dense cozero element c in L which is contained in I. Take $\gamma \in \mathcal{R}L$ such that $\operatorname{coz}\gamma = c$. Then $\gamma \in Q$, and γ is a nonzero divisor since $\operatorname{coz}\gamma$ is dense. Thus, $\gamma \notin P$ because minimal prime ideals consist entirely of zero-divisors. So $Q \neq P$, and therefore $Q = M$ since L is a quasi P-frame. We show from this that I is a maximal ideal in $\operatorname{Coz} L$. Consider any $d \in \operatorname{Coz} L$ with $d \notin I$. We must show that $\langle I, d\rangle$, the ideal generated by I and d, is the whole of $\operatorname{Coz} L$. Take $\gamma \geq 0 \in \mathcal{R}L$ with $d = \operatorname{coz}\gamma$. Then $\gamma \notin Q$. Since Q is a maximal ideal in $\mathcal{R}L$, the ideal $\langle Q, \gamma\rangle = \mathcal{R}L$, so $\mathbf{1} = q + \delta\gamma$, for some $q \in Q$ and $\delta \in \mathcal{R}L$. Thus,

$$1 = \operatorname{coz}(q + \delta\gamma) \leq \operatorname{coz} q \vee \operatorname{coz}\gamma \in \langle I, d\rangle.$$

Therefore $\langle I, d\rangle = \operatorname{Coz} L$, showing that I is a maximal ideal in $\operatorname{Coz} L$. □

Proposition 4.3.8. *Given a cozero complemented completely regular frame L, L is a quasi P-frame if for any prime ideal $I \subseteq \operatorname{Coz} L$ which contains a dense cozero element, I is maximal.*

Proof. Taking a non-minimal prime z-ideal Q then it contains some nonzero divisor γ because L is cozero complemented as seen in Lemma 4.3.6. Let

$$I = \{\operatorname{coz} \delta \mid \delta \in Q\}.$$

We need to show that I is a prime ideal of $\operatorname{Coz} L$. Then $\operatorname{coz} \gamma$ is dense since γ is a nonzero divisor. However $\operatorname{coz} \gamma \in I$ and therefore by hypothesis I is a maximal in $\operatorname{Coz} L$ and so Q is a maximal ideal in $\mathcal{R}L$ and it follows that L is a quasi P-frame. □

4.3.2 Characterizations of quasi F-frames

In this section we will focus on quasi-F frames, which are defined below, with our main references being [17, 24] and [36]. Moreover, Ball and Walters in [8] show element-wise characterization of quasi F-frames. Particularly, let L be a completely regular frame. Then L is said to be quasi-F if and only if when the join of two cozero elements is dense, then the pseudocomplements of the cozero elements are completely separated thereby giving us the definition below.

Definition 4.3.9. For a given frame L, if for all $x, y \in \operatorname{Coz} L$ such that $x \wedge y = 0$ and $x \vee y$ is dense and there exist $a, b \in \operatorname{Coz} L$ such that $x \wedge a = y \wedge b = 0$ and $a \vee b = 1$, then L is said to be a *quasi-F* frame.

We then characterize quasi F-frames as follows. The following proposition follows from [32, Corollary 3.3].

Corollary 4.3.10. *1. Every almost-P frame is quasi-F.*

2. Every F'-frame is quasi-F.

Proof. 1. Let L be an almost-P frame and for some $x, y \in \operatorname{Coz} L$, $x \vee y$ is dense. We then have $x \vee y = 1$, and therefore $x^{**} \vee y^{**} = 1$ showing that L is quasi-F.

2. Assume that L is an F'-frame. Now, suppose that for some $x, y \in \operatorname{Coz} L$, $x \vee y$ is dense and $x \wedge y = 0$, and therefore $x^* \wedge y^* = 0$ by density, and $x^* \vee y^* = 1$ since L is an F'-frame. This shows that x^* and y^* are complemented. This shows that they are cozero elements implying L is a quasi-F frame.

□

Recall that given a frame L, we say it is *basically disconnected* if $a^* \vee a^{**} = 1$ for all $a \in \operatorname{Coz} L$. We then have the corollary below from [32].

Corollary 4.3.11. *1. Every basically disconnected frame is quasi-F.*

2. Every nearly cozero-complemented quasi F-frame is basically disconnected.

Proof. 1. Assume that a frame L is basically disconnected and that for some $u, v \in \operatorname{Coz} L$, $u \vee v$ is dense. We then have that $u^* \wedge v^* = 0$ which shows that $u^* \leq v^{**}$. Because L is basically disconnected, then $u^* \vee u^{**} = 1$ and therefore $u^{**} \vee v^{**} = 1$ implying that L is quasi-F.

2. Assume that a frame M is nearly cozero-complemented quasi F and $s \in \operatorname{Coz} M$ and then choose a $t \in \operatorname{Coz} M$ to have it that $s \vee t$ is dense and $s \wedge t = 0$. We then have $s^{**} \vee t^{**} = 1$. However, $s^{**} \wedge t^{**} = 0$ because $s \wedge t = 0$ and therefore we have that s^{**} is complemented implying that M is basically disconnected. □

We now give a characterization in terms of rings of continuous functions such that for each $I \in \beta L$, the ideal $\boldsymbol{M}^I$ of $\mathcal{R}L$ is defined by

$$\boldsymbol{M}^I = \{\varphi \in \mathcal{R}L \mid r(\operatorname{coz} \varphi) \subseteq I\}.$$

We then have the following lemma from [32].

Lemma 4.3.12. *Let $S \subseteq \mathcal{R}L$ and put $a = \bigvee\{\operatorname{coz} \alpha \mid \alpha \in S\}$. Then* $\operatorname{Ann}(S) = \boldsymbol{M}^{r(a^*)}$. *In particular, for any $\delta \in \mathcal{R}L$,* $\operatorname{Ann}^2(\delta) = \boldsymbol{M}^{r(\operatorname{coz} \delta)^{**}}$.

Proof. For any $\tau \in \mathcal{R}L$ we have

$$\begin{aligned}
\tau \in \operatorname{Ann}(S) &\iff \tau\alpha = \mathbf{0} \text{ for each } \alpha \in S \\
&\iff \operatorname{coz} \tau \wedge \operatorname{coz} \alpha = 0 \text{ for each } \alpha \in S \\
&\iff \operatorname{coz} \tau \wedge a = 0 \\
&\iff \operatorname{coz} \tau \leq a^* \\
&\iff r(\operatorname{coz} \tau) \subseteq r(a^*) \\
&\iff \tau \in \boldsymbol{M}^{r(a^*)},
\end{aligned}$$

proving the first part. By the first part,

$$\operatorname{Ann}^2(\delta) = \operatorname{Ann}(\operatorname{Ann}(\delta)) = \operatorname{Ann}(\boldsymbol{M}^{r(\operatorname{coz} \delta)^*}).$$

By complete regularity, we have

$$\bigvee\{\operatorname{coz}\alpha \mid \alpha \in \boldsymbol{M}^{r(\operatorname{coz}\delta)^*}\} = \bigvee\{\operatorname{coz}\alpha \mid \operatorname{coz}\alpha \leq (\operatorname{coz}\delta)^*\} = (\operatorname{coz}\delta)^*.$$

Thus, by the first part, $\operatorname{Ann}^2(\delta) = \boldsymbol{M}^{r(\operatorname{coz}\delta)^{**}}$ proving the second part. □

Check that given a zero divisor δ of $\mathcal{R}L$, then because $\operatorname{coz}\delta$ misses some nonzero cozero element then it is not dense. On the flip side, if a nonzero element misses some nonzero element of a completely regular frame, then it misses some nonzero cozero element. As a consequence, we get the following from [24] and [32].

Lemma 4.3.13. *An element δ of $\mathcal{R}L$ is a nonzero divisor iff $\operatorname{coz}\delta$ is dense.*

Proposition 4.3.14. *A completely regular frame L is quasi-F iff for every $\delta, \gamma \in \mathcal{R}L$, $\operatorname{Ann}^2(\delta) + \operatorname{Ann}^2(\gamma) = \mathcal{R}L$ whenever $\delta + \gamma$ is a nonzero divisor.*

Corollary 4.3.15. *A completely regular frame L is quasi-F iff a^* and b^* are completely separated whenever $a, b \in \operatorname{Coz} L$ are such that $a \vee b$ is dense.*

Proof. ($\Rightarrow$): Assume L is quasi-F and $a \vee b$ is dense for some $a, b \in \operatorname{Coz} L$. Pick nonnegative $\alpha, \beta \in \mathcal{R}L$ such that $a = \operatorname{coz}\alpha$ and $b = \operatorname{coz}\beta$. Then $\operatorname{coz}(\alpha + \beta) = \operatorname{coz}\alpha \vee \operatorname{coz}\beta$, which is dense. Therefore $\alpha + \beta$ is a nonzero divisor and $\operatorname{Ann}^2(\alpha) + \operatorname{Ann}^2(\beta) = \mathcal{R}L$. Take $\delta \in \operatorname{Ann}^2(\alpha)$ and $\delta \in \operatorname{Ann}^2(\beta)$ such that $\mathbf{1} = \gamma + \delta$. Then

$$1 = \operatorname{coz}\mathbf{1} = \operatorname{coz}(\gamma + \delta) \leq \operatorname{coz}\gamma \vee \operatorname{coz}\delta.$$

Since $\gamma \in \operatorname{Ann}^2(\alpha) = \boldsymbol{M}^{r(a^{**})}$, then $\operatorname{coz}\gamma \leq a^{**}$ and similarly, $\operatorname{coz}\delta \leq b^{**}$. As such, $\operatorname{coz}\gamma$ and $\operatorname{coz}\delta$ are cozero elements of L such that $\operatorname{coz}\gamma \vee \operatorname{coz}\delta = 1$ and $a^* \wedge \operatorname{coz}\gamma = b^* \wedge \operatorname{coz}\delta = 0$ and so a^* and b^* are completely separated.

($\Leftarrow$): Given $c, d \in \operatorname{Coz} L$, let $c \vee d$ be dense. By assumption, there exist $p, q \in \operatorname{Coz} L$ to have $p \vee q = 1$ and $c^* \wedge p = d^* \wedge q = 0$. As such, $p \leq c^{**}$ and $q \leq d^{**}$, such that $c^{**} \vee d^{**} = 1$ and so L is quasi-F. □

We then characterize quasi-F frames in terms of d-ideals as in the following corollary.

Corollary 4.3.16. *A frame L is quasi-F if and only if whenever P, Q are d-ideals of $\mathcal{R}L$ such that $P + Q$ is regular, then $P + Q = \mathcal{R}L$.*

We then recall the following. First, if every finitely generated ideal that contains a nonzero divisor is principal in a given f-ring R, then R is called *quasi-Bézout*. We would like to show that given a completely regular frame L is quasi-F if and only if $\mathcal{R}L$ is quasi-Bézout and therefore freely use properties of f-rings and characterizations of quasi-Bézout f-rings and still have the property of being quasi-Bézout being preserved by ring isomorphisms.

Proposition 4.3.17. *A completely regular frame L is quasi-F iff $\mathcal{R}L$ is quasi-Bézout.*

Proof. Given a quasi-F frame L, then βL is quasi-F. Since by spatiality, $\mathcal{R}(\beta L)$ is quasi-Bézout then because $\mathcal{R}^*L$ is isomorphic to $R(\beta L)$, then $\mathcal{R}^*L$ is quasi-Bézout. We then let $\mathbf{0} \leq \delta \leq \gamma \in \mathcal{R}L$ with γ as a nonzero divisor. From $\mathbf{0} \leq \delta \leq \gamma$, it follows that $\delta + \delta\gamma \leq \gamma + \delta\gamma$, that is,

$$\delta(\mathbf{1} + \gamma) \leq \gamma(\mathbf{1} + \delta). \tag{4.3}$$

Taking an invertible element $\beta \geq \mathbf{0}$ of $\mathcal{R}L$, then $\beta = \zeta^2$ for some $\zeta \in \mathcal{R}L$. As such $\beta^{-1} = (\zeta^{-1})^2$, which shows that the inverse of β is nonnegative. We then multiply both sides of Equation 4.3 above with the nonnegative element $(\mathbf{1} + \delta)^{-1}(\mathbf{1} + \gamma)^{-1}$ which gives

$$\mathbf{0} \leq \frac{\delta}{\mathbf{1} + \delta} \leq \frac{\gamma}{\mathbf{1} + \gamma}. \tag{4.4}$$

But now both $\frac{\delta}{\mathbf{1}+\delta}$ and $\frac{\gamma}{\mathbf{1}+\gamma}$ are in $\mathcal{R}^*L$ since, for any $\beta \geq \mathbf{0} \in \mathcal{R}L$, $\mathbf{0} \leq \frac{\beta}{\mathbf{1}+\beta} \leq \mathbf{1}$. Note that $\frac{\gamma}{\mathbf{1}+\gamma}$ is nonzero divisor in $\mathcal{R}^*L$, otherwise γ will be a zero divisor. This means that there exists $\alpha \in \mathcal{R}^*L$ such that

$$\frac{\delta}{\mathbf{1} + \delta} = \alpha\frac{\gamma}{\mathbf{1} + \gamma}$$

when $\delta = \beta\gamma$ for some $\beta \in \mathcal{R}L$ since $\mathcal{R}^*L$ is quasi-Bézout.

Conversely, to show that $\mathcal{R}^*L$ is quasi-Bézout, assume that $\mathcal{R}L$ is quasi-Bézout and let $\mathbf{0} \leq \delta \leq \gamma \in \mathcal{R}^*L$ where γ is not a zerodivisor in $\mathcal{R}^*L$. From our assumption that $\mathcal{R}L$ is quasi-Bézout and because $\mathbf{0} \leq \delta \leq \gamma \in \mathcal{R}L$, then there exists $\beta \in \mathcal{R}L$ which gives that $\delta = \beta\gamma$. Note that we may assume that $\beta \geq \mathbf{0}$ since

$$\delta = |\delta| = |\beta\gamma| = |\beta||\gamma| = |\beta|\gamma.$$

Therefore, $\mathbf{0} \leq \beta \wedge \mathbf{1} \leq \mathbf{1}$ meaning $\beta \wedge \mathbf{1}$ is an element of $\mathcal{R}^*L$ where

$$(\beta \wedge \mathbf{1})\gamma = \beta\gamma \wedge \gamma = \delta \wedge \gamma = \delta.$$

Since $\mathcal{R}^*L$ is quasi-Bézout then $R(\beta L)$ is quasi-Bézout and L is quasi-F since βL is quasi-F by spatiality. $\square$

Chapter 5

On quotient rings of point-free function rings

Introduction

Let L be a completely regular frame, and $\mathcal{R}L$ be the ring of real-valued continuous functions on L. As shown in [25], associated with each $I \in \beta L$ are two ideals, which are $\boldsymbol{M}^I$ and $\boldsymbol{O}^I$, of the ring $\mathcal{R}L$ given by

$$\boldsymbol{M}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\} \quad \text{and} \quad \boldsymbol{O}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha)^* \vee I = 1_{\beta L}\},$$

where $r_L \colon L \to \beta L$ is the localic map that embeds L as a sublocale of βL, and the asterisk denotes the pseudocomplement. Since closed sublocales of any frame are in one-one correspondence with the elements of the frame, we can regard these ideals as being indexed by the closed sublocales of βL (rather than the elements). Using the notation that, for any element a in a frame, $\mathfrak{c}(a)$ denotes the closed sublocale associated with a, we write $\boldsymbol{M}^{\mathfrak{c}(I)}$ and $\boldsymbol{O}^{\mathfrak{c}(I)}$ in place of $\boldsymbol{M}^I$ and $\boldsymbol{O}^I$, respectively.

Since

$$r_L(\operatorname{coz}\alpha) \subseteq I \iff \mathfrak{c}_{\beta L}(I) \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha)),$$

and

$$r_L(\operatorname{coz}\alpha)^* \vee I = 1_{\beta L} \iff \operatorname{int}_{\beta L} \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha)).$$

We can define for any sublocale S of βL the ideals

$$\boldsymbol{M}^S = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha))\}$$

and

$$\boldsymbol{O}^S = \{\alpha \in \mathcal{R}L \mid S \subseteq \operatorname{int}_{\beta L}\mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha))\} = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*)\}.$$

Viewing L as a sublocale of βL, among the ideals just defined are those that are indexed by sublocales because a sublocale of a sublocale is a sublocale. Our interest is mainly in the characterizations of quotient rings $\mathcal{R}L/\boldsymbol{M}^A$ and $\mathcal{R}L/\boldsymbol{O}^A$ for a closed sublocale A of βL. Recall that if every element of L is complemented, then L is called a *Boolean frame*. We shall denote the collection of all complemented elements of L by BL, which is called the *boolean part of L*. Boolean frames are precisely the complete Boolean algebras.

5.1 The essential maximal ideals of $\mathcal{R}L$

A nonzero ideal I of the ring R is *essential* if it meets every nonzero ideal non-trivially. In any *reduced* ring (that is, a ring with no nonzero nilpotent element), it is known that an ideal is essential if and only if $\operatorname{Ann}(I) = 0$. Recall that the *socle* of a ring R, denoted by $\operatorname{Soc}(R)$, is the sum of minimal ideals. It is also expressible as an intersection of some ideals, namely:

$$\operatorname{Soc}(R) = \bigcap\{E \subseteq R : E \text{ is essential ideal of } R\}.$$

Taking the intersection of the essential maximal ideals of R yields the ideal

$$\operatorname{Soc}_m(R) = \bigcap\{E \subseteq R : E \text{ is essential maximal ideal of } R\},$$

which is studied in detail in [38].

We recall that a sublocale of a frame is said to be *nowhere dense* [62] if it misses the smallest dense sublocale of the frame. For any frame M, denote by $\operatorname{Nd}(M)$ the sublocale

$$\operatorname{Nd}(M) = \bigvee\{S \in \mathcal{S}(M) \mid S \text{ is nowhere dense}\}.$$

Since a sublocale is nowhere dense precisely if it misses the smallest dense sublocale, and since the closure of a nowhere dense sublocale is nowhere dense, it is clear that

$$\operatorname{Nd}(M) = M \smallsetminus \mathfrak{B}M = \bigvee\{\mathfrak{c}_M(x) \mid x \text{ is a dense element of } M\}.$$

The essential ideals of the ring $\mathcal{R}L$ are studied in [35] and they are characterized as follows.

Lemma 5.1.1. *The following are equivalent for a sublocale A of βL.*

1. $\boldsymbol{O}^A$ *is essential.*
2. $\boldsymbol{M}^A$ *is essential.*
3. A *is nowhere dense.*

It is also shown in [35, Theorem 4.9] that

$$\mathrm{Soc}(\mathcal{R}L) = \boldsymbol{O}^{\mathrm{Nd}(\beta L)}.$$

The socle of the essential maximal ideals has not been shown. Since we are interested in quotient rings $\mathcal{R}L/\boldsymbol{O}^A$ and $\mathcal{R}L/\boldsymbol{M}^A$, we denote by $\overline{I}$, the ideal $(I + \boldsymbol{O}^A)/\boldsymbol{O}^A$ (resp., $(I + \boldsymbol{M}^A)/\boldsymbol{M}^A$) in $\mathcal{R}L/\boldsymbol{O}^A$ (resp., $\mathcal{R}L/\boldsymbol{M}^A$), where I is an ideal of $\mathcal{R}L$. Also, for any $\alpha \in \mathcal{R}L$, we mean $\overline{\alpha} \in \mathcal{R}L/\boldsymbol{O}^A$ (resp., $\mathcal{R}L/\boldsymbol{M}^A$), is $\alpha + \boldsymbol{O}^A$ (resp., $\alpha + \boldsymbol{M}^A$). For use in the proof below and elsewhere, consider any essential ideal P of the ring $\mathcal{R}L$, and denote by $\theta(P)$ the closed sublocale of βL given by

$$\theta(P) = \mathfrak{c}_{\beta L}\Big(\bigvee\{r_L(\operatorname{coz}\alpha) \mid \alpha \in P\}\Big).$$

We can now describe the socle of the essential maximal ideals of the ring $\mathcal{R}L$.

Theorem 5.1.1. $\mathrm{Soc}_m(\mathcal{R}L) = \boldsymbol{M}^{\mathrm{Nd}(\beta L)}$.

Proof. To show that $\mathrm{Soc}_m(\mathcal{R}L) \subseteq \boldsymbol{M}^{\mathrm{Nd}(\beta L)}$, consider $\{A_\lambda \mid \lambda \in \Lambda\}$ to be the set of all nowhere dense sublocales of βL, such that $\mathrm{Nd}(\beta L) = \bigvee_\lambda A_\lambda$. It follows from Lemma 5.1.1 above that for each λ, $\boldsymbol{M}^{A_\lambda}$ is an essential maximal ideal and in light of $\mathrm{Soc}_m(\mathcal{R}L)$ being the intersection of all essential maximal ideals,

$$\mathrm{Soc}_m(\mathcal{R}L) \subseteq \bigcap_\lambda \boldsymbol{M}^{A_\lambda} = \boldsymbol{M}^{\bigvee A_\lambda} = \boldsymbol{M}^{\mathrm{Nd}(\beta L)}.$$

To show the other containment, consider any essential maximal ideal I of $\mathcal{R}L$. As shown in [25, Proposition 5.2], $\boldsymbol{O}^{\theta(I)} \subseteq I \subseteq \boldsymbol{M}^{\theta(I)}$. By maximality of I and hence prime, it turns out that $I = \boldsymbol{M}^{\theta(I)}$ is a unique essential maximal ideal (see [25]). By Lemma 5.1.1, there is an index λ_0 such that $I = \boldsymbol{M}^{A_{\lambda_0}}$. Since the ideal $\mathrm{Soc}_m(\mathcal{R}L)$ is the intersection of all essential maximal ideals, it follows that $\bigcap_\lambda \boldsymbol{M}^{A_\lambda} \subseteq \mathrm{Soc}_m(\mathcal{R}L)$. This concludes the proof. □

Notice that the ideal $\boldsymbol{M}^{\mathrm{Nd}(\beta L)}$ is essential precisely when $\boldsymbol{O}^{\mathrm{Nd}(\beta L)}$ is essential in $\mathcal{R}L$. This is so because $\boldsymbol{O}^{\mathrm{Nd}(\beta L)}$ is always contained in $\boldsymbol{M}^{\mathrm{Nd}(\beta L)}$. Combining this with Lemma 5.1.1, this gives rise to the following characterization.

Proposition 5.1.2. *The ideal* $\mathrm{Soc}_m(\mathcal{R}L)$ *is essential if and only if* $\mathrm{Soc}(\mathcal{R}L)$ *is an essential ideal.*

5.2 Characterizations of quotient rings of $\mathcal{R}L$

To prepare for our main results, we recall the following. For a ring A, $\mathrm{Max}(A)$ is the space of maximal ideals of A, endowed with the hull-kernel topology for which the closed subsets are $\mathfrak{M}(I) = \{M \in \mathrm{Max}(A) \,|\, I \subseteq M\}$ for an ideal I in A. We remind the reader that

$$\mathrm{int}_{\beta L}(\theta(I)) = \bigvee\{\mathfrak{o}_{\beta L}(a) \mid \mathfrak{o}_{\beta L}(a) \subseteq \theta(I)\}.$$

For a closed sublocale A of βL, $A = \mathrm{Max}(\mathcal{R}L/\boldsymbol{M}^A)$ and for an ideal I of $\mathcal{R}L$, $\mathfrak{M}(I) = \theta(I)$. The author in [65, Lemma 3.1] shows that an ideal I of a reduced ring R is essential if and only if $\mathrm{int}\,\mathfrak{M}(I) = \emptyset$. In the ring $\mathcal{R}L$, it is characterized in [27, Lemma 3.1] as follows. If I and J are ideals of $\mathcal{R}L$ with $I \subseteq J$, then

$$I \text{ is essential in } J \iff \Big(\bigvee\{\mathrm{coz}\,\alpha \mid \alpha \in I\}\Big)^* \leq \Big(\bigvee\{\mathrm{coz}\,\gamma \mid \gamma \in J\}\Big)^*.$$

The ring $\mathcal{R}L$ is reduced and it has *bounded inversion* property, which means that every $\alpha \geq 1$ is invertible. Recall that closed sublocales are complemented. A complemented sublocale is nowhere dense if it has empty interior [62].

Lemma 5.2.1. *Let A be a closed sublocale of βL, and let I and J be essential ideals of $\mathcal{R}L$ that contains the ideal $\boldsymbol{M}^A$. Then* $\mathrm{Ann}(I/\boldsymbol{M}^A) = \mathrm{Ann}(J/\boldsymbol{M}^A)$ *if and only if* $\mathrm{int}_A(\theta(I)) = \mathrm{int}_A(\theta(J))$.

Corollary 5.2.2. *Let A be a closed sublocale of βL, and I be an ideal of $\mathcal{R}L$ containing $\boldsymbol{M}^A$. Then $I/\boldsymbol{M}^A$ is an essential ideal in $\mathcal{R}L/\boldsymbol{M}^A$ if and only if* $\mathrm{int}_A(\theta(I)) = \emptyset$.

Proof. It is evident by Lemma 5.2.1 and that an ideal I in a reduced ring is essential precisely when $\mathrm{Ann}(I) = 0$. $\square$

As in [61], the frame $\mathcal{S}_{\mathfrak{c}}(L)$ is defined by

$$\mathcal{S}_{\mathfrak{c}}(L) = \{S \in \mathcal{S}(L) \mid S \text{ is a join of closed sublocales of } L\}.$$

A locale (frame) L is said to be *subfit* if for every $a, b \in L$ such that $a \not\leq b$, there exists $c \in L$ such that $a \vee c = 1 \neq b \vee c$ and *fit* if for every $a, b \in L$ such that $a \not\leq b$, there exists $c \in L$ such that $a \vee c = 1$ and $a \to b \not\leq b$. A locale is called *scattered* if every nonzero closed sublocale contains a nonzero open Boolean sublocale [62]. It has been shown in [61, Chapter IX, Theorem 2.4.1] that every sublocale of L is a join of closed sublocales if and only if L is scattered and subfit, if and only if L is scattered and fit.

As in [40, Definition 3.8], we call a locale *perfect* if each open sublocale is a join of countable many closed sublocales. Perfect locales are also called *F_σ-perfect*. We call a locale *G_σ-perfect* if each closed sublocale is an intersection of countable many open sublocales. Each G_σ-perfect locale is F_σ-perfect [40, Remark 3.2].

When we say two disjoint sublocales A and B of a locale L *can be separated* by open sublocales, we mean that there are open sublocales U and V of L such that $U \cap V = \mathsf{O}$, $A \subseteq U$, and $B \subseteq V$. A locale is *normal* if any disjoint closed sublocales of L can be separated by open sublocales. In terms of elements, recall that a locale L is normal whenever $a \vee b = 1$ implies that $a \vee u = 1 = b \vee v$ for some $u, v \in L$ satisfying $u \wedge v = 0$. It is shown in [41, Proposition 3.5] that the classes of F_σ-perfect locales and G_σ-perfect locales coincide under normality. For the purpose of the proof of the theorem below and elsewhere, recall from [34, Proposition 3.4] that for any sublocale A of βL, $\theta(\boldsymbol{O}^A) = \theta(\boldsymbol{O}^{\overline{A}}) = \theta(\boldsymbol{M}^{\overline{A}}) = \overline{A}$.

Theorem 5.2.1. *Let $A \subseteq \beta L$ be a closed sublocale. The following are equivalent.*

1. *A is perfect.*

2. *Every prime ideal of $\mathcal{R}L/\boldsymbol{M}^A$ is essential.*

3. *Every maximal ideal of $\mathcal{R}L/\boldsymbol{M}^A$ is essential.*

Proof. (1)⇒(2): Suppose that A is a perfect sublocale and let $P/\boldsymbol{M}^A$ be a prime ideal of $\mathcal{R}L/\boldsymbol{M}^A$. Since P is prime, it turns out that there is a unique prime element J such that $\boldsymbol{O}^J \subseteq P \subseteq \boldsymbol{M}^J$ (see the proof of [25, Proposition 5.4]). Since A is perfect, it follows that each open sublocale of A is a join of countably many closed sublocales, which implies that $\mathrm{int}_A(\theta(P)) = \mathrm{int}_A(J) = \emptyset$. By Corollary 5.2.2, it follows that $P/\boldsymbol{M}^A$ is an essential ideal.

(2)$\Rightarrow$(3): This is obvious.

(3)$\Rightarrow$(1): Suppose that every maximal ideal of $\mathcal{R}L/\boldsymbol{M}^A$ is essential and let $\mathfrak{p}$ be a prime element in A. Then $\boldsymbol{M}^{\mathfrak{p}}/\boldsymbol{M}^A$ is a maximal ideal of $\mathcal{R}L/\boldsymbol{M}^A$ and by hypothesis, it is an essential ideal. So by Corollary 5.2.2, $\mathrm{int}_A(\theta(\boldsymbol{M}^{\mathfrak{p}})) = \mathrm{int}(\mathfrak{p})$ has an empty interior which implies that $\mathfrak{p}$ is nowhere dense in A. Since $\mathfrak{p}$ is nowhere dense, it means that it misses the smallest dense sublocale of βL, that is $\mathfrak{p} \cap \mathfrak{B}(\beta L) = \mathsf{O}$. By [60, Chapter V, Proposition 1.4], it follows that every open sublocale is the join of the closed sublocales which includes the join of the countable closed sublocales. By this argument, it follows that $\mathfrak{p}$ is perfect. Thus A is a perfect sublocale. $\square$

Corollary 5.2.3. *The following statements are equivalent.*

1. $\mathrm{Nd}(\beta L)$ *is perfect.*
2. *Every maximal ideal of* $\mathcal{R}L/\operatorname{Soc}_m(\mathcal{R}L)$ *is essential.*
3. *Every prime ideal of* $\mathcal{R}L/\operatorname{Soc}_m(\mathcal{R}L)$ *is essential.*

The notion of a round quotient of the Stone-Čech compactification of a frame was defined in [25] as follows. A quotient map $h\colon \beta L \to M$ is *round* if $\boldsymbol{M}^h = \boldsymbol{O}^h$. Recall from [31] that a point I of βL is *sharp* if, for any $c \in \operatorname{Coz} L$, $r_L(c) \leq I$ implies $c \in I$. It is clear from [31] that a point $\mathfrak{p} \in \mathrm{Pt}(\beta L)$ is sharp if and only if $\boldsymbol{M}^{\mathfrak{p}} = \boldsymbol{O}^{\mathfrak{p}}$. We denote by $\mathrm{Sh}(L)$ the set of sharp points of L. A frame L is a *P-frame* if every cozero element is complemented. We leave it to the interested reader to check the following; see [25, Proposition 4.17] and [14, Proposition 3.9] if needed.

Lemma 5.2.4. *For a completely regular frame L, the following statements are equivalent.*

1. *L is a P-frame.*
2. *Every quotient map $\beta L \to M$ is round.*
3. *Every point of* $\mathrm{Pt}(\beta L)$ *is sharp.*

Proposition 5.2.5. *Let A be a closed sublocale of βL. The following statements hold.*

1. *For any $\mathfrak{p} \in A$, the maximal ideal $\boldsymbol{M}^{\mathfrak{p}}/\boldsymbol{O}^A$ is essential in $\mathcal{R}L/\boldsymbol{O}^A$ if and only if $\mathfrak{p} \notin \mathfrak{B}(A) \cap \mathrm{Sh}(\beta L)$.*

2. *Every maximal ideal of $\mathcal{R}L/\boldsymbol{O}^A$ is essential if and only if A does not contain a sharp point which is at the same time a supplement sublocale.*

Proof. (1) Suppose that the maximal ideal $\boldsymbol{M}^{\mathfrak{p}}/\boldsymbol{O}^A$ is an essential ideal and $\mathfrak{p} \in \mathfrak{B}(A) \cap \mathrm{Sh}(\beta L)$. This implies that $\boldsymbol{M}^{\mathfrak{p}}/\boldsymbol{O}^A = \boldsymbol{O}^{\mathfrak{p}}/\boldsymbol{O}^A$. The point $\mathfrak{p}$ has a supplement sublocale, which means that $\mathfrak{p} \vee \mathfrak{p}^{\#} = \beta L$. This implies that $\boldsymbol{O}^{\mathfrak{p}}/\boldsymbol{O}^A \cap \boldsymbol{O}^{\mathfrak{p}^{\#}}/\boldsymbol{O}^A = \{\mathbf{0}\}$. This says that $\boldsymbol{M}^{\mathfrak{p}}/\boldsymbol{O}^A$ is not essential maximal ideal, which is a contradiction.

(2) The proof of (2) follows from (1) above. $\square$

Corollary 5.2.6. *The sublocale* $\mathrm{Nd}(\beta L)$ *does not contain a sharp point which is at the same time a supplement sublocale.*

We remind the reader of the following. The frame βL is the lattice of all completely regular ideals of $\mathrm{Coz}\, L$. The mapping

$$j_L \colon \beta L \to L \quad \text{defined by} \quad j_L(I) = \bigvee I$$

is dense onto, and is the coreflection map to L from compact completely regular frames. Its right adjoint $r_L \colon L \to \beta L$ is given by

$$r_L(a) = \{c \in \mathrm{Coz}\, L \mid c \prec\!\!\prec a\}.$$

A *nucleus* is a mapping $\nu \colon L \to L$ satisfying (i) $a \leq \nu(a)$, (ii) $a \leq b$ implies $\nu(a) \leq \nu(b)$, (iii) $\nu\nu(a) = \nu(a)$, and (iv) $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$ for all $a, b \in L$. The set $\mathrm{Fix}(\nu) = \{a \in L \mid \nu(a) = a\}$ is a locale with meets in L. Every sublocale $S \subseteq L$ is associated with the quotient map $\nu_S \colon L \to S$ defined by

$$\nu_S(a) = \bigwedge \{s \in S \mid a \leq s\}.$$

For each $a \in L$ and $S \in \mathcal{S}(L)$, $\mathfrak{o}_S(\nu_S(a)) = S \cap \mathfrak{o}_L(a)$ and $\mathfrak{c}_S(\nu_S(a)) = S \cap \mathfrak{c}_L(a)$.

Recall that if A is a sublocale of L and $a \in A$, then for the open sublocale $\mathfrak{o}_A(a)$ of A, we have

$$\mathfrak{o}_A(a) = A \cap \mathfrak{o}_L(a)$$

and similarly for the closed sublocale because the associated quotient map $\nu_A\colon L \to A$ fixes elements of A. In the upcoming proof of the next lemma, and elsewhere, we shall use the following property of sublocales that, for each $a, b \in L$ and $S \in \mathcal{S}(L)$,

$$S \subseteq \mathfrak{o}(u) \subseteq \mathfrak{c}(a) \vee \mathfrak{c}(b) \qquad \text{implies} \qquad S \subseteq \mathfrak{o}(a^*) \vee \mathfrak{c}(b).$$

Lemma 5.2.7. *Let $\bar{e}$ be an idempotent of $\mathcal{R}L/\boldsymbol{O}^A$. Then $A \cap \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*) = A \cap \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e))$.*

Proof. It is known that $A \cap \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*) \subseteq A \cap \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e))$. We only need to show the converse. To do that, we have

$$A \subseteq \mathfrak{o}_{\beta L}\Big(\mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e)) \vee \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}(1-e)))\Big).$$

Hence there exists an open sublocale U such that

$$A \subseteq U \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e)) \vee \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}(1-e))).$$

This implies that

$$A \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*) \vee \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}(1-e)))$$

and similarly

$$A \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e)) \vee \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}(1-e))^*).$$

Thus

$$A \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*) \vee \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}(1-e))^*)$$

for

$$\mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e)) \cap \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}(1-e))) = 0.$$

Now

$$A \cap \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e)) \subseteq A \smallsetminus (A \cap \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}(1-e))) \subseteq A \cap \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*).$$

This concludes the proof. □

Recall that a sublocale is *clopen* if it is both closed and open.

Lemma 5.2.8. *Let A be a closed sublocale of βL. The following are equivalent.*

1. *$\mathfrak{p}$ is a clopen sublocale of A.*

2. $\mathfrak{p} = \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*) \cap A$ *for some idempotent* $\overline{e} \in \mathcal{R}L/\boldsymbol{O}^A$.

3. $\mathfrak{p} = \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} k)) \cap A$ *for some idempotent* $\overline{k} \in \mathcal{R}L/\boldsymbol{M}^A$.

Proof. (1)⇒(2): Let $\mathfrak{p}$ be a clopen sublocale in A. Then $\mathfrak{p}$ and $A \smallsetminus \mathfrak{p}$ are two disjoint clopen sublocales in A, and hence they are two disjoint closed sublocales in βL. There are $\alpha, \beta \in \mathcal{R}^*L$ such that

$$\mathfrak{p} \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\alpha)^*) \text{ and } A \smallsetminus \mathfrak{p} \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\beta)^*)$$

and

$$r_L(\operatorname{coz}\alpha)^* \cap r_L(\operatorname{coz}\beta)^* = 0.$$

Now define $e = \frac{1}{1+|e|}$ which is an element of $\mathcal{R}^*L$, hence an element of $\mathcal{R}L$. Then $r_L(\operatorname{coz}\beta)^* \leq r_L(\operatorname{coz}(1-e))^*$ and $r_L(\operatorname{coz}\alpha)^* \leq r_L(\operatorname{coz} e)^*$. Hence

$$\mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\beta)) \leq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}(1-e))) \text{ and } \mathfrak{c}_{\beta L}(r_L(\operatorname{coz}\alpha)) \leq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e)).$$

By these, we have

$$A \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*) \vee \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}(1-e))^*) \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}(e(1-e)))^*),$$

that is $\overline{e} \in \mathcal{R}L/\boldsymbol{O}^A$ is an idempotent.

To show the converse, we only need to show that $A \cap \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*) \subseteq \mathfrak{p}$ since it is known that $\mathfrak{p} \subseteq A \cap \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*)$. If $q \in \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*) \cap A$, then $q \in A \smallsetminus (\mathfrak{o}_{\beta L}(r_L(\operatorname{coz}(1-e))^*) \cap A)$ and so $q \notin A \smallsetminus \mathfrak{p}$ which implies $q \in \mathfrak{p}$. Thus $\mathfrak{p} = \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*) \cap A$.

(2)⇒(3): If $\overline{e}$ is an idempotent in $\mathcal{R}L/\boldsymbol{O}^A$ and $\mathfrak{p} = \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e)^*) \cap A$, then by Lemma 5.2.7, $\mathfrak{p} = \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e)) \cap A$ and so by Lemma 5.2.8, $\overline{e}$ is an idempotent of $\mathcal{R}L/\boldsymbol{M}^A$.

(3)⇒(1): $\mathfrak{p} = \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} k)) \cap A = A \smallsetminus \Big(\mathfrak{c}_{\beta L}(r_L(\operatorname{coz}(1-k))) \cap A\Big)$ is clopen sublocale in A for some idempotent $\overline{k} \in \mathcal{R}L/\boldsymbol{M}^A$. □

Recall from [60, Chapter XIII] that a locale L is *connected* if $BL = \{0, 1\}$. Thus a connected locale is supposed to be non-trivial. The following corollary is the consequence of the foregoing lemmas.

Corollary 5.2.9. *For a closed sublocale A of βL, the following are equivalent.*

1. $\mathcal{R}L/\boldsymbol{M}^A$ has no non-trivial idempotents.

2. The sublocale A is connected.

3. $\mathcal{R}L/\boldsymbol{O}^A$ has no non-trivial idempotents.

Recall that two sublocales S and T of a locale L are said to be *completely separated* [8] if they have disjoint zero-set sublocale neighbourhoods. For any $a, b \in L$, the sublocales $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ are completely separated if and only if there exist $c, d \in \operatorname{Coz} L$ such that $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$ [28]. Thus, if the closed sublocales $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ are completely separated, then there exists $d \in \operatorname{Coz} L$ such that $d \prec a$ and $d \vee b = 1$. Indeed, if c and d are cozero elements with $c \wedge d = 0$ and a $\vee c = 1 = b \vee d$, then $d \prec a$ (with c as a witness for this) and $d \vee b = 1$ [28]. A topological space X is called an *EF-space* if for any collections $\mathcal{U}$ and $\mathcal{V}$ of clopen subsets of X such that $\bigcup \mathcal{U} \cap \bigcup \mathcal{V} = \emptyset$, then $\bigcup \mathcal{U}$ and $\bigcup \mathcal{V}$ are completely separated [66]. These notions are extended to frames (see [58] for more details). A frame L is an *EF-frame* if for any collections $U = \{u_i \mid i \in I\}$ and $V = \{v_j \mid j \in J\}$ in BL such that $\bigvee U \wedge \bigvee V = 0$, the sublocales $\mathfrak{o}(\bigvee U)$ and $\mathfrak{o}(\bigvee V)$ are completely separated [58]. It is straightforward to check that

- $U, V \subseteq BL$ is equivalent to saying that $\mathcal{U} := \{\mathfrak{o}(u_i) \mid i \in I\}$ and $\mathcal{V} := \{\mathfrak{o}(v_j) \mid j \in J\}$ are collections of clopen sublocales of L, and
- $\bigvee U \wedge \bigvee V = 0$ means precisely that $\bigvee \mathcal{U} \cap \bigvee \mathcal{V} = \mathsf{O}$ in the lattice of sublocales.

We conclude that a frame L is an EF-frame precisely when for any collections $\mathcal{U}$ and $\mathcal{V}$ of clopen sublocales of L such that $\bigvee \mathcal{U} \cap \bigvee \mathcal{V} = \mathsf{O}$, then $\bigvee \mathcal{U}$ and $\bigvee \mathcal{V}$ are completely separated.

The concept of Essential Ikeda-Nakayama rings (abbreviated EIN-rings) was introduced and studied in [12]. We call a ring R an *EIN-ring* if for each two orthogonal ideals I and J of R which are generated by two subsets of idempotents, $\operatorname{Ann}(I) + \operatorname{Ann}(J) = R$. Every reduced ring is an EIN-ring [12, Theorem 2.2]. Recall that for any two orthogonal ideals I and J of a reduced ring R, $\operatorname{Ann}(I) + \operatorname{Ann}(J) = R$ if and only if for any two ideals I and J of R, $\operatorname{Ann}(I) + \operatorname{Ann}(J) = \operatorname{Ann}(I \wedge J)$ (see [58, Lemma 3.9]). A completely regular frame L is an *EF-frame* if and only if for any two ideals I and J of $\mathcal{R}L$ which are generated by two idempotent elements of $\mathcal{R}L$, $\operatorname{Ann}(I) + \operatorname{Ann}(J) = \operatorname{Ann}(I \wedge J)$ (see [58, Theorem 3.11]). Recall from the preliminary section that every $\mathcal{R}L$ is a reduced f-ring with bounded inversion. It is easy to see that the ring $\mathcal{R}L$ is an EIN-ring if and only if L is an EF-frame. For the purpose of the proof of the next result we recall the following. It has been shown in [58, Proposition 3.2] that a normal frame L is an EF-frame if and only if for any two collections U and V in BL such that $\bigvee U \wedge \bigvee V = 0$, then we have $\mathfrak{o}\big(\bigvee U\big) \vee \mathfrak{o}\big(\bigvee V\big) = 1$. In the next theorem, when we say that "the sublocale A is an EF-frame" we mean that A, as a frame in itself, belongs to the class of EF-frames, and similarly when we say that a sublocale of a frame is normal when it is normal as a frame in itself.

Theorem 5.2.2. *Let A be a closed sublocale of βL. The following statements are equivalent.*

1. *$\mathcal{R}L/\boldsymbol{O}^A$ is an EIN-ring.*

2. *The sublocale A is an EF-frame.*

3. *$\mathcal{R}L/\boldsymbol{M}^A$ is an EIN-ring.*

Proof. (1)⇒(2): Suppose that (1) is true. Consider $\{A_\alpha \mid \alpha \in S\}$ and $\{A_\beta \mid \beta \in K\}$ to be clopen sublocales in A such that $\bigvee A_\alpha \cap \bigvee A_\beta = \mathsf{O}$. By Lemma 5.2.8, there are idempotents $\overline{e}_\alpha$ and $\overline{e}_\beta$ in $\mathcal{R}L/\boldsymbol{O}^A$ such that $A_\alpha = A \smallsetminus \big(\mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e_\alpha)^*) \cap A\big)$ and $A_\beta = A \smallsetminus \big(\mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e_\beta)^*) \cap A\big)$. Then

$$\Big(\bigvee_{\alpha\in S} A \smallsetminus (\mathfrak{o}_{\beta L}(r_L \operatorname{coz} e_\alpha)^* \cap A)\Big) \cap \Big(\bigvee_{\beta\in K} A \smallsetminus (\mathfrak{o}_{\beta L}(r_L \operatorname{coz} e_\beta)^* \cap A)\Big) = \mathsf{O}.$$

This implies that

$$A \subseteq \Big(\bigwedge \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e_\alpha)^*)\Big) \vee \Big(\bigwedge(\mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e_\beta)^*)\Big). \tag{†}$$

Suppose that $I = \langle e_\alpha \mid \alpha \in S\rangle$ and $J = \langle e_\beta \mid \beta \in K\rangle$. It follows from † that $IJ \subseteq \boldsymbol{O}^A$ which implies that $\overline{IJ} = \overline{0}$. Since $\mathcal{R}L/\boldsymbol{O}^A$ is an EIN-ring, then $\operatorname{Ann}(\overline{I}) + \operatorname{Ann}(\overline{J}) = \mathcal{R}L/\boldsymbol{O}^A$. There are elements $\overline{\delta} \in \operatorname{Ann}(\overline{I})$ and $\overline{\eta} \in \operatorname{Ann}(\overline{J})$ such that $\overline{1} = \overline{\delta} + \overline{\eta}$. That is $1 - (\delta + \eta) \in \boldsymbol{O}^A$. We have

$$A \smallsetminus (\mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e_\alpha)^*)) \cap A) \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} \delta))$$

and

$$A \smallsetminus (\mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e_\beta)^*)) \cap A) \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} \eta)),$$

and there is sublocale, say $\mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e_\gamma)^*)$, in A such that $\mathfrak{o}_{\beta L}(r_L(\operatorname{coz} e_\gamma)^*) \cap \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}(\delta + \eta)^*)) = 0$. This means that

$$\bigvee_{\alpha \in S} A_\alpha \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} \delta)) \cap A,$$

$$\bigvee_{\beta \in K} A_\beta \subseteq \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} \eta)) \cap A$$

and

$$\big(\mathfrak{c}_{\beta L}(r_L(\operatorname{coz} \delta)) \cap A\big) \cap \big(\mathfrak{c}_{\beta L}(r_L(\operatorname{coz} \eta)) \cap A\big) = \mathsf{O}.$$

By normality of A, these two closed sublocales are completely separated in A since in a normal locale, disjoint closed sublocales are completely separated. Hence $\bigvee A_\alpha$ and $\bigvee A_\beta$ are completely separated.

(2)⇒(1): Suppose that $\overline{I}$ and $\overline{J}$ are orthogonal ideals (that is, $\overline{IJ} = \overline{0}$) of $\mathcal{R}L/\boldsymbol{O}^A$ generated by two idempotent elements $\{\overline{e_\alpha} \mid \alpha \in S\}$ and $\{\overline{e_\beta} \mid \beta \in K\}$ of $\mathcal{R}L/\boldsymbol{O}^A$. We claim that $\operatorname{Ann}(\overline{I}) + \operatorname{Ann}(\overline{J}) = \mathcal{R}L/\boldsymbol{O}^A$. Since $\overline{IJ} = \overline{0}$, it implies that $IJ \subseteq \boldsymbol{O}^A$. This means that $A \subseteq \theta(IJ) \subseteq \theta(I) \vee \theta(J)$. Thus

$$(A \smallsetminus \theta(I)) \cap (A \smallsetminus \theta(J)) = \left(\bigvee A_\alpha \smallsetminus \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e_\alpha)) \cap A\right) \cap \left(\bigvee A_\beta \smallsetminus \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e_\beta)) \cap A\right) = \mathsf{O}.$$

So we have two disjoint sublocales in A, by Lemma 5.2.8. Since A is an EF-frame, there are two disjoint clopen sublocales, say, $\mathfrak{o}_{\beta L}(r_L(\operatorname{coz} \eta)^*)$ and $\mathfrak{o}_{\beta L}(r_L(\operatorname{coz} \delta)^*)$ in A such that

$$\left(\bigvee_{\alpha \in S} A_\alpha \smallsetminus \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e_\alpha)) \cap A\right) \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} \eta)^*)$$

and

$$\left(\bigvee_{\beta \in K} A_\beta \smallsetminus \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e_\beta)) \cap A\right) \subseteq \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} \delta)^*).$$

We therefore have

$$A \smallsetminus \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} \eta)^*) \subseteq \bigwedge_{\alpha \in S} (\mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e_\alpha)) \cap A),$$

and

$$A \smallsetminus \mathfrak{o}_{\beta L}(r_L(\operatorname{coz} \delta)^*) \subseteq \bigwedge_{\beta \in K} (\mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e_\beta)) \cap A).$$

Therefore $\eta i \in \boldsymbol{O}^A$ and $\delta j \in \boldsymbol{O}^A$ for every $i \in I$ and $j \in J$. Thus $\overline{\eta}^2 + \overline{\delta}^2$ is invertible in $\mathrm{Ann}(\overline{I}) + \mathrm{Ann}(\overline{J})$ since $A \cap \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\eta)^*) \cap \mathfrak{o}_{\beta L}(r_L(\operatorname{coz}\delta)^*) = \mathsf{O}$. We have shown that $\mathrm{Ann}(\overline{I}) + \mathrm{Ann}(\overline{J}) = \mathcal{R}L/\boldsymbol{O}^A$.

(3)$\Leftrightarrow$(2): The proof of this is similar to (1)$\Leftrightarrow$(2) above. $\square$

Corollary 5.2.10. *The following are equivalent.*

1. $\mathcal{R}L/Soc(\mathcal{R}L)$ *is an* EIN*-ring.*

2. $\mathrm{Nd}(\beta L)$ *is an* EF*-frame.*

3. $\mathcal{R}L/Soc_m(\mathcal{R}L)$ *is an* EIN*-ring.*

A ring R is said to be *Baer* if for every $S \subseteq R$, $\mathrm{Ann}(S)$ is generated by an idempotent. We recall from [49] that a locale is said to be *extremally disconnected* if the second De Morgan law is satisfied in L, that is, if

$$(a \wedge b)^* = a^* \vee b^* \text{ for every } a, b \in L,$$

or equivalently if the relation

$$a^* \vee a^{**} = 1 \text{ for every } a \in L$$

holds in L. It is easy to show that a locale L is extremally disconnected if and only if the closure of any open sublocale is open.

Theorem 5.2.3. *For a closed sublocale A of βL, $\mathcal{R}L/\boldsymbol{M}^A$ is a Baer ring if and only if A is extremally disconnected.*

Proof. Suppose $\mathcal{R}L/\boldsymbol{M}^A$ is a Baer ring and S is a sublocale of A. Then S is closed in βL. We know from [37] that for each sublocale S of a locale βL, the closure is denoted as $\mathrm{cl}_L S = \bigcap \big\{ \mathfrak{c}(a) \mid S \subseteq \mathfrak{c}(a) \big\}$ and the interior is denoted as $\mathrm{int}_L S = \bigvee \big\{ \mathfrak{o}(a) \mid \mathfrak{o}(a) \subseteq S \big\}$. Set $J = \langle a_i \mid a_i \in S \rangle$. Then there is an idempotent element $\overline{e}$ in $\mathcal{R}L/\boldsymbol{M}^A$ such that $\mathrm{Ann}(\overline{J}) = \mathrm{Ann}(\langle \overline{e} \rangle)$. Now, Lemma

5.2.1 implies that

$$\begin{aligned}\operatorname{int}_A S &= \operatorname{int}_A\left(\left(\bigcap_i \mathfrak{c}_{\beta L}(a_i)\right)\cap A\right)\\ &= \operatorname{int}_A\left(\mathfrak{c}_{\beta L}\left(\bigvee_i a_i\right)\cap A\right)\\ &= \operatorname{int}_A(\theta(J)\cap A)\\ &= \operatorname{int}_A(\mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e))\cap A)\end{aligned}$$

where $\operatorname{int}_A(\mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e))\cap A)$ is clopen in A. Hence, for any open S, $S = \operatorname{int}_A S$ is clopen. Thus A is extremally disconnected.

Now suppose that A is extremally disconnected, which means that the closure of its of each open sublocale is open, and let $\overline{I}$ be an ideal of $\mathcal{R}L/\boldsymbol{M}^A$. Combining the hypothesis and Lemma 5.2.8, there exists an idempotent $\overline{e}$ of $\mathcal{R}L/\boldsymbol{M}^A$ such that

$$\operatorname{int}_A(\theta(I)) = \mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e))\cap A = \operatorname{int}_A(\mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e))\cap A). \tag{$\dagger$}$$

Since

$$\operatorname{int}_A(\theta(I)) = \operatorname{int}_A(\mathfrak{c}_{\beta L}(r_L(\operatorname{coz} e))\cap A)$$

from †, it follows from Lemma 5.2.1 that $\operatorname{Ann}(\overline{I}) = \operatorname{Ann}(\langle\overline{e}\rangle)$. Thus $\operatorname{Ann}(\overline{I})$ is generated by an idempotent, which then follows that $\mathcal{R}L/\boldsymbol{M}^A$ is a Baer ring. This concludes the proof. □

5.3 The classical quotient rings of $\mathcal{R}L$

We know that apart from βL, there are other three coreflective subcategories of **CRFrm** (category of completely regular frames) that we wish to say something about. Without going into definitions and constructions, we mention that they are comprised of (i) the realcompact frames, yielding the realcompactification υL of L; (ii) the Lindelöf frames, yielding the Lindelöfication λL of L; and (iii) the paracompact frames, yielding the paracompactification πL of L. They can be summarized by the commutative diagram below (See [47] for details).

$$\begin{tikzcd}
& & \upsilon L \arrow[dr, "\upsilon_L"] & \\
\beta L \arrow[r, "k_L"] & \lambda L \arrow[ur, "\ell_L"] \arrow[rr, "\lambda_L"] \arrow[dr, "p_L"'] & & L \\
& & \pi L \arrow[ur, "\pi_L"'] &
\end{tikzcd}$$

As in [22], for any maximal ideal M, the field $\mathcal{R}L/M$ always contains a subfield isomorphic to the field of real numbers. The mapping $r \to \mathbf{r}$ is a one–one ring homomorphism $\mathbb{R} \to \mathcal{R}L$. Thus, the map

$$\Psi\colon \mathbb{R} \to \mathcal{R}L/M \text{ given by } \Psi(r) = \mathbf{r} + M$$

is a one–one ring homomorphism. It is one–one because, for any $r \in \mathbb{R}, \Psi(r) = 0$ implies that $\mathbf{r} \in M$, such that $\mathbf{r}$ is not invertible, and hence $r = 0$. Now, as usual, M is *real* if $\mathcal{R}L/M \cong \mathbb{R}$, and *hyper-real* otherwise. An ideal Q of $\mathcal{R}L$ is *free* if $\bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\} = 1$. A completely regular frame L is realcompact if and only if every free maximal ideal of $\mathcal{R}L$ is hyper-real [22, Proposition 4.1]. The (free) maximal ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}^A$ for $A \in \mathrm{Pt}(\beta L)$. They are distinct for distinct points. Following the argument in [22, Section 4], it is clear that the free maximal ideals of $\mathcal{R}L$ are sets $\ker(\mathcal{R}\xi)$ for the characters ξ of L. They are different for different characters. A little reflection from the foregoing argument leads us to the following proposition.

Proposition 5.3.1. *Let $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ be a ring homomorphism, then the kernel of $\mathcal{R}h$ is of the form $\boldsymbol{M}^A$ for some $A \subseteq \upsilon L$. Moreover, every ideal of this form is the kernel of a ring homomorphism.*

Banaschewski [9] defines a function ring $\mathcal{R}L$ (resp. the ring $\mathfrak{Z}L$ of integer-valued continuous functions on a zero-dimensional frame L) to be *classical* if it is isomorphic to some $C(X)$ (resp. the ring $C(X, \mathbb{Z})$ of integer-valued continuous functions on a zero-dimensional Hausdorff space X). A frame L is *realcompact* if every maximal ideal of $\operatorname{Coz} L$ that has join equal to the top has a countable subset whose join is the top. Realcompact completely regular frames form a coreflective subcategory of the category **CRFrm** of completely regular frames. Recall that an element x of a subset S of a topological space X is said to be *isolated* in S if there is an open set U in X such that $U \cap S = \{x\}$. We shall denote the set of isolated points of a space X by $I(X)$.

For the purpose of the proof of the upcoming result, we recall from [26] that $\mathfrak{B}(\beta L) \cong \beta(\mathfrak{B}L)$ if and only if L is finite, and the rings $\mathcal{R}L$ and $\mathcal{R}(\upsilon L)$ are isomorphic as f-rings.

Theorem 5.3.1. *The following are equivalent for a locale L.*

1. $\mathcal{R}L/\operatorname{Soc}(\mathcal{R}L)$ *is embedded in a classical ring.*
2. $\mathfrak{B}L$ *is finite.*
3. $\mathcal{R}L/\operatorname{Soc}(\mathcal{R}L)$ *is classical.*

Proof. (1) $\Rightarrow$ (2): Assume that $\mathcal{R}L/\operatorname{Soc}(\mathcal{R}L)$ is embedded in a classical ring $\mathcal{R}(\mathfrak{O}X)$ for some space X. Recall that the rings $C(X)$ and $\mathcal{R}(\mathfrak{O}X)$ are isomorphic for any Tychonoff space X. Since the homomorphism $\upsilon_L\colon \upsilon L \to L$ is dense onto, $\mathfrak{B}(\upsilon L) \cong \mathfrak{B}L$. It suffices to show that $\mathfrak{B}(\upsilon L)$ is finite. We have $\mathcal{R}L/\operatorname{Soc}(\mathcal{R}L) \cong \mathcal{R}(\upsilon L)/\operatorname{Soc}(\mathcal{R}(\upsilon L))$. There is a ring homomorphism $\mathcal{R}(\upsilon L) \to \mathcal{R}(\mathfrak{O}X)$ such that the $\ker(\mathcal{R}h) = \operatorname{Soc}(\mathcal{R}(\upsilon L))$. By Proposition 5.3.1, it follows that there exists a sublocale $S \subseteq \upsilon(\upsilon L) = \upsilon L$ such that $\operatorname{Soc}(\upsilon L) = \boldsymbol{O}^{\mathrm{Nd}(\beta L)} = \boldsymbol{M}^S$. Hence $A \subseteq \upsilon L \smallsetminus \mathfrak{B}(\upsilon L)$. If $\mathfrak{B}(\upsilon L)$ is infinite, then it contains an infinite countable sublocale, say $Q = \{\alpha_n \colon n \in \mathbb{N}\}$. We define the function α as

$$\alpha(t) = \begin{cases} \frac{1}{t}, & \text{if } t \in Q, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that α is continuous on υL and $\alpha \in \boldsymbol{M}^S$. But $\upsilon L \smallsetminus \operatorname{coz} \alpha$ is infinite and so $\alpha \notin \operatorname{Soc}(\mathcal{R}(\upsilon L))$. This is a contradiction and hence $\mathfrak{B}(\upsilon L)$ must be finite.

(2) $\Rightarrow$ (3): It is straightforward that if $\mathfrak{B}L$ is finite, then $\mathcal{R}L/\operatorname{Soc}(\mathcal{R}L) \cong C(X \smallsetminus I(X))$, which means that $\mathcal{R}L/\operatorname{Soc}(\mathcal{R}L)$ is classical.

(3) $\Rightarrow$ (1): This is obvious. $\square$

The authors in [4] generalize the notions of C- and C^*-embedded sublocales of [8] to localic maps. A localic map $f\colon L \to M$ is a C-*map* (resp. C^*-*map*) if for every continuous (resp. bounded and continuous) real-valued function $g\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$, there exists a continuous (resp. bounded and continuous) function $\bar{g}\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(M)^{op}$ such that the diagram

$$\begin{tikzcd}
 & \mathfrak{L}(\mathbb{R}) \arrow[dl, "\bar{g}"'] \arrow[dr, "g"] & \\
\mathcal{S}(M)^{op} \arrow[rr, "f_{-1}[-]"] & & \mathcal{S}(L)^{op}
\end{tikzcd}$$

commutes. That is, f is a C-map precisely when g factors through $f_{-1}[-]$ for every $g \in \mathsf{C}(L)$.

By the isomorphism $\mathsf{C}(L) \cong \mathcal{R}L$, it follows that $f\colon L \to M$ is a C-map (resp. C^*-map) if and only if for every $g \in \mathcal{R}L$ (resp. $\mathcal{R}^*L$) there exists $\bar{g} \in \mathcal{R}L$ (resp. $\bar{g} \in \mathcal{R}^*L$) such that the diagram

$$\begin{tikzcd}
 & \mathfrak{L}(\mathbb{R}) \arrow[dl, "\bar{g}"'] \arrow[dr, "g"] & \\
M \arrow[rr, "f^*"] & & L
\end{tikzcd}$$

commutes. A sublocale S is *C-embedded* (resp. *C^*-embedded*) if for every continuous (resp. bounded and continuous) real function $g\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(S)^{op}$ there exists a continuous (resp. bounded and continuous) function such that the diagram

$$\begin{tikzcd}
 & \mathfrak{L}(\mathbb{R}) \arrow[dl, "\bar{g}"'] \arrow[dr, "g"] & \\
\mathcal{S}(L)^{op} \arrow[rr, "j_{-1}[-]"] & & \mathcal{S}(S)^{op}
\end{tikzcd}$$

commutes, where j is the localic embedding of S in L. Because of how preimages of localic embeddings are computed, the diagram above just means that $\bar{g}(a) \cap S = g(a)$ for every $a \in \mathfrak{L}(\mathbb{R})$.

In this case, we say that $\bar{g}$ is a *continuous extension* (resp. *continuous and bounded extension*) of g. Hence, a sublocale S of L is C-embedded (resp. C^*-embedded) if and only if the embedding $j\colon S \hookrightarrow L$ is a C-map (resp. C^*-map). These localic embeddings are precisely the right adjoints of the C-quotients (resp. C^*-quotients) of [8]. If $f\colon L \to M$ is a C-map (resp. C^*-map), then $f[L]$ is a C-embedded (resp. C^*-embedded) sublocale of L. The reader is referred to [4] for more details.

Proposition 5.3.2. *Let Q be an ideal of $\mathcal{R}L$. Then $\mathcal{R}L/Q$ is classical if and only if there exists a closed C-embedded $S \subseteq \upsilon L$ such that $Q = \boldsymbol{M}^S$.*

Proof. ($\Rightarrow$): Assume that X is a Hausdorff space such that $\mathcal{R}L/Q \cong C(X)$. But $\mathcal{R}(\mathfrak{O}X) \cong C(X)$. Then there is a ring homomophism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}(\mathfrak{O}X)$ such that $\ker(\mathcal{R}h) = Q$. We know that the frame homomorphism $h\colon \upsilon L \to \mathfrak{O}X$ induces the ring homomorphism $\mathcal{R}\bar{h}\colon \mathcal{R}(\upsilon L) \to \mathcal{R}(\mathfrak{O}X)$ (notice that $\mathcal{R}\bar{h} = \mathcal{R}h$ since $\mathcal{R}L \cong \mathcal{R}(\upsilon L)$ for any frame L). This implies that the localic map $\pi\colon \mathfrak{O}X \to \upsilon L$ is a C-map. By Proposition 5.3.1, we have that $\ker(\mathcal{R}h) = \boldsymbol{M}^{\pi[\mathfrak{O}X]}$. By the argument above, $\pi[\mathfrak{O}X]$ is C-embedded in υL. Just as in the case where $A \subseteq \beta L$, it is easy to show that $\boldsymbol{M}^A = \boldsymbol{M}^{\bar{A}}$ where $A = \pi[\mathfrak{O}X]$, and A is C-embedded in υL, and so is $\bar{A}$.

($\Leftarrow$) : If a closed sublocale A is C-embedded in υL, then the ring homomrphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}A$ is given by $\mathcal{R}h(\alpha) = \alpha|_A$. The ring homomorphism is surjective and $\ker(\mathcal{R}h) = Q = \boldsymbol{M}^A$. That is, $\mathcal{R}L/Q$ is a classical ring. □

For the purpose of the next corollary, recall from Sect. 5.1 that

$$\mathrm{Nd}(M) = M \smallsetminus \mathfrak{B}M = \bigvee\{\mathfrak{c}_M(x) \mid x \text{ is a dense element of } M\}$$

and that $\mathrm{Soc}_m(\mathcal{R}L) = \boldsymbol{M}^{\mathrm{Nd}(\beta L)}$ (Theorem 5.1.1).

Corollary 5.3.3. *$\mathcal{R}L/\mathrm{Soc}_m(\mathcal{R}L)$ is classical if and only if $\upsilon L \smallsetminus \mathfrak{B}L$ is dense in $\beta L \smallsetminus \mathfrak{B}L$ and C-embedded in υL.*

Proof. ($\Rightarrow$) : Assume that $\mathcal{R}L/\mathrm{Soc}_m(\mathcal{R}L)$ is a classical ring. We have that $\mathcal{R}L/\mathrm{Soc}_m(\mathcal{R}L) \cong \mathcal{R}(\upsilon L)/\mathrm{Soc}_m(\mathcal{R}(\upsilon L))$. Since $\mathcal{R}(\upsilon L)/\mathrm{Soc}_m(\mathcal{R}(\upsilon L))$ is classical by hypothesis, it follows from Proposition 5.3.2 that there is a closed C-embedded sublocale A of υL such that $\mathrm{Soc}_m(\mathcal{R}(\upsilon L)) = \boldsymbol{M}^A$. Considering Theorem 5.1.1, we deduce that

$$\mathrm{Soc}_m(\mathcal{R}(\upsilon L)) = \boldsymbol{M}^A = \boldsymbol{M}^{\mathrm{Nd}(\beta L)} = \boldsymbol{M}^{\beta L \smallsetminus \mathfrak{B}(\beta L)}.$$

But A is a C-embedded closed sublocale of υL which implies that

$$\mathrm{Soc}_m(\mathcal{R}(\upsilon L)) = \boldsymbol{M}^A = \boldsymbol{M}^{\mathrm{Nd}(\upsilon L)} = \boldsymbol{M}^{\upsilon L \smallsetminus \mathfrak{B}(\beta L)} = \boldsymbol{M}^{\mathrm{Nd}(\beta L)} = \boldsymbol{M}^{\beta L \smallsetminus \mathfrak{B}(\beta L)}.$$

The equality implies that $\bar{A} = \upsilon L \smallsetminus \mathfrak{B}(\beta L) = \beta L \smallsetminus \mathfrak{B}(\beta L)$. Hence $\upsilon L \smallsetminus \mathfrak{B}(\beta L)$ is dense in $\beta L \smallsetminus \mathfrak{B}(\beta L)$ and C-embedded in υL.

$(\Leftarrow)$: The converse follows from Proposition 5.3.2. $\square$

Comment: A reader might ask if the same results can be achieved in the ring $\mathfrak{Z}L$ of integer-valued continuous functions on a zero-dimensional frame L. We would first need to develop certain types of ideals of the ring $\mathfrak{Z}L$ and we are of the view that such pursuit belongs to a different discussion. We leave that for another occasion.

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