INVESTIGATING GRADE ELEVEN LEARNERS' MATHEMATICAL MODELLING COMPETENCIES IN ALGEBRAIC PROBLEM SOLVING

by

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DECLARATION

I, REUBEN BAFANA DLAMINI, hereby declare that the thesis titled **INVESTIGATING GRADE ELEVEN LEARNERS MATHEMATICAL MODELLING COMPETENCIES THROUGH PROBLEM SOLVING,** which I hereby submit for the degree of Doctor of Philosophy in Mathematics Education at the University of South Africa, is my work and has not previously been submitted by me for a degree at this or any other institution.

SIGNATURE **26/03/2026**
DATE

SUPERVISOR'S DECLARATION

I declare that this thesis has been prepared under my supervision and I certify that it meets the requirements to be presented for the award of the degree Doctor of Philosophy in Mathematics Education.

SIGNATURE: ___________________

NAME OF SUPERVISOR: PROF DEONARAIN BRIJLALL

DATE: __________________

EDICATION

I dedicate this thesis to my wife Duduzile; my children, Fisiwe, Simangaliso, Sisanda, Okuhle, Sinethemba and Mcebo who always believe in me. I also dedicate it to my late parents for bringing me to the academic world and nurturing me during my early years of schooling.

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ABSTRACT

The National Curriculum Statement (NCS), implemented in 2003, marked the formal introduction of mathematical modelling into the South African curriculum. This represented a significant shift, as mathematical modelling had previously been largely absent from the curriculum. This study investigated the mathematical modelling proficiency of Grade 11 learners from three high schools in the Pongola Circuit, KwaZulu-Natal Province, with a specific focus on determining their competency in solving non-routine problems.

The study was motivated by the observation that many teachers did not receive adequate training in mathematical modelling during their pre-service education. Existing research indicates that inadequately trained teachers can negatively affect learners' academic performance. Furthermore, the teacher is widely regarded as the most critical agent in the successful implementation of instructional reforms at the classroom level (Shepherd, 2019; Theophile et al., 2020). It was therefore necessary to examine learners' competencies in mathematical modelling.

Learners' competencies were assessed using the five-step modelling process proposed by Kaiser and Stender (2013), which includes: (1) understanding the problem, (2) formulating a mathematical model, (3) solving the model, (4) interpreting the results, and (5) validating the results. For a learner to be considered competent in mathematical modelling, all five stages of the process had to be correctly executed.

A total of 75 Grade 11 learners from three purposively selected schools participated in the study. Qualitative data were collected through document analysis. The data analysis process involved familiarisation with learners' written responses, followed by thematic analysis to interpret meaning, identify patterns, and generate insights related to the research questions.
The findings revealed that all learners demonstrated incomplete competency in mathematical modelling. Specifically:

- None of the learners successfully completed all five stages of the modelling process in any of the four test questions.
- Learners did not make or attempt to make assumptions, which are essential in solving real-life problems.

- Variables were used without clear definitions, and final solutions were often expressed in terms of unidentified variables. This indicated a lack of interpretation of results within the context of the real-world problems. Interpretation involves translating mathematical outcomes back into meaningful real-life conclusions.
- Learners did not verify or validate their solutions. Validation is critical for assessing the accuracy and appropriateness of both the mathematical model and its results in relation to the real-world context.

The findings suggest that Mathematics teachers should be encouraged to adopt a mathematical modelling approach in teaching and learning. It can be inferred that learners had limited or no exposure to mathematical modelling, as key processes—such as defining variables, making assumptions, interpreting results, and validating solutions—were consistently omitted. According to the Curriculum and Assessment Policy Statement (CAPS), mathematical modelling should serve as a central focus of the Mathematics curriculum (Department of Basic Education [DBE], 2011). Therefore, the Department of Education has a responsibility to ensure the effective implementation of mathematical modelling, emphasising the integration of real-life contexts across all aspects of the curriculum.

Keywords: mathematical competency, constructivism, models, mathematical modelling, mathematisation, modelling processes

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LIST OF ACRONYMS

CAPS:	Curriculum and Assessment Policy Statement
DBE:	Department of Basic Education
FET:	Further Education and Training
ICTMA:	International Community of Teachers of Mathematical Modelling and Applications
MMO:	Mathematical Modelling Outreach
NCS:	National Curriculum Statement
NDP:	National Development Plan
TIMMS:	Trends in International Mathematics and Science Study

CHAPTER ONE – INTRODUCTION TO THE STUDY

1.1 Overview of the study

In this study, I investigated the mathematical modelling competencies of Grade 11 learners through problem-solving tasks conducted in three secondary schools. Participants were required to complete a test comprising questions in which mathematical concepts were embedded within real-life contexts. The nature of these questions was aligned with the types of problems learners are expected to encounter at the Grade 12 level.

This thesis is organised into six chapters. Chapter One provides an overview of the study, including the introduction and background to the problem. It further outlines the problem statement, rationale for the study, research questions, aims and objectives, significance of the study, scope, and key concepts, and concludes with a summary.

Chapter Two presents a review of the relevant literature related to the study. Chapter Three outlines the theoretical framework underpinning the research. Chapter Four details the research methodology and design employed in the study. Chapter Five presents the data, along with the findings and their analysis. Finally, Chapter Six provides a summary of the study, discusses its limitations, and offers recommendations.

1.2 Introduction

The increasing demand for professionals in Science, Technology, Engineering, and Mathematics (STEM) disciplines has posed significant challenges for education systems, which are required to prepare learners who are competent in these fields (Damlamian et al., 2013, as cited in Kohen et al., 2021). In response to these evolving demands, there has been a growing recognition of the need to reform education. Consequently, substantial developments have taken place within education systems, with various strategies being introduced to align teaching and learning with societal needs and to improve educational outcomes.

According to Tasarib et al. (2025), mathematical modelling has been incorporated into Mathematics curricula in several countries as a response to contemporary societal and educational demands. Its integration into classroom practice has been shown to enhance learners' cognitive development, which is essential for meaningful learning. Furthermore, the inclusion of mathematical modelling in the teaching of Mathematics supports the development of application skills and promotes a deeper understanding of mathematical concepts.

Developments in the South African Mathematics curriculum are consistent with global trends, particularly in the shift towards mathematical modelling. Mathematical modelling was formally introduced into the South African curriculum in 2003 (Department of Education, 2002), marking a significant transformation, as it had previously been largely absent. The implementation of the Revised National Curriculum Statement (RNCS) in 2003 signified the official inclusion of mathematical modelling. Subsequent curricula—including the National Curriculum Statement (NCS) for Grades R–9, the NCS for Grades R–12, and the current Curriculum and Assessment Policy Statement (CAPS)—have all continued to incorporate mathematical modelling. CAPS emphasises modelling as a central feature of the curriculum, advocating for the integration of real-life problems across all content areas wherever possible (Department of Basic Education, 2011a).

1.3 Background of the problem

Mathematics is a valuable tool in everyday life, influencing activities such as financial planning, food preparation, navigation, and sports. Its applications enable individuals to make informed decisions, enhance daily functioning, and improve their overall quality of life. A sound understanding and application of basic mathematical concepts can significantly improve productivity and problem-solving abilities. Consequently, teachers have a responsibility to equip learners with the skills to apply Mathematics effectively in real-life contexts (Agbata et al., 2024).

Banerjee and Arshad (2025) further emphasise that Mathematics is inherently embedded in everyday life. Whether consciously or unconsciously, individuals engage with mathematical thinking daily. It is applied in diverse contexts such as finance (e.g., savings, interest calculations, and budgeting), cooking (e.g., measurements and timing), and travel planning (e.g., itinerary development and fuel efficiency). The integration of Mathematics into real-life contexts has become a key focus in educational research, with scholars advocating for context-based and authentic teaching approaches (Banerjee & Arshad, 2025).

Despite the importance of Mathematics in everyday life, learner performance in South African schools remains significantly below that of many other countries. For example, in the 2015

Trends in International Mathematics and Science Study (TIMSS), South Africa ranked 38th out of 39 participating countries at the Grade 8/9 level, outperforming only Saudi Arabia, and performing below Botswana, which ranked 35th (Zuze et al., 2017).

Zuze et al. (2017) further note that South Africa's performance in TIMSS, an international benchmarking assessment conducted every four years, has historically been poor, although some improvement has been observed over time. Between TIMSS 2003 and TIMSS 2015, South Africa showed an overall gain of approximately 90 points in Mathematics. However, despite this progress, performance remains below international benchmarks, largely because the country started from a relatively low baseline.

More recent results from TIMSS 2023 indicate continued challenges. South African Grade 5 learners ranked lowest in Mathematics among 59 participating countries. Although Grade 9 learners performed slightly better than in previous cycles, both Grade 5 and Grade 9 results remained below international averages, raising concerns about the country's future mathematical competency (Department of Basic Education, 2024).

It is also important to note that while most participating countries administered the assessment to learners in Grades 4 and 8, South Africa tested Grades 5 and 9 respectively (Department of Basic Education, 2024). This means that South African learners were, on average, a year older than their international counterparts. Despite this apparent advantage, their performance remained comparatively low.

The distribution of South African learners' achievement in TIMSS 2023 is summarised in Table 1.1. For Grade 5 Mathematics, 2% of learners performed at the advanced level, 6% at the high level, 17% at the intermediate level, and 35% at the low level, while 47% performed below the low benchmark. Similarly, for Grade 9 Mathematics, 1% of learners achieved the advanced level, 4% the high level, 15% the intermediate level, and 45% the low level, with 35% performing below the low benchmark. These results highlight persistent challenges in mathematical achievement among South African learners.

Table 1.1: Learners' achievements in Mathematics in TIMMS 2023

Grade 5	Benchmark	Grade 9
2	Advance Benchmark (625)	1
7	High Benchmark (550)	4
16	Intermediate Benchmark (475)	15
28	Low Benchmark (400)	45

Source: DBE (2023)

Learners' performance in TIMSS is considered relevant to this study because the assessment is structured around three cognitive domains, namely knowing, applying, and reasoning. In order to respond effectively to TIMSS items, learners must not only demonstrate familiarity with mathematical content but also draw on a range of cognitive skills, including selecting and executing procedures, applying knowledge to unfamiliar contexts, making reasoned decisions, and justifying solutions. In this framework, knowing refers to understanding mathematical concepts, applying involves using these concepts to solve real-world problems, and reasoning encompasses critical thinking throughout the problem-solving and modelling process (Department of Basic Education, 2024).

Reasoning and applying together constitute a significant proportion of the TIMSS assessment, accounting for 60% at Grade 4 and 65% at Grade 8 (Department of Basic Education, 2024). These cognitive demands are closely aligned with the requirements of mathematical modelling. Consequently, South African learners' poor performance in TIMSS may suggest insufficient development of mathematical modelling competencies at the high school level. Given that mathematical modelling is a central feature of the current South African Mathematics curriculum, it was therefore important for this study to investigate Grade 11 learners' modelling competencies.

Although this study focused on Grade 11 learners, it is also important to consider performance trends in the lower grades, as TIMSS assesses Grade 4 and Grade 8 internationally, while South Africa assesses Grade 5 and Grade 9 (Department of Basic Education, 2024). The consistently low performance of South African learners at these levels suggests that many learners may enter high school with significant learning gaps.

According to Torres (2021), learning gaps refer to the discrepancy between what learners have already mastered and what is expected at a particular grade level. If these gaps are not addressed timeously, they tend to widen over time as missing foundational knowledge accumulates. This compounding effect poses a serious challenge for both teaching and learning in higher grades. The TIMSS results for South Africa remain a major concern for educators and policymakers, raising questions about the effectiveness of curriculum reform in the democratic era. In this regard, Ndlovu and Mji (2012) examined the alignment between the TIMSS 2003 Grade 8 Mathematics assessment framework and the Revised National Curriculum Statement (RNCS), later revised into the Curriculum and Assessment Policy Statement (CAPS). Their study sought to identify areas of misalignment that could help explain South Africa's persistent underperformance.

Their findings indicated that while the RNCS performed relatively well in promoting factual knowledge and procedural fluency, it was weaker in developing reasoning skills. This suggests that the South African Mathematics curriculum is not fully aligned with TIMSS expectations in key cognitive domains. It therefore becomes necessary for curriculum reforms to be more responsive to both the content domains assessed in TIMSS (number, algebra, geometry, and data) and its cognitive domains, namely knowing, applying, and reasoning. In this regard, TIMSS should serve not only as an assessment tool but also as a source of feedback for curriculum improvement.

Furthermore, the findings of Ndlovu and Mji (2012) suggest that South African learners are particularly underperforming in the cognitive domains of application and reasoning. This is significant, as mathematical application is a critical component of effective Mathematics teaching and learning. Asli and Zsoldos-Marchis (2021) argue that teaching mathematical applications is essential for connecting Mathematics to everyday life contexts. Such applications support learners' personal development and serve as a meaningful tool for motivation by demonstrating the relevance of Mathematics to daily life, future careers, and other academic disciplines.

Similarly, Nkhase (2002, cited in Asli et al., 2020) highlights that a major challenge in Mathematics education is learners' inability to connect classroom learning with real-life experiences. Asli et al. (2020) further note that many learners complete schooling without recognising the importance of mathematical knowledge and skills for life, further studies, and career choices. This is often attributed to teaching approaches that fail to emphasise relevance, application, and critical thinking.

Rangel et al. (2016) also emphasise that Mathematics has always been an integral part of human daily life. Across historical civilisations, mathematical knowledge has evolved in response to practical needs arising from fields such as physics, chemistry, astronomy, religion, art, and music. However, in contemporary education, these real-world connections are often neglected in classroom practice. This disconnect limits learners' ability to appreciate the usefulness of Mathematics, whereas linking real-life problems to mathematical concepts can make learning more engaging and meaningful.

In their study, Rangel et al. (2016) demonstrate the value of teaching Mathematics through everyday contexts to enhance learners' understanding and engagement. This approach requires learners to interpret real-world situations, construct mathematical models, perform computations, interpret results, and validate their solutions—processes that closely align with mathematical modelling competencies.

Rangel et al (2016) gave an example of an integration exercise pertaining to a Calculus course where learners are asked to calculate the integral $\int_0^1 \sqrt{1+x}dx$. In this example, learners are required to solve an activity that has no connection to real-life situations. The authors argue that when learners fail to perceive a link between Mathematics and its context, they may begin to question its relevance, asking their teachers questions such as: "What is the purpose of Mathematics?" and "How is it used in day-to-day life?" In recent years, approaches to teaching Mathematics through real-world problems have increasingly been adopted to bridge this gap by explicitly connecting mathematical concepts to everyday experiences.

The findings of Rangel et al. (2016) suggest that Mathematics is often taught in ways that are disconnected from real-world applications. This lack of contextualisation may lead to reduced learner interest and difficulty in transferring classroom knowledge to practical situations. In this regard, mathematical modelling is identified as one pedagogical approach that teachers can use to foster learners' interest and engagement in Mathematics.

Similarly, Antao et al. (2025) found that engaging learners through practical and real-world applications of Mathematics can significantly enhance their interest in the subject. When learners recognise the direct relevance of Mathematics to their everyday lives, they are more likely to participate actively in learning activities. Furthermore, authentic learning tasks reinforce the usefulness of Mathematics in real-life contexts and promote deeper engagement and critical thinking. These findings highlight the importance of positioning Mathematics not merely as an abstract discipline, but as a practical tool for solving everyday problems.

Building on these findings, it is necessary to clarify the concept of learner engagement. According to Thomas et al. (2013, cited in Fung et al., 2018), engagement is increasingly recognised as a key prerequisite for achievement in both national and international assessments. Fung et al. (2018) describe engagement as a multidimensional construct comprising affective, behavioural, and cognitive dimensions. Affective engagement refers to learners' emotional responses to school, teachers, peers, and academic tasks, which influence their willingness to participate. Behavioural engagement relates to learners' observable actions in the classroom, such as effort, participation in activities, and discussions of mathematical problems. Cognitive engagement involves mental investment in learning, including perseverance, the use of problem-solving strategies, and sustained effort when working through challenging tasks.

Collectively, the studies discussed above emphasise the value of integrating real-world contexts into Mathematics teaching. They demonstrate that contextualised learning enhances learner motivation, engagement, and achievement. It is therefore evident that learners are more likely to understand and retain mathematical concepts when these are linked to real-life applications.

In this study, I investigated the mathematical modelling competencies of Grade 11 learners through problem-solving tasks in three secondary schools. Participants completed a test in which mathematical concepts were embedded in real-life contexts. The tasks were aligned with the types of problems learners are expected to encounter in Grade 12. However, learners often perform poorly in such tasks, particularly those involving the application of Differential Calculus. It was anticipated that exposure to these types of problems would provide learners with alternative strategies for solving similar problems in Grade 12.

Given that mathematical modelling is a relatively recent addition to the South African Mathematics curriculum, there is a need to examine learners' competencies in this area. To assess these competencies, learners' written responses to the test were analysed. A rubric was used to determine the extent to which learners demonstrated the key processes of mathematical modelling. In addition, modelling competencies as described by Maaß (2006) were used as an analytical framework.

This chapter provides an outline of the study. It begins with the background and purpose of the research, followed by the problem statement, research questions, motivation for the study, significance, aims and objectives, and definitions of key concepts. It concludes with a brief overview of the structure of the thesis, outlining each chapter.

In South African schools, external examinations are typically administered only at Grade 12 level. In Grades 1 to 11, learners complete school-based or district-based assessments that are marked internally. As a result, the Grade 12 examination serves as the primary standardized measure of educational achievement in the country. Furthermore, admission to tertiary institutions is largely determined by Grade 12 performance. For instance, entry into science-, technology-, or accounting-related degree programmes requires strong performance in Mathematics. At the University of KwaZulu-Natal, for example, applicants to Engineering programmes such as Agricultural, Computer, Civil, Electrical, Electronic, Mechanical, and Land Surveying Engineering are required to achieve at least 70% (Level 6 or higher) in Mathematics (University of KwaZulu-Natal, 2021).

Despite the Department of Basic Education's efforts to promote Mathematics enrolment, participation in the subject has declined since 2017, even though Mathematics remains essential in everyday life and future careers. The total number of learners writing the National Senior Certificate Mathematics examination declined from 265,912 in 2016 to 222,034 in 2019, representing a decrease of 43,878 candidates. In South Africa, learners are not permitted to take both Physical Sciences and Mathematical Literacy simultaneously in Grades 10–12, which further influences subject selection patterns (KwaZulu-Natal Department of Education, 2013). Given the demands of the Fourth Industrial Revolution, the decline in Mathematics enrolment is concerning. This technological era requires a workforce with strong competencies in Mathematics, Science, and Technology. In response, the South African government has implemented several initiatives, including the National Development Plan (NDP), which aims to increase participation and improve performance in MST (Mathematics, Science, and Technology) subjects by 2030 (National Planning Commission, 2011). One such initiative involves the development of specialised MST schools that focus on strengthening learner achievement in these disciplines. In these schools, all learners take Mathematics rather than Mathematical Literacy, and additional funding is provided to support MST education. Furthermore, technical subjects such as Technical Mathematics and Technical Science have been introduced to enhance learners' practical skills and responsiveness to technological change (National Planning Commission, 2011).

Despite these interventions, South Africa continues to perform poorly in Mathematics across both primary and secondary levels. As previously noted, learners underperform in TIMSS assessments at Grades 5 and 9, and Grade 12 results also reflect ongoing challenges. Although the pass rate improved from 63.5% in 2023 to 69.1% in 2024, the number of learners enrolled in Mathematics has continued to decline.

According to the NDP, the Department aims to increase the number of learners qualifying for Mathematics- and Science-related university programmes to 450,000 by 2030 (National Planning Commission, 2011). However, current trends suggest that this target may be difficult to achieve, as Mathematics enrolment has not exceeded 270,000 in recent years, and the number of learners achieving 40% and above has remained below 125,000. These trends indicate that South Africa remains far from achieving its national educational aspirations in Mathematics.

Table 1.2: Learners' achievements in Mathematics in the past 5 years

Year	No. wrote	No. achieved at 30% and above	% achieved at 30% and above	No. achieved at 40% and above	% achieved at 40% and above
2020	233 315	125 526	53,8	82 964	35,6
2021	259 143	149 177	57,6	97 561	37,6
2022	269 734	148 346	55,0	97 041	36,0
2023	262 016	166 337	63,5	114 311	43,6
2024	251 488	173 774	69,1	120 430	47,9

Source: DBE (2024)

Table 1.3 presents the Mathematics performance of all 25 MST schools in the Gert Sibande District of Mpumalanga for the period 2019 to 2023. This district was selected because of my familiarity with all MST schools within it. The analysis was conducted to determine the extent to which these schools have achieved the intended objectives of the MST programme.

Table 1.3: MST schools Mathematics performance from 2019 – 2023

Name of School	**2019**		**2020**		**2021**		**2022**		**2023**	
	No. Wrote	Pass %	No. Wrote	Pass %	No. Wrote	Pass %	No. Wrote	Pass %	No. Wrote	Pass %
AD Nkosi Sec	40	**72,5**	38	**73,3**	91	**64,8**	64	**71,9**	133	**58,6**
Amadlelo Sec	88	**39,8**	173	**20,2**	193	**31,1**	157	**34,4**	84	**52,1**
Camden Comb	24	**70,8**	22	**90,9**	35	**85,7**	35	**77,1**	41	**97,6**
Dlomodlomo	111	**39,6**	56	**71,4**	61	**83,6**	40	**75,0**	41	**78,0**
Ekulindeni Sec	44	**34,1**	20	**80,0**	44	**70,5**	25	**52,0**	21	**76,2**
Elangwane Sec	42	**38,1**	77	**41,6**	129	**28,7**	118	**43,2**	105	**38,1**
Highveld Sec	80	**86,3**	97	**66,0**	165	**61,2**	107	**75,7**	186	**77,4**
Ithafa High	93	**40,9**	146	**19,2**	252	**12,3**	157	**22,9**	111	**43,2**
Izimbali CBS	37	**64,9**	27	**88,9**	39	**74,4**	51	**64,7**	47	**80,9**
Khunjuliwe	52	**63,5**	90	**48,9**	82	**42,7**	85	**72,9**	96	**59,4**
Kiriyatswane	72	**44,4**	185	**43,8**	212	**31,6**	226	**27,4**	185	**35,7**
Lindile Sec	67	**43,3**	144	**17,4**	186	**13,4**	114	**13,2**	110	**37,3**
Mayflower	50	**64,0**	68	**51,5**	90	**55.6**	47	**74,5**	64	**85,9**
Nalithuba Sec	26	**57,7**	37	**40,5**	59	**54,2**	62	**58,1**	63	**52,3**
Ndlela Sec	446	**12,3**	384	**15,1**	246	**30.9**	256	**22,3**	242	**34,3**

Nthoroane Sec	31	**45,2**	73	**8,2**	60	**18,3**	24	**25,0**	19	**68,4**
Osizweni High	116	**72,4**	147	**45,6**	164	**50,6**	149	**43,0**	195	**46,2**
Simtfolile Sec	170	**61,2**	142	**51,4**	111	**78,4**	122	**76,2**	101	**68,3**
Siyabonga Sec	56	**51,8**	52	**26,9**	60	**65,0**	75	**60,0**	64	**60,9**
Takheni Sec	90	**61,1**	98	**46,9**	126	**54,0**	122	**41,8**	106	**55,7**
Thistle Grove	74	**98,1**	129	**57,4**	107	**54,2**	99	**45,0**	187	**41,7**
T Nhlabathi	110	**75,5**	166	**57,0**	221	**47,5**	167	**59,3**	187	**66,3**
Warburton	12	**50,0**	32	**46,9**	33	**45,5**	40	**55,0**	40	**80,0**
Zikhetheleni	14	**35,7**	12	**75,0**	11	**72,7**	9	**88,9**	23	**78,3**
Zinikeleni H	39	**64,1**	85	**30,6**	50	**46,0**	84	**51,2**	65	**58,5**

Source: Department of Basic Education (2019, 2021, 2023)

A cursory examination of these 25 MST schools indicates that several still have a considerable way to go in achieving national MST objectives, particularly in improving the proportion of learners attaining high achievement levels and increasing overall participation rates. As shown in Table 1.4, a number of these schools failed to achieve a minimum pass rate of 50% over the period 2019–2023. In contrast, some schools with higher pass rates recorded relatively low enrolment figures. Overall, these patterns suggest that the MST initiative by the Department of Basic Education has not yet yielded the desired improvements in Mathematics performance. The persistently low achievement in Mathematics prompted the then Minister of Basic Education, Angelina Motshekga, to advocate for reforms in the teaching and learning of the subject. This led to the development of the Teaching and Learning Framework, which aimed to establish a strong foundation for innovative approaches to Mathematics instruction. Importantly, this Framework was not intended to replace the existing Curriculum and Assessment Policy Statement (CAPS), but rather to complement it by offering alternative and innovative perspectives on the teaching, learning, and assessment of Mathematics (Department of Basic Education, 2018a).

Mathematics is widely recognised as a gateway subject that opens pathways to various fields of study. Learners who perform well in Mathematics are afforded a broader range of career opportunities, whereas poor performance may limit access to many academic and professional pathways. The current South African Mathematics curriculum is designed to equip learners with the competencies required for success in the twenty-first century. In their article, *Mathematical Practices that Promote 21st Century Skills*, Suh and Seshaiyer (2013) identify these competencies as the "4Cs," namely collaboration, critical thinking, creativity, and communication. Within the context of the Mathematics classroom, these skills are integral to effective teaching and learning and are further elaborated in Table 1.4, which summarises the key knowledge, skills, and instructional strategies required to develop these competencies.

Table 1.4: Skills and knowledge needed for the 21st Century

4Cs	**Skills and knowledge needed**	**CCSSM and Strategies**
Communication or interaction	Interchange of thoughts, concepts, questions and answers	"Model with Mathematics" and "use appropriate tools strategically" through a multi-representational approach with concrete manipulatives, tables, text, images, diagrams, and numbers
Collaboration or teamwork	Combining skills, knowledge, and intelligence to achieve a common objective	"Make sense of problems and persevere in solving for gallery walks them" by using group poster proof exercises
Critical Thinking or analysing attentively	Taking an innovative approach to problems and connecting knowledge from other fields and theme	Make convincing cases as you assess the views of others using the Strategy Venn Diagram and Questioning Prompt Cards
Creativity or inventiveness	Attempting new techniques to accomplish tasks equates to invention and innovation.	Translate an everyday situation into a practical math situation with problem solving and problem posing using Math Happenings

Source: Suh & Seshaiyer (2013)

The Common Core State Standards for Mathematics (CCSSM) are a national set of standards developed in the United States, according to Ives et al. (2017). These standards outline the knowledge and skills that learners are expected to demonstrate at each grade level. Their primary purpose is to ensure that all learners acquire the mathematical knowledge and competencies necessary for higher education, employment, and everyday life. In addition, they specify not only what should be taught but also how learners should engage with Mathematics. The emphasis of the current South African Mathematics curriculum on a mathematical modelling approach suggests that curriculum developers aimed to design a system that responds to the demands of the twenty-first century. This is because the "4Cs"—collaboration, critical thinking, creativity, and communication—are essential skills for successful engagement in mathematical modelling. Borda (2024) explains that problem-based learning typically involves learners working collaboratively to solve complex, real-world problems, making it an ideal foundation for teaching mathematical modelling. In such activities, learners analyse and define the problem, identify what they already know and what they need to learn, allocate responsibilities, conduct independent research, and collaborate to develop and evaluate solutions. This approach promotes essential twenty-first century workplace skills, including collaboration, critical thinking, problem-solving, creativity, innovation, and communication (Poláková et al., 2023, as cited in Borda, 2024). Furthermore, studies indicate that collaborative learning is particularly effective when learners work in small groups to solve mathematical problems, with Rustanuarsi and Karyat (2019) finding that it significantly improves learners' mathematical problem-solving abilities.

Mathematical modelling is a cyclical process that involves mathematising real-world situations and evaluating results to represent, analyse, and interpret them using Mathematics. Unlike conventional word problems, modelling tasks require learners to make assumptions, define quantities, apply mathematical procedures, interpret results, and validate solutions in context. Learners begin with a real-world problem, construct a mathematical representation, solve it, and then return to the original context to assess the validity of their solution (Taite et al., 2025). Tasarib et al. (2025) note that mathematical modelling is incorporated into curricula as a learner-centred instructional approach. This shifts the focus from passive knowledge acquisition to active engagement, inquiry, and knowledge construction. According to Borromeo-Ferri and Blum (2019, as cited in Tasarib et al., 2025), mathematical modelling is grounded in constructivist learning theory, which emphasises learner participation and knowledge construction through problem-solving and experience. This approach also promotes higher-order thinking skills as learners formulate, analyse, interpret, and validate mathematical models rather than simply solve equations.

Vasuki et al. (2016) argue that the implementation of constructivist principles is essential for developing both procedural fluency and conceptual understanding, which are necessary for long-term success in Mathematics. In this approach, teachers act as facilitators rather than lecturers, guiding learners through problem-solving while encouraging discussion, reflection, and the application of mathematical concepts. Stillman et al. (2017, as cited in Tasarib et al., 2025) further emphasise that integrating mathematical modelling into curricula enhances learners' practice and deepens their understanding of mathematical concepts.

To improve learners' achievement in Mathematics, there is a need for changes in classroom instructional practices, particularly those that position learners as passive recipients of knowledge. Teachers should adopt strategies that promote active learning, which has been shown to enhance understanding (Al-Odwan, 2016). Similarly, Arends et al. (2017) found that classroom practices such as collaborative learning have a positive impact on learners' achievement in Mathematics. Active learning is therefore central to this study. As noted by Tasarib et al. (2025), mathematical modelling is rooted in constructivist theory, which emphasises active learner engagement and knowledge construction through experience and problem-solving.

The Curriculum and Assessment Policy Statement (CAPS), which represents South Africa's current national curriculum, was developed as part of curriculum reforms aimed at improving learner performance and preparing learners for the demands of the modern world. One of its key principles is the promotion of active learning and critical thinking. This focus makes the South African Mathematics curriculum relevant to twenty-first century educational demands and aligns with the views of Al-Odwan (2016) regarding learner participation in Mathematics learning.

As indicated earlier, mathematical modelling is positioned as a central feature of the South African Mathematics curriculum. This implies that real-life problems should be integrated across all topics whenever appropriate. These problems should reflect issues related to political, economic, health, environmental, social, cultural, and scientific contexts (DBE, 2011a). Saxena et al. (2016) argue that learners' achievement in Mathematics is generally not at a satisfactory level, partly because learners perceive Mathematics as an abstract subject. Learners often struggle with mathematical problem-solving and rely heavily on memorised theorems, formulas, and procedures. Therefore, it is important to make Mathematics more engaging and meaningful so that learners can enjoy the learning process. Mathematical problem-solving should not only focus on academic performance but also on developing strategies that help learners connect Mathematics to real-life situations. This requires significant changes in educational practice, particularly the integration of everyday contexts into mathematical learning. Mosimege (2017) supports this view by suggesting that real-world contexts should include social, political, economic, cultural, health, scientific, and environmental issues, and should be incorporated regularly into classroom activities.

Heiliö et al. (2016) explain that mathematical modelling is widely used to solve complex real-world problems in fields such as bioscience, environmental science, social sciences, technology, and industry by translating them into mathematical language. For example, biological systems such as cells can be modelled mathematically to simulate internal processes. Similarly, mathematical models are used to understand large-scale systems such as atmospheric processes, ocean dynamics, space exploration, and underground geological layers.

Modern society also depends on vast systems such as energy distribution, transportation, and public services. In economics, simulation and system modelling are used to understand behaviour in markets and logistics systems. Industry and technology rely heavily on mathematical methods, including computational algorithms, in manufacturing and innovation processes. Simulation techniques are also widely used in technology to model real-world systems (Heiliö et al., 2016).

He Yanxuan (2024) further notes that mathematical modelling is widely applied in the social sciences, particularly economics, psychology, and sociology. In economics, it is used to analyse phenomena such as trade, market behaviour, supply and demand, and economic growth through methods such as regression analysis and time series modelling. In psychology, mathematical models help explain complex human behaviour by examining relationships between variables such as motivation, personality, and emotion. Structural equation modelling, for instance, allows researchers to analyse the combined effects of psychological factors on behaviour. In sociology, mathematical modelling and data analysis are used to study social structures, social change, and social problems. Social network analysis enables researchers to examine relationships among individuals and their influence on social outcomes. These methods improve the accuracy and predictive power of sociological research and provide deeper insights into social systems.

During the COVID-19 pandemic, mathematical modelling played a crucial role in predicting the spread of the disease and evaluating intervention strategies. Deng et al. (2022) note that mathematical models were used to forecast the progression and dynamics of the epidemic, thereby supporting decision-making processes in public health. There is a growing emphasis on the use of mathematical modelling at all levels of education, from primary to tertiary level. In Mathematics education, it serves as a bridge between classroom content and real-life applications. Through mathematical modelling, learners can develop mathematical solutions to real-world problems. When learners are able to connect textbook problems to everyday contexts, their interest and engagement in Mathematics tend to increase. However, a key challenge is that designing meaningful contextual problems requires additional effort from educators (Saxena et al., 2016).

Rangel et al. (2016) observe that in the past, Mathematics teaching often failed to connect classroom content to everyday life, resulting in less engaging instruction. Teaching was largely algorithmic, focusing on procedures rather than meaningful understanding or context. This limited learners' ability to see the relevance of Mathematics in their daily lives. Contextualised problems, however, have been shown to benefit learners significantly. Pratiwi and Widjajanti (2020) argue that such problems increase learners' interest because they can relate learning to their own experiences, making lessons more meaningful and relevant. Similarly, Rusminia and Sury (2017) found that contextual learning helps learners understand the purpose and usefulness of what they are learning, ultimately improving their ability to connect mathematical concepts to real-life situations.

The above discussion informed the decision to conduct this study, which seeks to investigate learners' competencies in mathematical modelling. The study is grounded in the need to determine whether learners' mathematical competencies are effectively embedded in real-life problem contexts. This investigation is particularly important given that mathematical modelling is increasingly regarded as a central feature of the current South African Mathematics curriculum.

1.4 Problem statement

TIMSS assessments indicate that South African learners' performance in Grade 9 Mathematics is consistently below that of many participating countries. In addition, underperformance in Mathematics is also evident at Grade 12 level, as shown in Table 1.1.

Learners' performance further varies across different topics. In general, the poorest performance in Paper 1 is consistently observed in the topic *Application of Differential Calculus*, which typically involves practical and real-life problem situations (Department of Basic Education, 2014, 2015, 2016, 2017, 2018b, 2019, 2022). Figure 1.1 further indicates that Question 9, which focused on the application of calculus, was the worst-performing question. Importantly, questions involving mathematical modelling are closely related to those in the application of calculus.

Several studies have highlighted the strong relationship between calculus and mathematical modelling, particularly in the area of optimisation. Brijlall and Ndlovu (2013) argue that learners can deepen their understanding of Mathematics and extend their problem-solving tools by using mathematical modelling to represent and analyse real-world situations through mathematical language and symbols. In Grade 12, this is commonly achieved through tasks involving the application of differential calculus, especially optimisation problems.

Similarly, Karaman Dündar et al. (2025) emphasise that optimisation and mathematical modelling are closely interconnected. Optimisation involves determining the best possible solution under given constraints, typically by maximising or minimising an objective function. Their study shows that optimisation tasks can improve learners' motivation, strengthen modelling abilities, and develop metacognitive and problem-solving skills. They further suggest that integrating mathematical modelling with optimisation tasks enhances learners' abstraction and reasoning abilities. Since optimisation problems require mathematical models to represent relationships between real-world situations and mathematical structures, modelling becomes essential in their solution.

Karaman Dündar et al. (2025) further recommend that future research should focus on integrating mathematical modelling with optimisation tasks to strengthen conceptual understanding and problem-solving skills. They also highlight the importance of examining how modelling activities support learners in making assumptions, constructing models, and validating solutions.

In a similar vein, Taranto et al. (2024) argue that solving optimisation problems strengthens learners' ability to construct and analyse mathematical models while also promoting critical thinking and problem-solving skills. They emphasise that mathematical modelling is inherently linked to real-life optimisation problems and can foster learners' interest in Mathematics by demonstrating its practical relevance in fields such as engineering, economics, logistics, and transportation. Engaging learners in both modelling and optimisation processes enhances their understanding of real-world applications and improves their mathematical reasoning skills.

In relation to this study, both TIMSS Grade 9 results and Grade 12 performance data reflect weaknesses in learners' foundational algebraic and geometric skills. These results also highlight difficulties in applying Mathematics in real-life contexts, particularly in mathematical modelling tasks. This suggests that learners' ability to engage with mathematical modelling at Grade 11 level may also be underdeveloped (Dizha, 2021).

In Grade 12, the application of Mathematics is commonly assessed through optimisation problems. These problems typically involve maximising desirable outcomes, such as profit, or minimising undesirable outcomes, such as production costs. In this study, examples of such problems include determining the maximum area that can be enclosed by a fence of a given length and finding the maximum area of a garden given a fixed perimeter.

Examples of optimisation questions drawn from previous National Senior Certificate examinations illustrate this approach. Like mathematical modelling tasks, these questions are presented in word-problem format. Consequently, learners are required to carefully read, interpret, and understand the problem before applying appropriate mathematical techniques to solve it.

QUESTION 8 (November 2019)

After flying a short distance, an insect came to rest on a wall. Thereafter, the insect started crawling on the wall. The path that the insect crawled can be described by where the height (in cm) above the floor is and is the time (in minutes) since the insect started crawling

8.1 At what height above the floor did the insect start to crawl? (1)

8.2 How many times did the insect reach the floor? (3)

8.3 Determine the maximum height that the insect reached above the floor (4)

[8]

QUESTION 10 (November 2024)

A cyclist rode from town P and stopped at town T. The speed (in km/h) at which this cyclist rode, is represented by the equation

NOTE: Speed is the rate of change in distance with respect to time.

10.1 Calculate the maximum speed that the cyclist reached on this ride (3)

10.2 Calculate the distance between town P and town T (5)

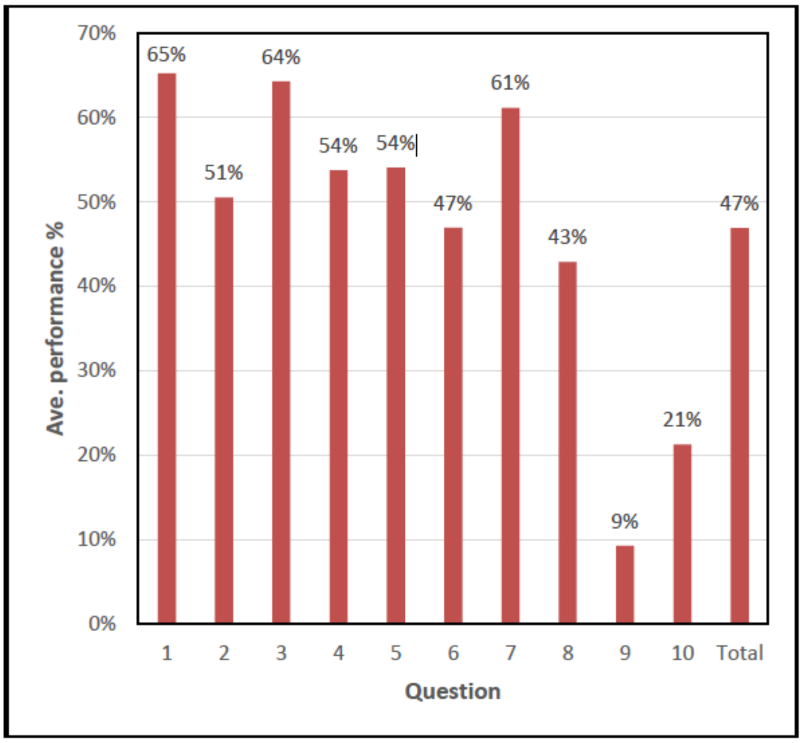
[8]

QUESTION 11 (November 2021)

After travelling a distance of 20 km from home, a person suddenly remembers that he did not close a tap in in his garden. He decided to turn around immediately and return home to close the tap.

The cost of the water, at the rate at which water is flowing out of the tap, is R1,60 per hour. The cost of petrol is rands per km, where is the average speed in km/h. Calculate the average speed which the person must travel home to keep his cost as low as possible [7]

The study focuses on Grade 11 learners. However, the inclusion of Grade 12 performance data serves to highlight that learners often progress to Grade 12 while still experiencing difficulties with problems involving real-life contexts. In particular, it is evident that optimisation problems are not straightforward for many learners and continue to present challenges at this level.



Q	Topics
1	Equations, Inequalities and Algebraic Manipulation
2	Number Patterns and Sequences
3	Number Patterns and Sequences
4	Functions and Graphs
5	Functions and Graphs
6	Functions and Graphs
7	Finance
8	Calculus
9	Calculus
10	Probability and Counting

Figure 1.1 Average performance per question in Paper 1

Source: Department of Basic Education (2022)

The strong link and interdependence between mathematical modelling and the application of calculus informed the inclusion and discussion of calculus applications in this study. In most cases, both mathematical modelling and calculus application problems are presented in verbal form. To solve such problems, learners must first read and understand the given context before translating it into a mathematical representation. These initial stages correspond closely to the first two phases of mathematical modelling, namely problem comprehension and mathematisation (the process of interpreting a situation mathematically). Failure to interpret the problem correctly often results in an incorrect mathematical model and, consequently, an incorrect solution. Once the mathematical representation has been constructed, calculus techniques are then used to solve the resulting equations, often through differential calculus to determine maximum or minimum values. For example, learners may be required to use derivatives to determine maximum areas in optimisation problems. Mudaly and Narriadoo (2024) emphasise that both mathematical modelling and calculus application depend heavily on learners' ability to read and comprehend word problems. However, their research also shows that word problems remain a significant challenge for learners in general. In the South African context, this challenge is further compounded by the multilingual nature of the education system, where learners are taught and assessed in multiple officially recognised languages.

The inability to comprehend word problems is therefore a major concern in both mathematical modelling and Mathematics learning more broadly. The importance of understanding word problems is reflected in the aims of the current South African Mathematics curriculum, which seeks to develop learners who can:

• identify and solve problems and make decisions using critical and creative thinking
• communicate effectively using visual, symbolic and/or language skills in various modes (Department of Basic Education, 2011)

Despite these intentions, evidence shows that learners continue to struggle with solving word problems. Teachers can support improvement in this area by exposing learners to a wide variety of non-routine problems. In this regard, Pongsakdi et al. (2016) conducted a study involving elementary school learners (98 in the experimental group and 72 in the control group) to determine whether Word Problem Enrichment (WPE) could improve learners' problem-solving skills compared to traditional Mathematics instruction. The findings revealed that learners in the WPE group performed better in solving non-routine and application-based problems than those in the control group. This study supports the view that structured exposure to word problem strategies can significantly enhance learners' problem-solving performance and should be integrated into Mathematics instruction and textbooks.

This study therefore sought to identify the difficulties faced by secondary school learners when solving non-routine, real-life problems. Such problems are closely aligned with those used in mathematical modelling. In Grade 12, these types of questions are typically found in the application of calculus, where learners are often taught specific procedures to follow when solving optimisation problems. This is one of the reasons Grade 11 learners were selected for this study, as they have not yet been formally introduced to differential calculus and are therefore required to develop their own approaches to problem-solving.

Although Grade 12 learners are taught procedural methods for solving real-life optimisation problems, their performance in this area remains poor. Table 1.4 presents the average performance in optimisation questions over several years (DBE, 2014, 2015, 2016, 2017, 2018b, 2019), based on National Senior Certificate diagnostic reports. The lowest performance was recorded in 2017, with an average achievement of 9%, while the highest performance occurred in 2019 at 39%. Over the six-year period, learners' achievement in optimisation questions never exceeded 50%. These findings suggest that learners continue to struggle significantly with this topic, indicating the need for targeted interventions to improve performance in mathematical modelling and related calculus applications.

Table 1.5: Average achievement in subtopics of Calculus

Average learners' achievement in percentage for the past 6 years in questions involving Application of Calculus						
YEAR	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**
Average Performance in %	32	22	38	9	18	39

Source: DBE (2014, 2015, 2016, 2017, 2018b, 2019a)

As shown in Table 5, learners' performance in questions involving real-life problems remains a concern. Although there was a slight improvement in 2019, overall achievement in these types of questions did not exceed 40% over the six-year period. This persistent underperformance motivated an interest in exploring mathematical modelling as an alternative approach to solving non-routine, real-life problems. It is therefore argued that if learners are encouraged to engage with real-life problems without relying solely on procedural steps, their problem-solving skills may be enhanced.

1.5 Rationale for the study

The use of mathematical modelling in South African schools is still relatively new compared to countries such as the Netherlands and Germany (Vos, 2015; Greefrath et al., 2016). It was only after the revision of the curriculum, with the introduction of the National Curriculum Statement (NCS) about a decade ago, that mathematical modelling and the integration of real-life contexts into Mathematics teaching were explicitly emphasised (Mosimege, 2017). This development motivated the present study, which seeks to determine learners' competencies in mathematical modelling, now regarded as a central feature of Mathematics education. Furthermore, learners' persistent poor performance in questions requiring the application of differential calculus provided additional motivation to explore alternative approaches to solving real-life problems, particularly those involving maximisation and minimisation without exclusive reliance on procedural calculus techniques. In this study, mathematical modelling is understood as the translation of real-world situations into mathematical representations. Bliss et al. (2016) define mathematical modelling as the use of Mathematics to represent, analyse, and predict aspects of real-life situations, emphasising the relationship between mathematical descriptions and the real-world context as a key feature of modelling.

Since the incorporation of mathematical modelling into the South African school curriculum in 2003, many teachers who were already in service at the time may not have received formal training in this area. Mabena et al. (2021) note that teachers with limited subject matter competence often struggle to effectively present content, which in turn negatively affects learners' performance. The lack of adequate training in mathematical modelling during pre-service teacher education further motivated this study. Several studies have shown that learners' academic performance is significantly influenced by teacher competence, with the teacher playing a central role in implementing curriculum reforms at classroom level (Shepherd, 2019; Theophile et al., 2020). It is therefore important to investigate learners' competencies in mathematical modelling within this context.

Several studies have shown that when teachers consistently use problem-solving approaches in Mathematics instruction, learners' performance improves (Albay, 2019; Ortiz, 2016; Yuan, 2013). Similarly, research indicates that mathematical modelling enhances learners' achievement in Mathematics (Sokolowski, 2015; Ciltas & Isik, 2013; Nguyen, 2016). Although mathematical modelling is closely related to problem-solving, its distinctive feature is the emphasis on real-life contexts. In this study, modelling was explored through algebraic problem-solving tasks. It is hoped that the findings will encourage teachers to incorporate mathematical modelling more intentionally into their classroom practice.

This study also examined learners' proficiency in Algebra. Algebra forms a foundational component across many areas of Mathematics. For the purposes of this study, the focus was narrowed to questions involving perimeter and area. This choice was informed by the fact that optimisation questions in Grade 12 commonly involve quantities such as area, perimeter, volume, and surface area. The intention was to expose Grade 11 learners to similar problem types before they encounter formal optimisation in Grade 12, while also demonstrating that such problems can be solved using methods other than differential calculus.

When learners are required to model measurement-based problems, a strong foundation in Algebra is essential for formulating equations. Mastery of Algebra is therefore critical in Mathematics and mathematical modelling, as it provides the language used to represent real-world situations such as population growth, measurement, motion, cost, flow rates, and relationships between variables. Since modelling problems are typically presented in word format, learners must first translate verbal descriptions into mathematical expressions or equations. This involves assigning variables to unknown quantities and defining relationships between them. In problems involving perimeter, area, and volume, learners must correctly assign variables to the relevant dimensions based on the geometric figure in question.

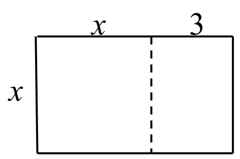
Example 1

Mr Sabina wants to plant seedlings in his rectangular garden with an area of 54 m^2. The length of the garden is longer than its width by 3m. What are the dimensions of his garden?

Solution

Since this problem involve real life problem, one must make assumptions. In real life, it is possible that a portion of this area is swampy or is covered by rocks. If this is the case, this means the arable land will be less than the 54 m^2. Most real-world problems do have more than one solution, largely dependent on the fundamental assumptions made by of those attempting to solve them. To solve this problem simpler, I have assumed that the land is flat, no large rocks and all the available land is suitable for growing vegetables.

Algebraically, the dimensions of the garden can be calculated by first making a sketch using the given information. The sketch of the rectangular field is shown below:



$$\text{Let the width be } x$$
$$\therefore \text{The length} = x + 3$$

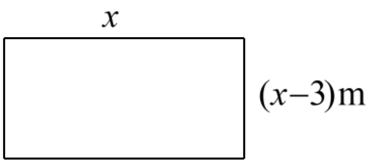
Formula for the area of a rectangle = length $\times$ width. It is given that the area is 54 m^2.

$\therefore Area = (x+3)(x) = 54$

$$x^2 + 3x = 54$$
$$x^2 + 3x - 54 = 0$$
$$(x+9)(x-6) = 0$$
$$x = -9 \text{ or } x = 6$$

Since we are solving the real-life situation, we only consider the positive value of x. Therefore $x = 6$ is the only solution. This means that the width is 6 m. Given that the length is longer than its width by 3m, this means to get the length one must add 3m to the already calculated width. Therefore, the length is 6m+3m= 9 metres. The answer or solution must be stated clearly. To verify the solutions, one must multiply the length and breadth, which is the way of calculating an area of a rectangular figure. If the calculated dimensions are correct, they must give an area of $54m^2$. The area of the garden = $9m \times 6m = 54m^2$ which suggest that the obtained solutions are correct

Similarly, the same questions can be resolved by assuming x to be the length. This time the width will be given as $(x-3)$ m. The diagram below represents this new situation.



Let the length be x m

$\therefore$ the width = $(x - 3)$ m

$\therefore$ Area $= x(x-3) = x^2 - 3x$

$\therefore x^2 - 3x = 54$

$$x^2 - 3x - 54 = 0$$
$$(x-9)(x+6) = 0$$

$\therefore x = 9$ or $x = -6$

Again, the value of x to be considered is the positive value because the value of the distance cannot be negative. Therefore, the value of the length = 9m. Since we were given that the length of the garden is longer than its width by 3m, this means the width= 9 – 3= 6m. It will be easier for learners to understand Mathematics if they are using their own methods and not adhering to the procedures supplied to them by a teacher. Relying mainly on procedures taught to them by a teacher may have a detrimental effect on learners' achievement in Mathematics, especially in the cases where a learner has forgotten the steps to be followed in resolving a problem. Also, when learners are depending on algorithms taught to them, we cannot be sure that they are developing the necessary understanding of Mathematics. If they can use an investigative method in form of a table, possible length and width can be calculated as follows:

Width (w)	**Length (l)**	**Area (A)**
$1m$	$4m$	$1m \times 4m = 4m^2$
$2m$	$5m$	$2m \times 5m = 10m^2$
$3m$	$6m$	$3m \times 6m = 18m^2$
$4m$	$7m$	$4m \times 7m = 28m^2$
$5m$	$8m$	$5m \times 8m = 40m^2$
$6m$	$9m$	$6m \times 9m = 54m^2$
$7m$	$10m$	$7m \times 10m = 70m^2$

From the table above, it can be observed that an area of 54 m^2 is obtained when the length is 9 m and the width is 6 m. When the width increases beyond 6 m, for example to 7 m, the resulting area becomes greater than 54 m^2.

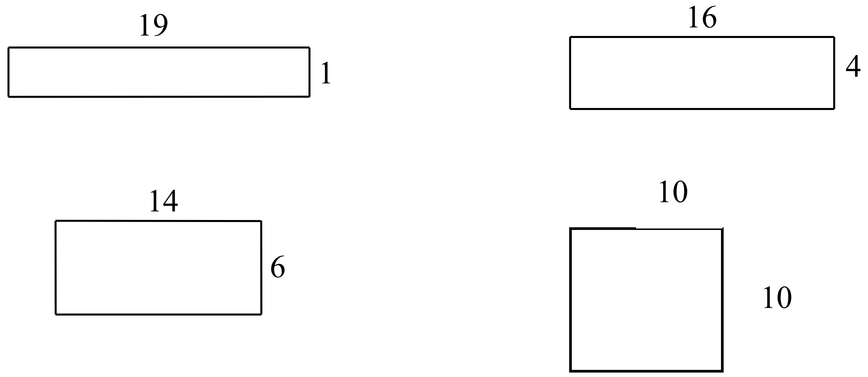
Some mathematical modelling problems may require the use of tables as part of the solution process. In such cases, it is also important to include a sketch to represent the given information visually. This approach can help learners develop a deeper understanding of problem-solving strategies, rather than relying solely on procedural methods that may be easily forgotten. However, applying this method during tests or examinations may be time-consuming. For this reason, it is particularly useful as an introductory strategy when teaching optimisation. Once learners have been introduced to differential calculus in Grade 12, the same problems can be revisited and solved using calculus methods to verify whether the solutions obtained are consistent with those derived from the table-based approach.

Example 2

A fence that is 40 metres long has to enclose a rectangular garden. Determine the dimensions of the garden that will give the maximum area.

Solution

In such a case, it is wise to draw a table and list possible dimensions that can be made from a fence that is 40m long. Numerous sketches can be made but just to show how a fence that is 4om long can be utilised to fence a garden, only four have been shown.



Length(m)	Breadth(m)	Area(m^2)
19	1	$19 \times 1 = 19$
18	2	$18 \times 2 = 36$
17	3	$17 \times 3 = 51$
16	4	$16 \times 4 = 64$
15	5	$15 \times 5 = 75$
14	6	$14 \times 6 = 84$
13	7	$13 \times 7 = 91$
12	8	$12 \times 8 = 96$
11	9	$11 \times 9 = 99$
10	10	$10 \times 10 = 100$
9	11	$9 \times 11 = 99$
8	12	$8 \times 12 = 96$
7	13	$7 \times 13 = 91$

From the table, the maximum area can be obtained when both the breadth and width are equal to 10 metres. The maximum area occurs when the garden is a square in terms of shape. Therefore, the maximum area = 10m $\times$ 10m = 100m^2

If the same question was given to the Grade Twelve learners, they were going to use the application of calculus. One possible method would be:

$P = 2l + 2b$ ……... (1)

Given that the perimeter is 40, they will substitute P by 40 and get $2l + 2b = 40$, which simplifies to $l + b = 20$ …... (2)

The next step would be to make l the subject of the formula: $l = 20 - b$ ……. (3)

Area of the rectangle = length $\times$ breadth

$A = (20 - b)(b)$

$A = 20b - b^2$

To find the maximum area, they were going to differentiate A with respect to b to get:

$A' = 20 - 2b$. Then equating the derivative to zero gives :

$20 - 2b = 0$

$20 = 2b$

$b = 10$

From equation (3), $l = 20 - b$

$l = 20 - 10$

$l = 10$

Therefore, the length is 10 metres, and the breadth is also 10 metres. This means the maximum area will be obtained at these dimensions. The maximum area = 10m $\times$ 10m = 100m^2. Both methods gave the same answer.

1.6 Objectives and aims

The aims of the study were to:

(a) Establish learners' mathematical modelling competencies in algebraic problem solving.

(b) Assess the Grade 11 learners' competencies in applying mathematical modelling process: understanding the problem, formulating a model, solving a mathematical model, interpretation and verification

Research question:

- What is the nature of competencies of Grade Eleven learners in mathematical modelling when resolving real world problems?

Research sub-questions

- What kinds of success do Grade Eleven learners in have solving non-routine realistic and practical problems using the mathematical modelling approach?
- What kinds of challenges do Grade Eleven learners experience when solving non-routine realistic and practical problems using the mathematical modelling approach?
- Are learners able to apply mathematical modelling procedures?

Both problem-solving and mathematical modelling are important components of Mathematics instruction and learning in South Africa. This is reflected in the following specific aims of the Mathematics curriculum:

- Mathematical modelling is an essential feature of the curriculum. Wherever possible, real-life problems should be incorporated into teaching and learning. The contexts used in modelling tasks should be realistic and naturally occurring rather than artificially constructed. These problems should, where appropriate, reflect social, health, environmental, economic, political, scientific, and cultural contexts (Department of Basic Education, 2011:8).

- The curriculum also aims to promote problem-solving and the development of cognitive skills. Learners should not only focus on the "how" of solving problems, but also the "when" and "why." Teaching should therefore go beyond procedures and include conceptual understanding and reasoning, including the use of proofs. Failure to support learners in understanding the purpose and application of procedures may result in superficial learning and inadequate preparation for future use of knowledge (DBE, 2011:8).

As indicated above, mathematical modelling and problem-solving are central features of the South African Mathematics curriculum. Considering this, the present study investigated Grade 11 learners' competencies in solving non-routine problems using a mathematical modelling approach. The study was guided by the need to explore the following:

- learners' ability to apply mathematical modelling in solving real-life problems;
- learners' understanding of the mathematical modelling process;
- challenges learners experience when using mathematical modelling to solve real-life or non-routine Mathematics problems; and
- barriers learners face when translating real-life situations into mathematical models.

1.6.1 Research questions

This research investigated learners' competencies in mathematical modelling to address the following research question:

- What is the nature of competencies of Grade Eleven learners in mathematical modelling when resolving real world problems?

1.6.2 Sub-questions

To unpack the main research question, the following sub-questions were formulated:

- What kinds of success do Grade Eleven learners in have solving non-routine realistic and practical problems using the mathematical modelling approach?
- What kinds of challenges do Grade Eleven learners experience when solving non-routine realistic and practical problems using the mathematical modelling approach?
- Are learners able to apply mathematical modelling procedures?

1.7 Significance of the study

This study may contribute to improving learners' performance in Mathematics, particularly in optimisation and real-life problem-solving. This is because, when learners reach Grade 12, they should ideally be able to solve optimisation problems using algebraic reasoning in addition to, and not solely dependent on, differential calculus. Özer-Demir and Bukova-Güzel (2024) support this view by arguing that mathematical modelling has a positive impact on learners' achievement, as it enables them to apply mathematical knowledge to real-world contexts. In this way, learners can recognise the relevance and usefulness of Mathematics in their everyday lives.

The findings of this study may provide insights into learners' low levels of competence in mathematical modelling when solving measurement-related word problems aimed at determining maximum areas. Learners' difficulty may be linked to limited exposure to the full mathematical modelling cycle. Since mathematical modelling competence is assessed through learners' ability to engage in this cyclical process, lack of familiarity with it may hinder performance. Open-ended modelling tasks require learners to translate real-world situations into mathematical forms, solve the resulting mathematical problems, and interpret the solutions in relation to the original context.

In addition, learners may struggle to evaluate whether their mathematical solutions are reasonable in the real-world context or to express final answers using appropriate units. These difficulties further highlight gaps in learners' understanding of the modelling process. The results of this study may also shed light on whether language presents a barrier to mathematical modelling. Since modelling tasks are typically presented in word-problem format, language comprehension plays a crucial role. Peter (2016) notes that learning Mathematics in a second or foreign language presents significant linguistic, cognitive, and pedagogical challenges that can negatively affect learners' understanding and performance. In South African secondary schools, most learners learn Mathematics in English. As a result, some learners may struggle to interpret word problems accurately, which can lead to incorrect mathematical representations and flawed solutions.

The Department of Basic Education may also benefit from this study, as it provides insight into learners' competencies in solving real-life and practical Mathematics problems. The study further highlights the importance of developing Grade 11 learners' conceptual understanding of mathematical modelling in preparation for Grade 12. The findings may therefore assist the Department in designing targeted interventions to support learners earlier in their schooling, ensuring better preparedness for higher-level Mathematics.

As noted earlier, one of the key aims of the Mathematics curriculum is to prepare learners for further education and the workplace (DBE, 2011). There is therefore a need to strengthen learners' exposure to mathematical modelling while they are still at school, particularly for those intending to pursue Mathematics at tertiary level. Early engagement with modelling tasks can help learners develop a deeper appreciation of Mathematics and its real-world applications, thereby improving their motivation and interest in the subject.

Mathematical modelling is also offered at higher education institutions. For example, at the University of South Africa, students enrolled for a Bachelor of Science degree with Mathematics as a major take the module Mathematical Modelling (APM1514) in their first year. This module is designed to develop students' understanding of basic optimisation and applications. Learners who participated in this study are therefore expected to be better prepared for such modules if they are exposed to mathematical modelling at school level.

However, learners' achievement in Mathematics is often below expectations, partly because they perceive Mathematics as abstract. A key goal of mathematical modelling is therefore to make Mathematics more meaningful, engaging, and relevant by connecting it to real-life situations. Considerable effort is required to promote positive attitudes towards Mathematics from the lower grades; otherwise, learners may develop negative perceptions of the subject. It is hoped that this study will contribute towards improving learners' attitudes and, in turn, enhancing their performance in Mathematics.

1.8 Key concepts, terms and phrases used in this study

The terminology used in this research is defined in this section. These terms are further elaborated on in the study.

1.8.1 Constructivism

This is a learning theory which proposes that learners construct their own knowledge. It holds that individuals actively build or create subjective representations of reality. New information is therefore understood by linking it to prior knowledge.

1.8.2 Curriculum and Assessment Policy Statements (CAPS)

This refers to a single comprehensive document that specifies the content to be taught in all subjects outlined in the NCS Grades R–12. This document has replaced previously used documents such as Learning Area Statements, Subject Assessment Guidelines, and Learning Area Programmes.

1.8.3 Department of Basic Education (DBE)

This is the name given to the Department of Education in South Africa that is responsible for primary and secondary education. It was established in 2009 following the division of the former Department of Education into two entities, namely the Department of Basic Education (DBE) and the Department of Higher Education and Training (DHET).

1.8.4 Mathematisation

This is the process by which a real model is converted to a mathematical model.

1.8.5 Mathematical modelling

This is a process in which real-life situations and relationships within those situations are represented using mathematical language and concepts.

1.8.6 National Curriculum Statement

The National Curriculum Statement (NCS) is the South African curriculum framework that aims to ensure that learners acquire and apply knowledge and skills in ways that are meaningful to their own lives.

1.8.7 National Senior Certificate

The National Senior Certificate (NSC) is the highest qualification awarded after the completion of twelve years of formal schooling. In South Africa, it is commonly referred to as the matriculation (matric) certificate.

1.9 Thesis alignment

This section outlines the structure of the thesis, indicating how it is organised and summarising the content of each chapter.

Chapter 1: Introduction and Background

Chapter 1 provided an overview of the study. In this chapter, the researcher outlined the research design, background, problem statement, purpose of the study, objectives, research questions, significance of the study, scope, key concepts, and the structure of the thesis. In other words, it clearly defined what the study was about, why it was conducted, and who the intended beneficiaries were.

Chapter 2: Literature Review

In this chapter, various perspectives and viewpoints were discussed to help explain the phenomenon under investigation. The overall structure and focus of the study were also outlined. In addition, key concepts such as mathematical modelling, its advantages and disadvantages in classroom practice, and the skills required for its effective implementation were explained.

Chapter Three: Theoretical Framework

This chapter presented the theoretical and conceptual framework underpinning the study. Constructivism and the strands of mathematical learning were discussed as the main theoretical foundations. In addition, various learning theories, mathematical modelling processes, modelling competencies, and problem-solving models were also examined.

Chapter Four: Research Methodology and Design

This chapter outlined the research methodology and the approaches used to conduct the study. It explained how the research was carried out, including the research paradigm, design, site, population, and sample. Issues of trustworthiness and ethical considerations were also discussed in this chapter.

Chapter Five: Data Collection Technique, Analysis and Interpretation

This chapter described the methods used for data collection and outlined the strategies and procedures followed in gathering the data. It also addressed the classification, organisation, and interpretation of the collected information. Furthermore, the chapter explained how the data were processed, presented, and reported. The analysis involved examining and evaluating the findings in order to draw meaningful conclusions.

Chapter Six: Recommendations, Conclusions and Limitations of the Study

This chapter provided a summary of the study's findings and outlined the key outcomes. It also presented suggestions for improving the research and offered recommendations for future researchers to refine and extend the study. In addition, the chapter highlighted potential limitations and challenges that may have affected the attainment of the intended results, including constraints such as time, participant responses, and possible shortcomings in the sampling methods or research procedures used.

1.10 Conclusion

This chapter provided an introduction and orientation to the study. It outlined the background, problem statement, purpose, and significance of the research. It is hoped that the findings and recommendations will contribute to the development of learners' competencies in mathematical modelling and foster positive attitudes towards Mathematics. The next chapter presents the literature reviewed for the study. It provides a comprehensive overview of previous research related to the teaching of Mathematics in South Africa, different definitions of mathematical modelling, and the advantages and challenges associated with its implementation. It also explores the role of creativity and critical thinking in mathematical modelling. Furthermore, the chapter examines problem-solving, including the distinction between routine and non-routine approaches, and highlights the importance of active learning in both problem-solving and mathematical modelling. It also discusses metacognition and its significance and concludes with examples of countries that effectively incorporate mathematical modelling in their curricula.

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CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

This chapter provides a comprehensive review of literature on mathematical modelling and competencies in algebraic problem-solving. It begins by outlining the goals of Mathematics education in South Africa and what these aims seek to achieve. The chapter further explores definitions of mathematical modelling, the modelling process, and the distinction between mathematical modelling and applied Mathematics. It also examines different models used in mathematical modelling, as well as the advantages and disadvantages of implementing mathematical modelling in practice, including strategies for addressing associated challenges. In addition, the chapter discusses the importance of critical thinking, creativity, communication, teamwork, metacognition, and problem-solving within mathematical modelling. Examples of countries that successfully implement mathematical modelling in their curricula are also presented. The chapter concludes with a summary of the key ideas discussed.

2.2 Aims of teaching Mathematics in South Africa

The CAPS document outlines the qualities of learners that should emerge after being exposed to the Mathematics curriculum. Among its key objectives are the development of critical thinking, logical reasoning, and problem-solving skills, which are essential for functioning effectively in the modern world. The Mathematics curriculum therefore aims to produce learners who can solve problems, making informed decisions, collaborating with others, and applying critical and creative thinking. In addition, learners are expected to communicate effectively using language, symbols, and visual representations, as well as to collect, analyse, organise, and critically evaluate data. They should also be able to apply science and technology in ways that promote safety, well-being, and societal benefit (DBE, 2011a).

However, these intended outcomes are unlikely to be achieved if teachers continue to rely on traditional instructional approaches. Bhardwaj (2025) notes that although teacher-centred methods are effective for transmitting basic knowledge, they are often criticised for limiting the development of creativity, critical thinking, and problem-solving skills. In contrast, learner-centred approaches emphasise active participation, thereby promoting the development of skills needed to address real-world challenges.

In traditional classrooms, the teacher is regarded as the expert and primary source of knowledge, while learners are expected to listen, memorise, and reproduce information. To develop the competencies outlined in the curriculum, educators therefore need to adopt pedagogical approaches that position learners as active participants in the learning process. Envisaged learner outcomes cannot be achieved through rote learning. Pliusheh et al. (2022) argue that conventional approaches often fail to engage learners actively, leading to disengagement and limiting the development of higher-order thinking skills. In contrast, Muniandy et al. (2023) highlight that contemporary approaches such as collaborative learning and project-based learning promote active engagement and foster critical thinking, creativity, and problem-solving abilities.

One instructional approach that supports the development of these competencies is mathematical modelling, as introduced in Chapter One and further elaborated in Section 2.3. Mathematical modelling requires the use of real-life contexts in classroom problem-solving. Learners are expected to represent information using graphs, tables, symbols, and verbal descriptions (DBE, 2011a). These skills are fundamental to modelling, as learners must translate real-world situations into mathematical expressions and equations. It is therefore essential that mathematical modelling be emphasised across all grade levels to ensure that learners develop proficiency as they progress through the schooling system.

Suh et al. (2017) investigated how two elementary teachers implemented mathematical modelling in their classrooms and how this influenced learner engagement and the development of creativity, critical thinking, teamwork, and communication skills. Learners were required to determine how to collect relevant information, identify variables, and decide which elements of the situation would change or remain constant in constructing their models. The findings of Suh et al. (2017) showed that critical thinking and problem-solving skills were enhanced as learners explained their reasoning, justified their decisions, and evaluated possible solutions. Creativity was fostered through individual reflection, whole-class discussions, and small-group collaboration. The study also highlighted that mathematical modelling encourages collaboration, as learners must work together to develop and refine solutions. These findings are consistent with Bliss et al. (2016) in the GAIMME report, which emphasises that mathematical modelling develops creativity, critical thinking, communication, and teamwork. Mathematical modelling is therefore an effective pedagogical approach that connects Mathematics to real-life situations. It encourages learners to begin with authentic contexts, make appropriate assumptions, and represent situations mathematically using equations, graphs, and other forms of representation. After solving the mathematical problem, learners interpret and validate their results within the original real-world context, thereby strengthening understanding and engagement.

This approach aligns with learner-centred pedagogy, where learners actively construct knowledge through exploration and collaboration, while teachers act as facilitators of learning. For Grade 11 learners in particular, mathematical modelling is valuable as it bridges the gap between abstract theory and practical application, thereby enhancing conceptual understanding and promoting critical thinking. By engaging with real-world problems, learners are also able to see the relevance of Mathematics in everyday life, while simultaneously developing communication, collaboration, and problem-solving skills.

2.3 What is mathematical modelling?

There is no single universally accepted definition of mathematical modelling, as different scholars describe it in varying ways. Asempapa (2018) notes that although definitions differ, they consistently present mathematical modelling as a process in which a real-world situation is identified, assumptions and decisions are made, and a mathematical representation is developed to generate a solution that can be interpreted in the real-world context.

Similarly, Arseven (2015) describes mathematical modelling as the application of Mathematics to explain and interpret real-world phenomena, test ideas, and make predictions. Stohlmann and Albarracin (2016) view it as an iterative process involving open-ended, authentic problems through which learners connect Mathematics to real-life contexts by making assumptions, estimations, and using multiple representations to construct meaning.

Taile and Dinapoli (2025) further define mathematical modelling as a cyclical process in which Mathematics is used to represent, investigate, and deepen understanding of real-world situations by formulating a problem mathematically and validating the results. Unlike traditional word problems, modelling tasks require learners to identify relevant quantities, apply mathematical procedures, make assumptions, interpret outcomes, and refine solutions within authentic contexts.

In addition, Taile and Dinapoli (2025) explain mathematisation as the process of translating real-world situations into mathematical structures. This process requires strong algebraic competencies, including identifying relationships between variables, formulating equations or diagrams, and selecting appropriate representations or tools. After solving the model, learners must critically evaluate whether the results are meaningful and interpret them within the original context.

In simple terms, mathematical modelling can be understood as the process of using mathematical objects such as equations, graphs, and diagrams to represent real-life situations. In classroom practice, learners are presented with real-world problems, which they must translate into mathematical form, solve, and then interpret back into the original context.

Haines et al. (2007) note that the growing importance of mathematical modelling contributed to the establishment of the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) in 1983. This organisation brings together academics and Mathematics educators who promote research, teaching, and application of mathematical modelling through international conferences and publications. According to Haines et al. (2007), ICTMA focuses on the exploration, research, teaching, implementation, and promotion of mathematical modelling across educational levels, from schools to universities and workplaces. The organisation has held biennial conferences since 1983, with proceedings published in various academic outlets.

Bliss et al. (2016) define mathematical modelling as "the utilisation of Mathematics for the representation, analysis, prediction, and other aspects of real-life events" (p. 8). They further argue that most definitions emphasise the relationship between real-world problems and their mathematical representation. Common elements include the use of mathematical language to quantify and analyse real-world phenomena, the application of Mathematics to understand authentic problems, and the iterative nature of modelling as an ongoing problem-solving process.

Neumaier (2003) similarly describes mathematical modelling as the process of transforming real-world problems into mathematical structures, whose analysis leads to understanding, solutions, and practical guidance. Stohlmann and Albarracin (2016) also emphasise its iterative and authentic nature, highlighting that learners use assumptions and multiple representations to make sense of real-world problems through Mathematics.

Cheng (2001) distinguishes between teaching learners to construct mathematical models and teaching mathematical modelling as a process. He argues that while model construction often focuses on obtaining correct answers, mathematical modelling emphasises developing appropriate representations of real-world situations. In modelling, learners begin with a real problem and proceed step by step towards a solution, continuously interpreting and refining their work.

Importantly, solving a word problem does not necessarily constitute mathematical modelling. English (2003) explains that earlier approaches to Mathematics education often focused on routine word problems that required direct translation into arithmetic or algebraic expressions based on key linguistic cues. For example, in a problem such as "Suzzie put $12 in her savings while Lillian banked three times that amount. How much did Lillian save?", learners typically apply a straightforward multiplication operation.

English (2003) argues that such tasks are not modelling activities because they involve a single, predetermined method of solution and rely heavily on recognising linguistic triggers such as "times," "less," or "twice." While these problems are useful for basic skill development, they do not adequately promote higher-order mathematical reasoning or authentic modelling competencies. The role of language in mathematical modelling is further discussed in Section 2.8.

Spandaw and Zwaneveld (2010) observe that many Mathematics teachers lack experience in modelling or allocate insufficient time to its teaching. They therefore emphasise the importance of integrating modelling into the Mathematics curriculum. They define modelling as the application of Mathematics to non-mathematical, real-world problems, often used in fields such as science and engineering to analyse variable quantities and relationships. They further illustrate that real-life situations can generate meaningful modelling problems, such as determining whether it is worth driving to a neighbouring country to purchase cheaper fuel, considering additional costs such as travel expenses. They also emphasise that different models may emerge from the same situation depending on the question being asked. For instance, a steel ball may be used to model different phenomena such as a sphere, a conductor, or a gas particle, depending on the context.

In practice, mathematical modelling requires guidance, especially for learners who are new to the approach. Just as directions are necessary for reaching an unfamiliar destination, structured guidance helps learners navigate the modelling process and avoid unnecessary errors. The next section therefore discusses the mathematical modelling process as a sequence of steps used to solve real-world problems systematically.

2.4 Mathematical modelling process

In most cases, the mathematical modelling process is presented graphically in the form of flow charts that outline the steps to be followed. Different researchers have proposed various modelling cycles for use in research and classroom practice. According to Perrenet and Zwaneveld (2012), the modelling process can be described as a series of cyclical actions that begin with a real-world problem situation and end with a mathematical solution, which is then interpreted back in the real-world context. They further note that this cycle has undergone several refinements and modifications over time.

Some modelling cycles are presented in earlier work such as Houston (2007). Perrenet and Zwaneveld (2012) explain that many of these variations are derived from the foundational models developed by Kaiser (1995) and Blum (1996). Despite differences in structure, most modelling cycles share four core steps, as illustrated in Figure 2.2. However, some models include additional stages. For example, the modelling process used in this study consists of five steps and is adapted from Kaiser and Stender (2013), while Blum and Leiß's (2007) seven-step modelling cycle, which captures the key components of the modelling process, is presented in Figure 2.1.

Typically, mathematical modelling begins with a real or practical situation, which may involve observed conditions in a laboratory or everyday contexts such as grasslands, forests, factories, or other familiar environments. Once the situation has been identified, the next step is to formulate the problem more precisely and unambiguously. This stage involves selecting and defining the key concepts relevant to the situation, which must be clearly described and carefully justified. It is also at this point that assumptions and approximations are made in order to simplify the problem and make it mathematically tractable (Perrenet and Zwaneveld, 2012).

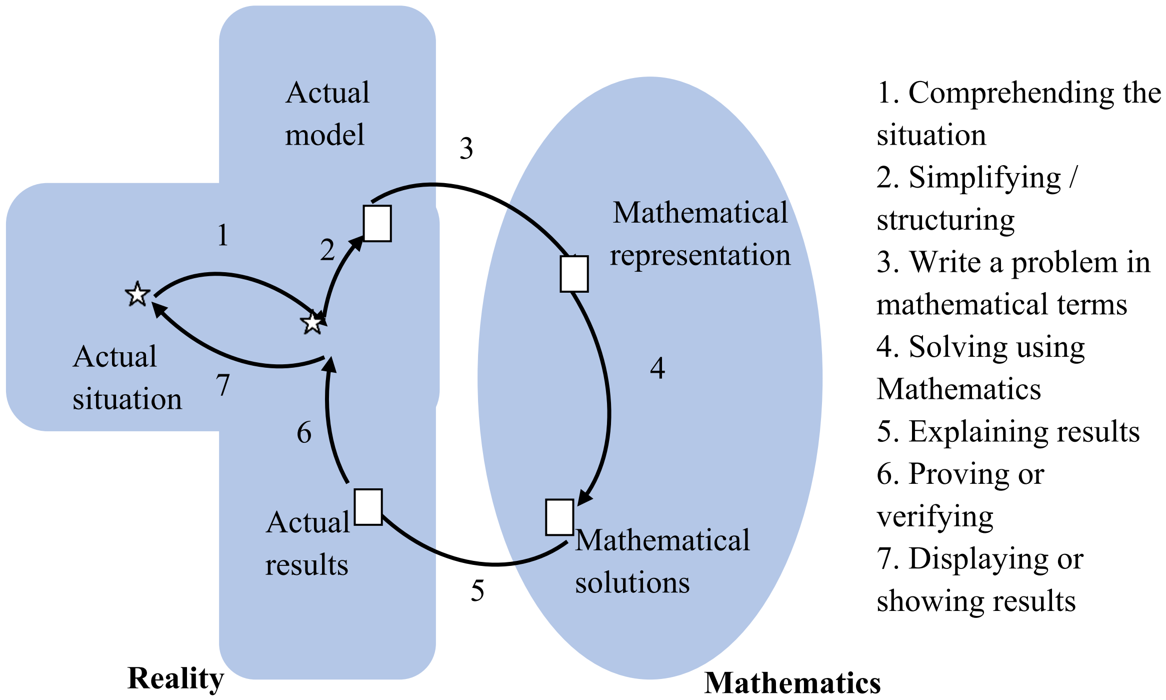

Figure 2.1: Modelling process consisting of seven steps

Source: Blum & Leiß (2007)

The third step involves a high level of creativity, as the model must be carefully analysed and the relevant processes identified for use in solving the problem. The main purpose of this stage is to translate the real-world situation into mathematical form. Consequently, the situation is represented using mathematical symbols, where real quantities and processes are expressed through equations, functions, formulas, expressions, and other mathematical relationships.

The final stage of the modelling process involves interpreting and validating the results by comparing the mathematical outcomes with the real-world situation. At this point, the solution is assessed in relation to the original context, and any predictions made by the model are checked against actual observations or expected behaviour. In many cases, a thorough evaluation is necessary to determine whether the model's predictions are consistent with reality and whether the assumptions made remain valid.

In summary, the modelling process can be outlined as follows: (a) describing the real-world situation or problem, (b) formulating a mathematical question or model, (c) solving the mathematical problem, and (d) interpreting and validating the results by comparing them with the real-world context.

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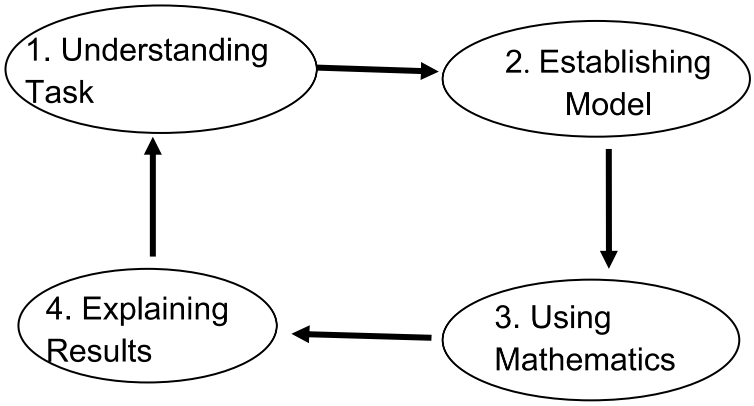


Figure 2.2: Four steps to solve a modelling task

Source: Blum & Ferri (2009)

Kaiser (2005) presented a mathematical modelling cycle that is particularly relevant to modelling in school contexts. In describing this process, the cycle begins with a real-life situation or problem, which serves as the starting point of the modelling activity. Such a problem is regarded as an ideal or authentic representation of an everyday challenge, as indicated in Figure 2.3(a).

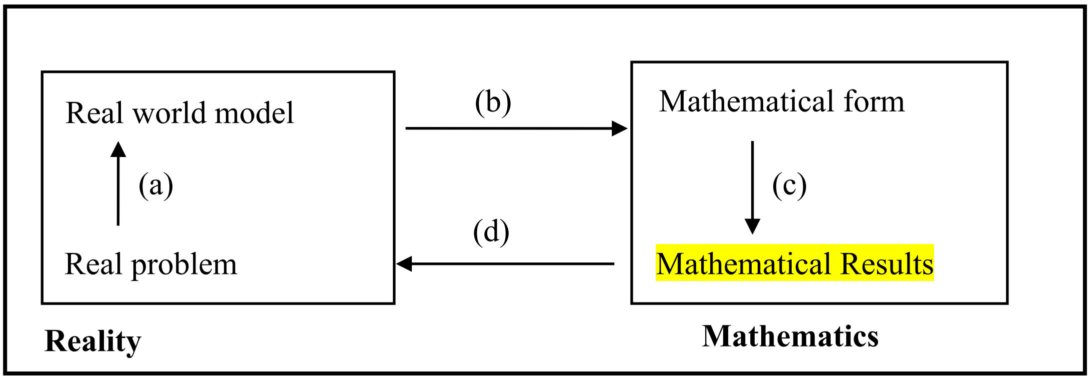


Figure 2.3: Modelling process based on ideal typical procedure

Source: Kaiser (2005)

The real-life problem is then translated into mathematical form, referred to as step (b) in the cycle. According to Blum and Ferri (2009), as cited in Yilmaz et al. (2016), mathematisation is the process of converting a real-world situation into a mathematical model, often expressed through equations or mathematical structures.

This translation results in a mathematical representation of the original problem, such as an equation or set of expressions. Mathematical results (step c) are then generated through mathematical reasoning and procedures applied to these representations. These solutions must subsequently be interpreted in relation to the real-world context, as indicated in step (d). At this stage, the reasonableness and applicability of the solution are assessed. If the solution is found to be inadequate or unrealistic, the modelling process is revisited and refined accordingly (Kaiser, 2004).

Another modelling framework that outlines the steps involved in solving mathematical modelling problems is that of Blomhøj and Højgaard (2003). They propose a six-step model that can also be used to assess competence in mathematical modelling. The steps are as follows:

(a) Clearly formulating a task, which involves identifying key features of the real-life situation to be modelled.

(b) Selecting relevant elements and relationships from the real-world context and idealising them so that the situation can be represented mathematically.

(c) Translating the information from its original form into mathematical representations.

(d) Applying mathematical methods and procedures to obtain solutions and make reasoned conclusions.

(e) Interpreting the results in relation to the original real-world context.

(f) Evaluating the validity and reliability of the model by comparing it with observed or expected data.

Blomhøj and Højgaard (2003) illustrate this six-step modelling process using the context of anaesthesia in surgery, as shown in Figure 2.4. They explain that administering anaesthetic requires careful modelling to ensure the correct dosage—sufficient to eliminate pain during surgery but not so high as to cause overdose. Based on this context, the modelling process can be summarised as follows:

1. Formulation of the task
2. Systemisation
3. Mathematisation process
4. Mathematical analysis and solution
5. Interpretation and evaluation of results
6. Validation and testing of the model

Each of these steps is further explained below:

- **Formulation of the task**

Blomhøj and Højgaard (2003) explain that this stage involves questions such as: “What factors may influence the concentration of the medical drug in a patient during surgery?”

- **Systemisation process**

This step involves identifying the structure of the real situation. A patient, for example, may be viewed as a system consisting of interacting components such as blood volume and drug concentration, where exchange processes occur within the body (Blomhøj and Højgaard, 2003).

- **Mathematisation process**

At this stage, the situation is represented using mathematical expressions, often resulting in linear or differential equations that describe relationships between physical quantities.

- **Investigation of the mathematical procedure**

Mathematical techniques are applied to analyse the model, approximate parameters where necessary, and derive possible solutions.

- **Interpretation and evaluation of results**

The results are interpreted in relation to the original context and checked against practical or observed outcomes to determine their relevance and usefulness, for example in determining appropriate anaesthetic dosage.

- **Analysis and testing of model validity**

Finally, the model is evaluated in terms of its scope, accuracy, and applicability, including its limitations and suitability for use in other patients or with different medications (Blomhøj and Højgaard, 2003).

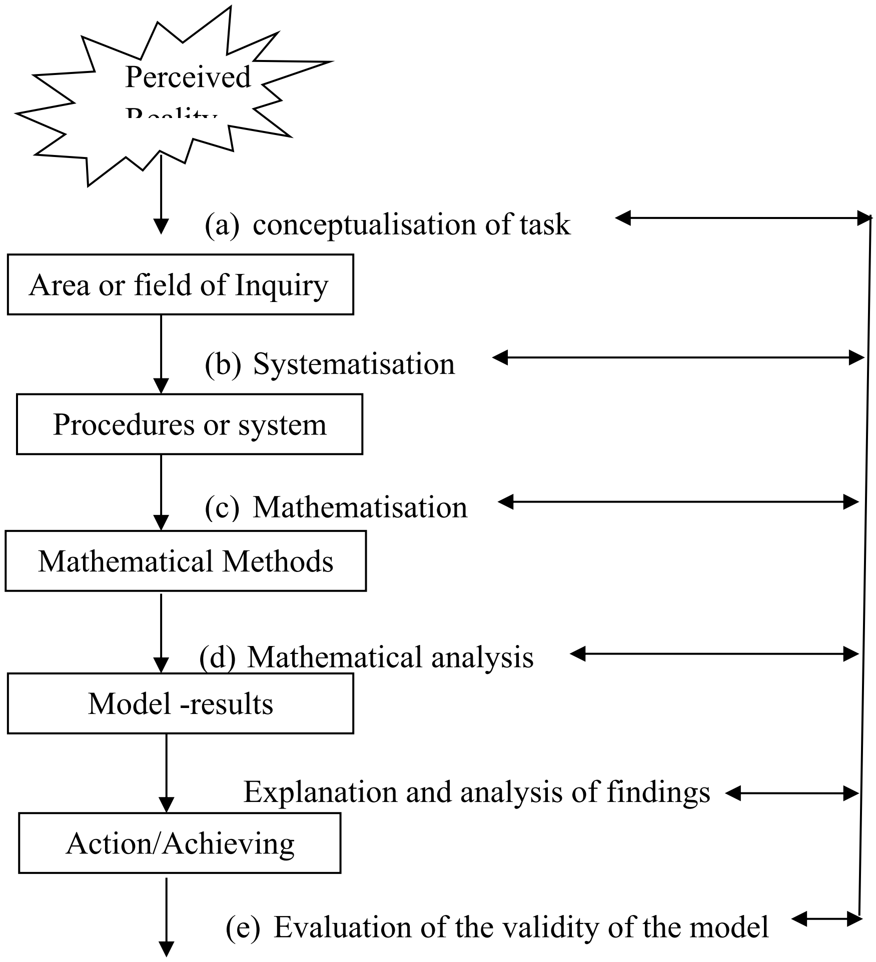


Figure 2.4: Process of mathematical modelling

Source: Blomhoej & Højgaard (2003)

Bliss et al. (2016) state that the sequence of actions involved in mathematical modelling consists of the steps illustrated in Figure 2.5. Like other modelling processes, this approach begins with the identification of a problem to be solved and proceeds through various stages until the final phase, where results are communicated. The modelling process presented by Bliss et al. (2016) indicates that not all stages follow a strictly linear progression. Some of the arrows are bidirectional, showing that movement between stages is often iterative rather than one-way. In addition, the components of the model are not numbered, which avoids the implication of a rigid, sequential order. Instead, the process emphasises that certain steps may need to be revisited and refined before a satisfactory model is implemented and results are reported.

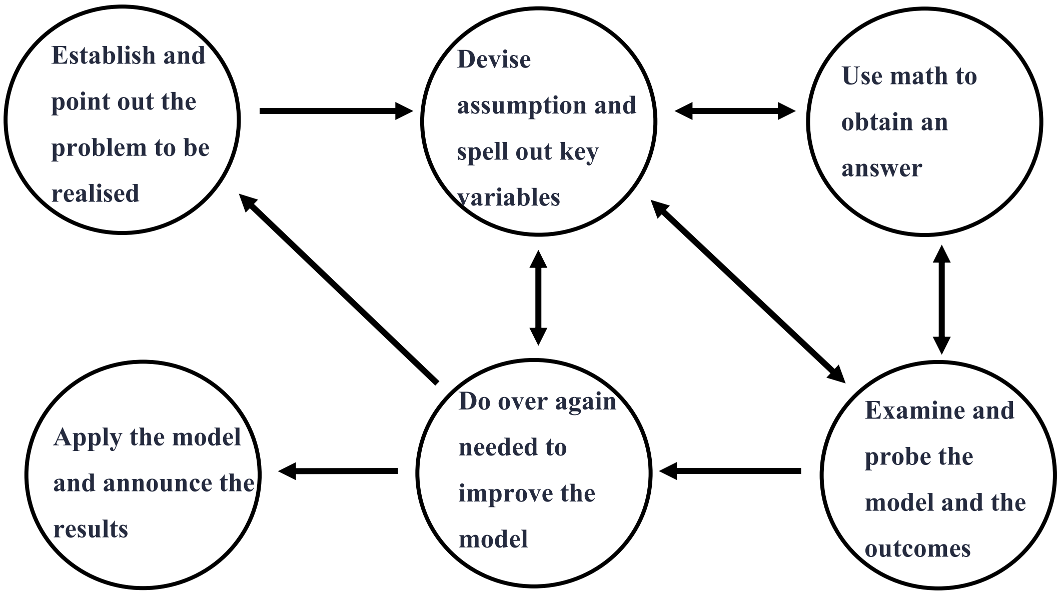


Figure 2.5: The modelling process

Source: Bliss et al., (2016:13)

The basic features of the modelling process illustrated in Figure 2.4 are as follows:

(a) **Establish the problem:** A real-life or community-based problem is identified and clearly described.

(b) **Create assumptions and identify variables:** Relevant aspects of the real situation are selected, and relationships between them are established. At this stage, decisions are made regarding which elements to include and which to simplify or exclude, resulting in an idealised representation of the problem.

(c) **Use mathematical representation:** The idealised situation is translated into mathematical form, producing equations or expressions that represent the model. The model is then solved using appropriate mathematical techniques to obtain results.

(d) **Analyse and evaluate the solutions:** The obtained solutions are interpreted and assessed to determine whether they are reasonable and meaningful in relation to the original real-world context. Their feasibility and validity are also considered.

(e) **Iterate:** The process is repeated where necessary to refine, improve, or extend the model.

(f) **Implement the model:** Once a satisfactory solution has been obtained and validated, the model is applied to the original problem situation.

According to Spandaw and Zwaneveld (2010), the modelling process is cyclical in nature, meaning that the steps are not strictly linear but are often repeated. The process begins with a real-life problem that must be solved. This problem is first described in words, and simplifying assumptions are made to construct a conceptual model. This conceptual understanding is then translated into a mathematical representation, such as an equation or system of equations, which is solved using appropriate mathematical techniques.

The resulting solution is then interpreted within the original context, a stage known as interpretation of results. Finally, the solution is verified or validated to determine its appropriateness and accuracy. If necessary, the modeller may return to earlier stages of the cycle to refine assumptions, adjust the model, or improve the solution.

2.5 Difference between applied Mathematics and mathematical modelling

It is often assumed that the terms "application of Mathematics" and "mathematical modelling" have the same meaning; however, they are distinct concepts with different definitions. Stillman et al. (2007) emphasise that a clear distinction exists between the two, and this distinction is consistent with that supported by the International Community of Teachers of Mathematical Modelling and Applications (ICTMA), which promotes the teaching and use of modelling and applications in Mathematics education.

In simple terms, mathematical application begins with Mathematics and seeks contexts in which existing mathematical knowledge can be applied in real-life situations. In this case, the direction of movement is from Mathematics to reality (Mathematics → reality). In contrast, mathematical modelling begins with a real-world problem and then develops or selects appropriate Mathematics to solve it. In this case, the direction of movement is from reality to Mathematics (reality → Mathematics).

According to Spandaw and Zwaneveld (2010), the modelling process is cyclical in nature, meaning that its steps are repeated and refined rather than followed in a strictly linear sequence. The process begins with a real-life problem that needs to be solved. This problem is first described in words, and simplifying assumptions are made to construct a conceptual model. This conceptual understanding is then translated into a mathematical representation, such as an equation or model, which is solved using appropriate mathematical procedures. The results are then interpreted in relation to the original context, a process known as interpretation. Finally, the solution is validated to determine its accuracy and suitability. If necessary, earlier stages of the cycle may be revisited and adjusted to improve the model.

From the above discussion, it is evident that successful engagement in mathematical modelling requires learners to possess a range of skills. Firstly, strong language proficiency is essential, as modelling problems are typically presented in word format. For example, if English is the language of instruction, modelling tasks will also be written in English, requiring learners to read, interpret, and fully understand the given scenario. This understanding is crucial, as it informs the assumptions that learners make, and incorrect assumptions may lead to inaccurate solutions. For instance, consider the following simple mathematical problem:

A car travelled 100km in the first hour. What total distance will be travelled by car after 2 hours?

Generally, one might solve this type of problem by simply multiplying the distance by two, resulting in 200 km, since the time has been doubled. However, in authentic real-world contexts, this approach is often overly simplistic and may lead to incorrect conclusions. For instance, suppose a car travels on a flat, straight road during the first hour and then has to ascend a slope on a winding road in the second hour. In such a case, the vehicle is unlikely to maintain the same speed in the second hour due to the change in terrain. Similarly, if the first hour involves travelling during free-flowing traffic conditions, while the second hour coincides with peak traffic congestion, the travel speed will differ significantly. In another scenario, the second hour may involve travelling through villages with multiple speed restrictions, requiring frequent deceleration. Additionally, road construction may introduce stop-and-go conditions, further disrupting uniform motion.

These examples demonstrate that real-life situations are influenced by multiple variables that affect speed and distance, making simplification and the formulation of assumptions essential first steps in mathematical modelling. Learners must therefore be able to clearly state and justify assumptions. In the example above, the car can only be assumed to travel 200 km in 2 hours if conditions remain constant across both hours and the speed is uniform throughout. In other words, the model assumes constant speed over equal intervals of time. The role of assumptions is further discussed in Section 2.6.

The second important skill in mathematical modelling is the ability to formulate a mathematical representation from a given real-life scenario. This involves assigning variables to relevant quantities and developing equations or expressions that represent the situation. Once a mathematical model has been constructed, the next step is to solve it using appropriate mathematical procedures. This is followed by interpreting the obtained results in the context of the original problem. The final step is the verification or validation of the solution to ensure its appropriateness and realism. For example, if the solution of a distance problem yields both positive and negative values, the negative value would be rejected, as distance is inherently non-negative.

2.6 Importance and role of assumptions in mathematical modelling

To ensure that real-life problems are solved correctly using Mathematics, it is essential to formulate appropriate assumptions. This involves identifying and accounting for all relevant factors that may influence or distort the solution if they are not considered. For example, consider the following problem:

A fuel consumption of a car is 8 litres per 100km travelled. How many litres of petrol can be used by the car to travel 300km?

This problem may initially be treated as a direct proportion problem, which could lead one to assume that multiplying 8 litres by 3 would yield 24 litres, since the distance has also been multiplied by 3. However, in problems of this nature, several assumptions are required to ensure a valid solution. Two relevant assumptions are: (1) the car travels at a constant speed throughout the journey so that fuel consumption remains uniform (noting that higher speeds generally result in higher fuel consumption), and (2) the road conditions for the 300 km journey are similar to those of the 100 km journey, particularly in terms of slope and terrain. If, for example, the vehicle spends most of the longer journey climbing steep slopes, while the shorter journey is on a flat road (or vice versa), it would be incorrect to conclude that the car will consume 24 litres of petrol.

Krawitz et al. (2022) emphasise that solutions become inaccurate when inappropriate or unrealistic assumptions are made. It is therefore essential, when engaging in mathematical modelling, to identify and apply relevant and contextually appropriate assumptions. In teaching mathematical modelling, it is equally important for teachers to ensure that learners understand and appreciate the role of assumptions.

According to Krawitz et al. (2022), teachers should actively guide learners to become aware of assumptions as they work through real-world word problems. Learners must develop a clear understanding of the purpose and function of assumptions in modelling. The study by Krawitz et al. (2022) aimed to demonstrate, through empirical evidence, that fostering learners' ability to formulate assumptions is one of the most effective instructional strategies in mathematical modelling. The participants were Grade 9 learners from Kanagawa, Japan. Their framework was structured around three perspectives:

A. Assumptions or presumptions identified or proposed during the formulation phase.
B. Assumptions or presumptions recognised or inferred during the mathematical solving phase.
C. Assumptions established or implied during the interpretation and evaluation of results in relation to the real-world context.

The modeller applies Perspective A when constructing the problem situation, thereby linking the real-world context to its mathematical representation. Perspective B supports the simplification and execution of mathematical procedures during the solving process. Perspective C is used when interpreting and evaluating the mathematical solution within its original context.

In their study, Krawitz et al. (2022) used a textbook-based problem involving two brothers who left their home for a station 2 km away. The younger brother walked to the station 10 minutes earlier than his elder brother, who followed on the same route by bicycle. The younger brother's walking speed was 80 metres per minute, while the elder brother's cycling speed was 240 metres per minute. Learners were required to determine the time it took for the two brothers to meet, given that the elder brother started 10 minutes later, using the information provided.

5 Distance and time data following the little sibling's departure

Distance (m)	0	55	98	144	190	231	277	341	398
Time (sec)	0	41	68	98	126	152	181	221	257

6 Distance and time data following the senior sibling's departure

Distance (m)	0	55	98	144	190	231	277	341	398
Time (sec)	0	20	30	40	50	60	70	83	97

After learners had completed their calculations and submitted their solutions, the teacher (who was also the researcher in this study) engaged them in a plenary discussion in which they were explicitly asked to reflect on implicit assumptions. These included, among others, the assumption that speed remained constant throughout the journey and that neither of the brothers stopped along the way. These assumptions correspond to Perspective C in the framework, as they emerge during the interpretation and evaluation of results. Learners were further guided to consider why it is necessary to explicitly state assumptions when solving such problems. In this regard, it was emphasised that a correct solution requires the assumption of constant speed, which aligns with Perspective A in the framework, where assumptions are established during the formulation of the problem.

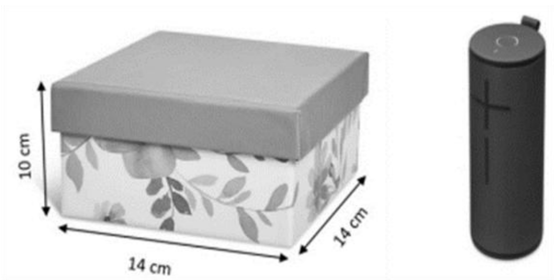
Through classroom discussions on the role and importance of assumptions in solving real-life mathematical problems, learners collaboratively generated a range of additional assumptions that could be considered in solving the given problem. These included:

- Absence of traffic signals along the route.
- No interaction with friends or other pedestrians along the way.
- Constant speed throughout the journey.
- Negligible or no resistance (e.g., wind resistance).
- No fatigue or exhaustion affecting performance.
- A straight and uninterrupted road.
- No traffic congestion.
- No reduction in speed when encountering road bends or curves.

Krawitz et al. (2022) propose "awareness or realisation of assumptions" as an effective approach to helping learners understand the role of assumptions in mathematical modelling. In this approach, learners are encouraged to identify or infer assumptions during the formulation, solution, and interpretation stages of the modelling process. The study suggests that developing this awareness is a practical and valuable teaching strategy in mathematical modelling.

Furthermore, Krawitz et al. (2022) examined the challenges learners encounter when making assumptions in open-ended modelling tasks, particularly in recognising the openness of problems and inferring missing information. The participants in their study were Grade 4 learners who were given modelling tasks, including a scenario requiring them to determine whether a speaker would fit into a box with specified dimensions.

Speaker
Maria bought the ***Ultimate Ears BOOM*** Speaker for 149.95 €, It has 360º sound with deep and precise bass. The speaker is 18, 4 cm high.
Maria looks for a box with a cover for her speaker. On the web, she found a beautiful box. It is 14 cm wide, 10 cm high, and 14 cm deep. Will the speaker fit in the box?



Since no dimensions of the speaker were provided, learners were required to estimate its diameter to make informed decisions. This estimation served as a substitute for missing information and was essential for completing the modelling task. It was observed that some learners based their estimates on the visual representation in the diagram. For example, learners may have approximated the diameter to be around 5 cm, reasoning that the diagram suggests the diameter is roughly a quarter of the speaker's height (18,4 cm).

Even if learners assumed that the speaker should be placed diagonally in the box because its height is shorter than the diagonal length, the solution would still be invalid if the diameter was not appropriately considered. In fact, the diameter must be 10 cm or less; otherwise, the lid of the box would not close properly. As illustrated in Figure 2.6, the diameter of the speaker directly determines how it fits into the box: the thicker the diameter, the shorter the effective length that can fit inside. Consequently, the configuration shown in Diagram A accommodates a shorter effective length than that shown in Diagram B. Furthermore, there is no feasible way to place the speaker as illustrated in Diagram C, since the speaker height of 18,4 cm exceeds the box height of 14 cm.

Therefore, it was essential for learners to make appropriate assumptions regarding the diameter of the speaker to reach a valid and meaningful solution to the problem.

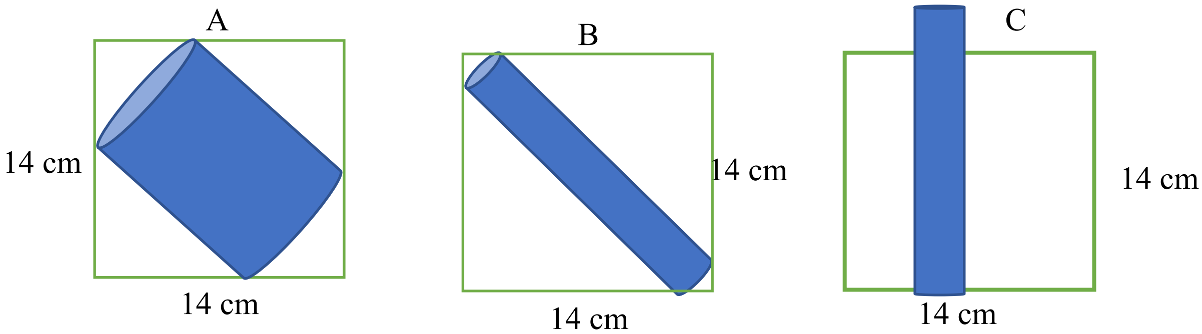


Figure 2.6: Illustration of possible positions of the speaker

Of the four participants, only two made explicit assumptions regarding the missing value of the diameter. One participant did not estimate the diameter of the speaker and, during the interview, explained that she assumed the maximum possible diameter would be 14 cm, which she regarded as relatively narrow. The fourth participant recognised that the diameter was an important variable; however, he was uncertain how to use the given information to construct a solution. Instead of estimating or incorporating the diameter into a model, he concluded his problem-solving process with a speculative response that the speaker would not fit into the box. This approach suggests that when essential missing information is not incorporated into the mathematical model, learners may struggle to proceed, particularly when the inclusion of additional variables increases the complexity of the problem, as was the case in the Speaker problem.

The findings of Krawitz et al. (2022) are useful in identifying barriers that learners encounter when making assumptions in open-ended problems. The study also found that simply informing learners about the openness of a problem—that is, that it may be solved using multiple approaches—was helpful in improving their immediate performance on the task. However, this awareness did not necessarily translate into improved performance in subsequent problems. The study therefore concluded that learners' inability to estimate or make assumptions about missing quantities significantly hinders their problem-solving processes.

Similarly, findings from Seino (2005) and Krawitz et al. (2022) demonstrate that solving real-life mathematical problems requires the formulation of appropriate assumptions. In open modelling tasks, assumptions play a critical role by substituting missing information and simplifying complex situations. The present study similarly required learners to determine either maximum dimensions or areas based on a fixed length of fencing used to enclose a garden or campsite. For example, one of the questions used in the study was formulated as follows:

A farmer has 600m of fence to create a pasture shaped like a rectangle which has to be divided into three camps all with the same dimensions. What is the maximum area that can be enclosed with this fence?

To obtain a realistic solution, learners should also have considered the size of the poles used at the corners of the camps. In the case of cylindrical poles, a larger circumference would require additional fencing material to account for the space occupied by the poles at the corners. If this is not considered, the effective enclosed area of the camp may need to be reduced to ensure that it can be fully enclosed by the available length of fencing. Another important assumption in this context is that the land to be enclosed is flat and level. If the terrain is uneven, such as containing valleys or being generally undulating, additional fencing would be required to accommodate changes in elevation and contour. The illustrations in Figures 2.7 and 2.8 further clarify these considerations.

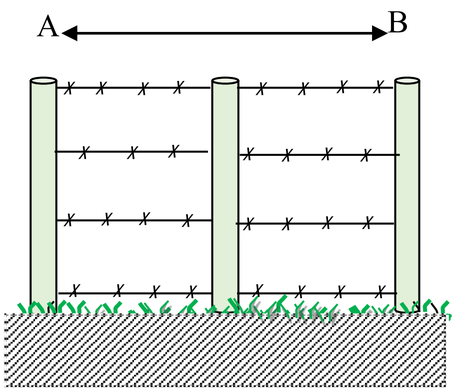


Figure 2.7: Fencing flat land

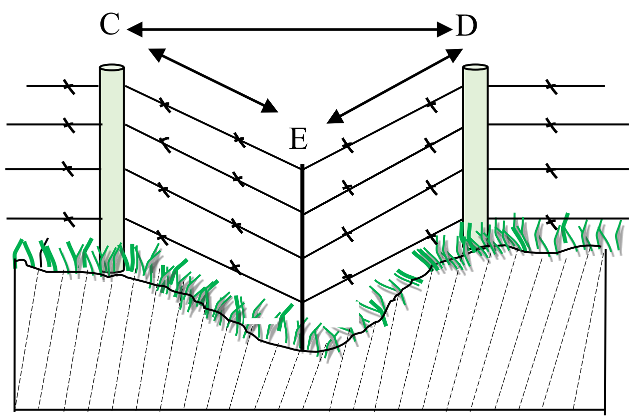


Figure 2.8: Fencing slope land

To fence the distance marked AB in Figure 2.7 and CD in Figure 2.8, one would require fencing of equal length, since the two straight-line distances are the same. However, in Figure 2.8, the presence of a valley along the boundary changes the actual distance that must be fenced. As a result, the fencing follows the path CE and ED, which is longer than the straight-line distance CD. This illustrates that if appropriate assumptions are not made in solving problems involving area and perimeter, incorrect solutions may be obtained.

The questions used in the test focused on area and perimeter, where learners were sometimes given the length of fencing and required to determine either the dimensions of the enclosed region or the maximum area that could be enclosed.

Through an exploration of learners' competencies in mathematical modelling, the study examined assumptions made by learners, their understanding of the problem, their use of appropriate mathematical methods, their interpretation of results, and the validation of their solutions based on learners' written work.

2.7 Models used in mathematical modelling

As discussed earlier in this chapter, one of the essential stages in the mathematical modelling process is model construction. It is therefore important to clarify what is meant by the term "model" within a mathematical context. According to the Cambridge Dictionary, a model is a simplified description of a system or mechanism used to predict or calculate possible outcomes. Similarly, Muthuri (2009) describes a model as a representation of part of the real world that has been simplified to make it easier to understand. Models may take different forms, including physical representations such as dolls, scale models of cars or aeroplanes, and globes, as well as visual representations such as maps or diagrams. In addition, models may also be expressed in mathematical or algebraic form. In this regard, mathematical models are defined as "a variety of equations that denote relationships within a setup and could be ascertained either by a computer or by hand" (Muthuri, 2009: 233).

In most classroom contexts, mathematical modelling problems are presented as word problems. Learners are required to translate these verbal descriptions into mathematical representations or systems. Such word problems may be presented in different languages, such as English, German, or Mandarin, depending on the language of instruction in each country (Phillips, 2015). Regardless of the language used, the key requirement is the ability to convert the written problem into an appropriate mathematical model. At school level, the most common forms of mathematical models include equations, tables, and graphs. In this study, which focused on area and perimeter, the models constructed mainly involved equations used to calculate perimeter and area.

Learners are expected to draw on prior knowledge when engaging with such tasks. For example, learners who have previously learned how to calculate the area and perimeter of a rectangle are better able to connect new problem situations to existing knowledge structures. This prior knowledge supports conceptual understanding and facilitates the construction of new meanings. Ningsih and Retnowati (2020) emphasise that teachers should be proficient in designing learning activities that explicitly link learners' prior knowledge with new content, thereby enabling the development of new understanding.

Equations

Simply stated, an equation is an algebraic expression that shows that two quantities are equal. In other words, an equation is a mathematical statement consisting of a left-hand side and a right-hand side, separated by an equal sign. For instance, $2l+2b=32$ is an equation where $2l+2b$ and 32are two expressions that are separated by an equal sign. Equations are formulated in terms of mathematical entities that correspond directly to physical quantities. When these quantities change within the context of the situation being modelled, they are referred to as variables, whereas quantities that remain fixed are generally referred to as parameters (Muthuri, 2009).

Muthuri (2009) notes that, in most cases, a model consists of formulae that connect essential quantities, usually referred to as inputs, which are used to determine a quantity of interest known as the output. She further illustrates this using a simple model that represents changes in soil moisture, as shown in Figure 2.9. In this model, the soil moisture (W) at time *t* is represented by W_t, rainfall represented by R_t, water uptake by plants from the soil illustrated by U_t and D_t stands for water drained from the soil. In mathematical terms, the model is given as $W_{t+1} = W_t + R_t - U_t - D_t$

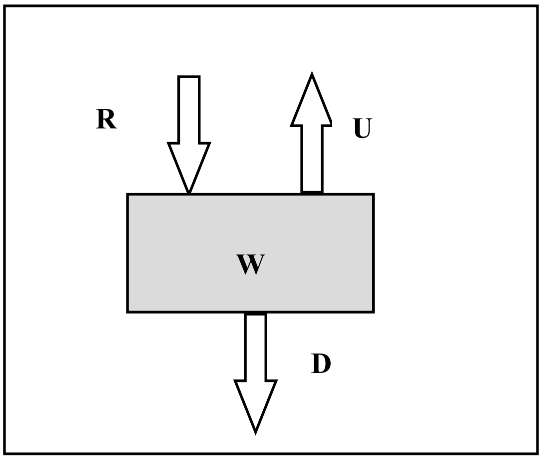


Figure 2.9: Straightforward model of soil moisture

This model implies that, to determine Wat any given moment, one must know the initial amount of water in the soil (W_0) as well as the corresponding values of rainfall (R), drainage (D), and plant uptake (U). The model is relatively simple and straightforward, as it excludes other factors such as water loss through evaporation. While this omission may not pose a problem in contexts where evaporation is negligible or deliberately disregarded, it becomes a limitation in situations where such losses are significant. Another important challenge associated with the use of this model is that it includes variables that may be difficult to measure in practice, such as plant uptake (U) and drainage (D).

Graphs

At school level, learners are expected to draw and interpret various types of graphs, including linear, parabolic, hyperbolic, and exponential functions. For example, they should be able to describe how the values of $y$change as the values of $x$increase. Table 2.1 presents different types of functions and the corresponding relationships between x and y.

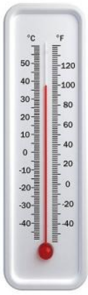
Directly proportional

In direct proportion, as one quantity increases, the other increases at a constant rate. In other words, two quantities are directly proportional if one is a constant multiple of the other. We indicate that $y$is directly proportional to $x$by $y \propto x$. In real-life contexts, this relationship can be illustrated using the cost of fuel, since petrol or diesel is typically charged per litre. If the cost per litre of petrol is $p$and the number of litres purchased is n, then the total cost $c$is obtained by multiplying the price per litre by the number of litres purchased, that is $c = pn$. For example, if the inland price of 95-octane unleaded petrol in South Africa is R22,46 per litre, then the cost of purchasing 20 litres would be calculated as $c = 22.46 \times 20$. A direct proportion relationship can also be represented graphically. When two quantities are directly proportional, the resulting graph is always a straight line that passes through the origin.

Linear relation

A linear relation is said to be linear if a link between two distinct variables x and y form a straight line on a graph. A linear relation is represented by $y = mx + c$. A linear relation is used in real life to do measurement conversion. For example, formula $C = \frac{5}{9}(F - 32)$ is used to rework units of temperature from Fahrenheit to Celsius using the formula $C = \frac{5}{9}(F - 32)$. This formula can be written as $y = \frac{5}{9}x - \frac{160}{9}$ which has taken the form $y = mx + c$. The formula can be rearranged to become $F = \frac{9}{5}C + 32$ to change Celsius degrees to Fahrenheit. It can also be written $y = \frac{9}{5}x + \frac{288}{5}$ which is the indication that the formulae concur with universal equation $y = mx + c$ that describes a linear relationship.

The figure on the right-hand side illustrates a thermometer calibrated in both degrees Celsius and Fahrenheit. Such a thermometer can be used in different countries because it displays both temperature scales, thereby eliminating the need for manual conversion between units. This is particularly useful, given that different countries around the world use different measurement systems. In this context, Celsius is the metric system unit for measuring temperature.



Parabolic relation

A relation is a quadratic function if the highest power of the variables in the relation is two. Examples of quadratic functions are $y = ax^2$, $y = -3x^2 + 5$, and $y = x^2 + 5x + 6$. Examples of things that have parabolic shapes include satellite dishes, rainbow and surface of concave mirror. Torches and headlights of cars also have parabolic shapes so that they can reflect light rays that will appear as a thick focused beam of light from a torch or headlight.

Satellite dish

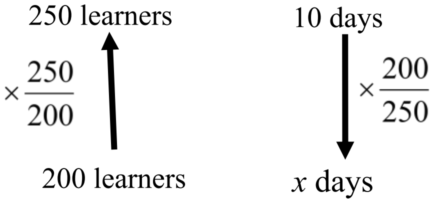
car headlamp

Inverse proportion

In the inverse proportion, the relationship is such that, if one quantity is increased, the other quantity decreases and vice versa. The equation for inversely proportion is $y = \frac{k}{x}$. This means as x increases, y decreases. The answer you get when multiplying the two variables result in a constant value, meaning $xy = k$. In inverse proportion, when the magnitude of one variable increases, the other decreases in such a way that their product remains constant. A real-life example of inverse proportionality is the relationship between speed and time: the higher the speed, the less time it takes to cover a fixed distance.

Another example is the relationship between the number of learners in a school and the duration for which food supplies last. If more students are admitted, the food will be consumed more quickly. Conversely, if the number of learners decreases due to dropouts, the food supplies will last for a longer period.

Suppose the number of learners in a school is 200 and the available food is sufficient to last for 10 days. If the school suddenly admits an additional 50 learners, the total number of learners increases, and the same quantity of food will no longer last for 10 days. Instead, the duration of the food supply will be reduced to approximately 8 days.



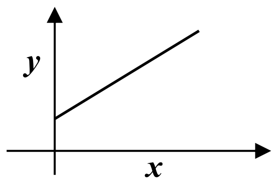
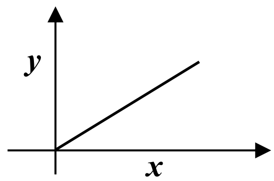
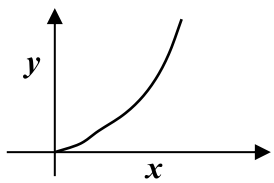
The arrows indicate that increasing the number of pupils' results in the decrease of the number of days the food would last. Therefore, the number of days food will last can be calculated as:

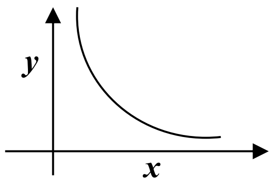
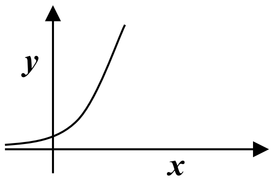
The food will last 250 learners 8 days $x = 10 \text{ days} \times \frac{200}{250} = 8 \text{ days}$

Exponential relation

The formula for an exponential relationship is generally expressed in the form $y = ab^x$, where $a$is a constant and $b$is the base, while $x$is the variable (exponent). As the value of the exponent increases, the rate of change of the function increases, resulting in a progressively steeper graph. Exponential relationships are widely applied in real-life contexts such as finance, particularly in the calculation of compound interest, as well as in modelling population growth, where quantities increase at an accelerating rate over time.

Table 2.1: Mathematical models, shapes and relationship between variables

Mathematical Model (Equation)	**Shape of the graph**	**Relationship between x and y**
$y = kx + c$		**Linear Relationship** Making the magnitude of x bigger, the magnitude of y also becomes bigger.
$y = kx$		**Direct Proportion** An increase in the size of x results in a corresponding increase in y. The ratio of y to x is exactly the same.
$y = kx^2$		**Parabolic Relation** A proportionate increase in y occurs when the amount ofx^2 increases. x^2 and y are directly proportional.

$y=k\left(\frac{1}{x}\right)$		**Inverse Proportion** When the magnitude of x becomes bigger, the magnitude of y reduces. The relationship between x and y is inverse.
$y=b^x$ where $b>0$ $b\neq1$		**Exponential Relationship** When the size of x is increased, y increases by a factor of b

Tables

Tables consist of rows and columns and are used to organise and present data in a structured manner. In classroom practice, learners often use tables as a preliminary step in graphing functions. By substituting values into a given equation, they generate ordered pairs (coordinates), typically at least three, which can then be plotted to construct a graph. Similarly, researchers use tables to present and summarise collected data for analysis. In this study, tables were employed during data analysis to organise the collected information. The data presented in tables can further be used to develop both equations and graphical representations. This is illustrated in the example below, where tabular data was used to derive an equation, which subsequently informed the construction of a graph.

Example

The charges of staying in Dusk to Dawn Guesthouse up to 5 days are given in the next table. There is a fixed amount, included in the cost, which is given back to you if there were no breakages.

Days (d)	**Cost in Rands (R)**
1	600
2	1050
3	1500
4	1950
5	2400

(a) How much do they charge per day?
(b) How much is refunded to a person if there were no damages during his or her stay?
(c) Determine the equation that will predict the cost for staying at the guesthouse any number of days?
(d) What do the numbers in your equation represent?

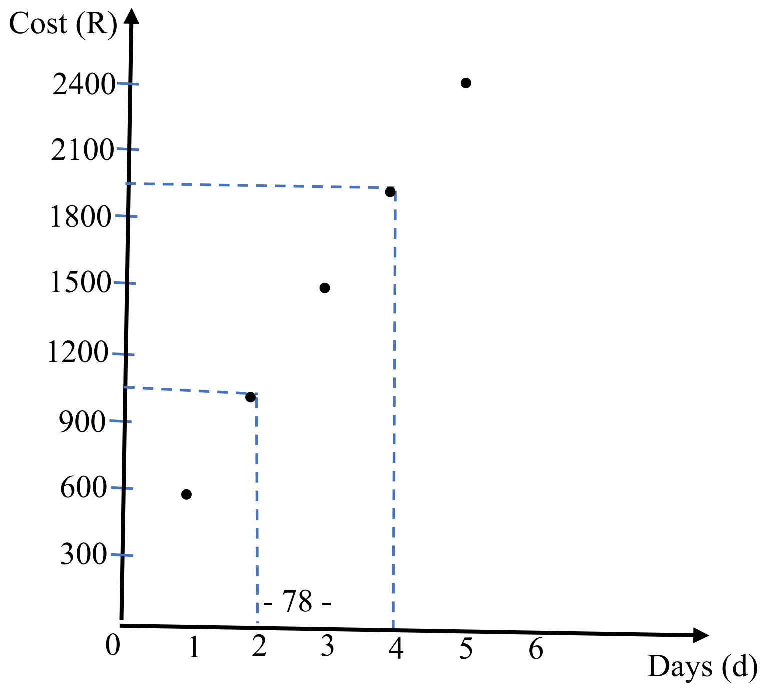
Solution

(a) There is a constant difference of R450 each day so that must be the charge per day.
(b) The refund would be R600 – R450 = R150.

(c)The equation is $c = 450d + 150$. (Variable c represent the total cost) If this can be drawn as a graph, it would give a straight line with the gradient or slope of 450 because it is the constant rate of change. The graph would start at point (0; 150), which is the y- intercept.

By using the equation derived in (c), it is possible to predict the cost of staying at Dusk to Dawn Guesthouse for any number of days. For example, the cost of staying for 15 days can be calculated by substituting 15 into the equation.

Alternatively, the same information can be obtained from the graph. To determine the cost for 2 days, one locates 2 on the horizontal axis (number of days), then moves vertically upwards until the graph is reached. From that point, a horizontal line is drawn to the left, parallel to the x-axis, to read the corresponding cost on the vertical axis. These steps are illustrated by the dashed lines on the graph on the opposite page.



The points were not joined because the Guesthouse charges per full day rather than for parts of a day. This means that even if a guest stays for one and three-quarter days, they will still be charged for two full days. The data is therefore considered discrete, as it takes only specific, separate values rather than a continuous range. The information presented in the table, together with the derived equation, can be used to construct a graph as shown above. However, real-world problems are often more complex than this example, as they may have multiple possible solutions. In such cases, learners are required to evaluate different options and select the most appropriate one. Problems of this nature are known as optimisation problems. In the current South African Mathematics curriculum, optimisation problems are assessed under Calculus in Grade 12. As indicated in Chapter One, learners' performance in questions involving optimisation (applications of calculus) has consistently been poor over the years, as illustrated in Table 1.3.

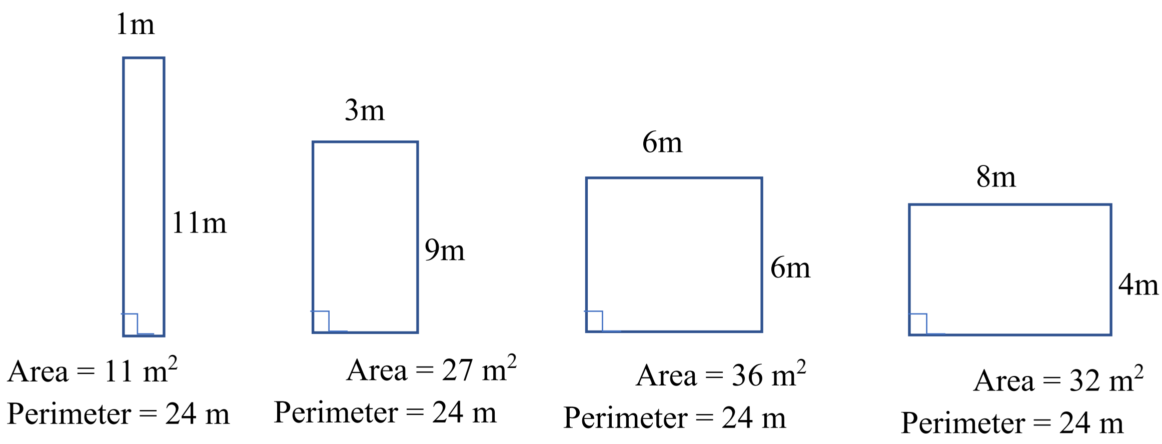
Optimisation Problems

When solving optimisation problems, one aims to determine the maximum or minimum value that a given mathematical expression can attain. Calculus is therefore used as a tool to find realistic solutions in situations where a measurable property of an object must be either maximised or minimised. Problems of this nature are referred to as optimisation problems.

For example, one may be required to determine the dimensions of a rectangular garden that will yield the maximum area, given a fixed perimeter of 24 metres. In such a case, there are many possible combinations of length and breadth that satisfy the perimeter condition. Since length and breadth are continuous variables, all positive real numbers must be considered, not only integers. For instance, a rectangle with a length of 11 m and a breadth of 1 m satisfies the perimeter condition but results in an area of only 11 m^2.

To illustrate the method further, consider a situation where 24 m of fencing is used to enclose a rectangular plot. The fence can be arranged in multiple ways, each producing a different area. The number of possible rectangles is not limited to a few discrete cases, as the dimensions may take any positive real values. After examining all possible combinations, it becomes evident that the maximum area is achieved when all four sides are equal, that is, when the rectangle becomes a square. In this case, the maximum area is 36 m^2.

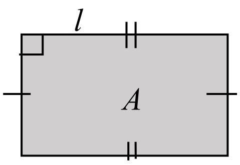
The solution can then be obtained using systematic mathematical procedures, as outlined in the following steps.



(a) Determine the part to be maximised, in this case it is the area

(b) Write this part as an equation $A = l \times b$ ……. (1)

(c) Represent the information in a diagram



(d) Formulate a suitable equation after establishing constraint on the variables. The constraint should indicate that the perimeter of the garden is 24 metres.

$2l + 2b = 24$ …. …... (2)

(e) From equation (2), make l the subject of the formula and express equation (1) as an equation with only one variable.

$2l + 2b = 24$
$l + b = 12$
$l = 12 - b$ ....(3)

Substitute l in equation (1)

$A = l \times b$
$A = (12 - b)b$
$A = 12b - b^2$

(f) Plot the above graph with *b* on the horizontal axis and *A* on the vertical axis.

From equation (3), the domain is $0 \le b \le 12$. This is because a length cannot be a negative value. Once the value of b is greater than 12 m, the value of l, the length will be negative. Since $A(b) = -b^2 + 12b$ is a quadratic function, its graph is a parabola. The parabola opens down and has a maximum value. By taking *b* to represent the *x* –values and area, A, to be the y –values, we can complete the table below to get the coordinates of the quadratic function.

Breadth(x)	0	1	2	3	4	5	6	7	8	9	10	11	12
Area (y)	0	11	20	27	32	35	36	35	32	27	20	11	0

The *x* values (breadth) and y values (area) were plotted on the grid to produce a graph illustrated in Figure 2.10

(g) Identify maximum value of the graph

From the graph and the equation (6; 36) is a maximum point. Examining the value of *A* at each endpoint of the domain indicates that the value of A is zero:

When $b = 0$, $A(0) = 0$.

When $b = 12$, $A(12) = 0$.

This reaffirms that (6; 36) is a maximum point.

Figure 2.10 shows a graph of the function $A(b) = -b^2 + 12b$

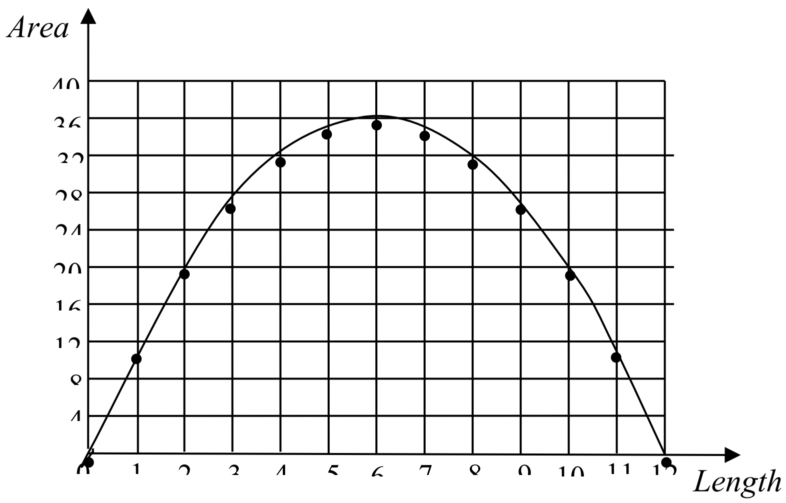


Figure 2.10: A quadratic function with the maximum value

(h) Declare or mention the results

Determine the length when the breadth is 6. Substitute b = 6 in equation (3).

$l = 12 - 6$

$= 6$

Hence, the dimensions of the garden are equal when the rectangle attains its maximum area. In this case, the garden has its largest possible area when both the width and length are 6 metres. In other words, the maximum area is obtained when the rectangle becomes a square.

In the current South African Mathematics curriculum, problems of this nature are addressed under Calculus in the subtopic of optimisation. Although the focus of this exploratory study was to investigate the mathematical modelling competencies of Grade 11 learners, it also exposed participants to optimisation problems that required the application of algebra and functions to determine solutions.

Another topic that is closely related to optimisation is linear programming. However, it is no longer included in the current secondary school Mathematics curriculum in South Africa, having been removed with the introduction of the revised curriculum in 2011. Linear programming involves the use of graphs of linear equations and inequalities to determine the best possible outcome in each situation.

It is commonly applied to determine the most efficient way of carrying out a particular task. For instance, a farmer may use linear programming to decide how much land to cultivate to maximise yield, while a company may use it to determine the optimal number of employees required to minimise costs while maximising profit. Thus, linear programming is essentially concerned with optimisation, where the objective is to maximise or minimise a particular quantity within given constraints. The following question illustrates an application of linear programming.

A baker makes two types of pies, chicken and beef. Labour and ingredients are available to bake 150 chicken pies and 130 beef pies per week. However, the bakery can only bake a total of 200 pies per week. There is a fixed order for 40 chicken and 10 beef pies per week. If the profit on chicken pie is R3 and the profit on a beef pie is R6, what number of each type should be made each week to make the maximum profit? (Shuters Mathematics Grade Twelve)

Solution

Step 1: The first step is to identify the constraints (restrictions placed on the given data and expressed them as inequalities or equations). Suppose x represents the number of chicken pies made and y represents the number of beef pies made. Least number of chicken pies is 40 and most is 150.

$\therefore 40 \leq x \leq 150$

Least number of beef pies is 10 and most is 130.

$\therefore 10 \leq y \leq 130$

Total number of pies made is 200.

$x + y \leq 200$

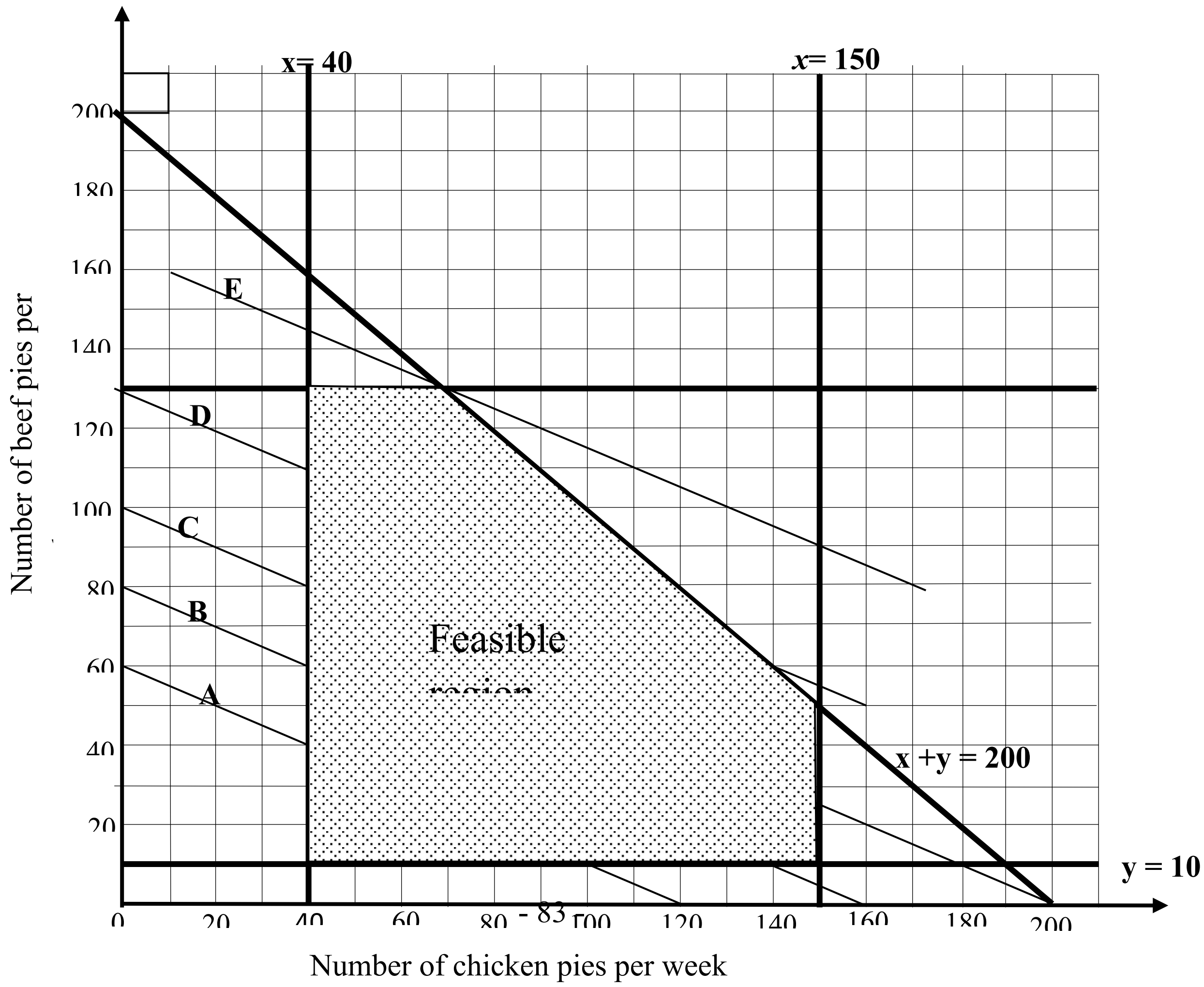


Figure 2.11: Feasible region of a linear programming

Step 2: Draw the graphs of the inequalities on a single set of axes to determine the feasible region. The feasible region is the area on the graph that satisfies all the given inequalities simultaneously.

Step 3: Determine the objective function. This involves identifying the quantity that must be maximised or minimised. In this problem, the aim is to determine the combination of pies that yields the maximum profit. It is given that the profit on a chicken pie is R3 and on a beef pie is R6. Therefore, the objective function for profit $P$is given by …

Step 4: Determine the optimal solution using the objective function. This is done by first rearranging the equation so that one variable (typically y) is expressed in terms of the other variables, allowing the function to be analysed graphically or evaluated at the boundary points of the feasible region.

$$P=3x+6y$$
$$6y=-3x+P$$
$$y=-\frac{1}{2}x+\frac{P}{6}$$

The maximum profit is found by drawing the objective function on the same axes with those functions given earlier by constraints. This is done by drawing a line with gradient $-\frac{1}{2}$ and any y – intercept. Take a ruler and move it parallel to the objective function that has been drawn, shifting it outwards until it reaches the final point of the feasible region. The different positions of the objective function are illustrated from line A through to line E. Line E represents the position at which the objective function touches the last point of the feasible region. The coordinates of this point correspond to the maximum profit. The maximum profit is achieved when 70 chicken pies and 130 beef pies are produced. Figure 2.11 illustrates the feasible region that satisfies all the given inequalities.

From the steps outlined above and the nature of the problem, it is evident that linear programming is a form of mathematical modelling that uses linear functions, including both linear equations and inequalities, to solve real-life problems. It is used to obtain optimal solutions by making effective use of available resources. Despite its relevance to everyday decision-making, linear programming was removed from the South African Mathematics curriculum when the National Curriculum Statement (NCS) was introduced in 2011.

According to the National Senior Certificate (NSC) Diagnostic Report of 2011, a significant number of candidates incorrectly formulated a particular constraint inequality due to a lack of understanding of the context. Language was identified as a contributing factor to this misinterpretation. For example, the statement “a school is planning a trip for 500 learners” was frequently misinterpreted by candidates as implying “at most 500 seats are required,” whereas the correct interpretation was “at least 500 seats are required” (DBE, 2011b). Such misinterpretations often lead to incorrect solutions and, ultimately, loss of marks.

Grade 12 learners in South Africa continued to experience difficulties with linear programming, as reflected in the National Senior Certificate Examination Diagnostic Reports of 2012 and 2013, which were the last years in which linear programming was assessed at this level (DBE, 2012; 2013). The 2012 report indicated that many candidates struggled to formulate and represent constraints using symbolic notation, even though these could be derived from given graphical representations rather than only from textual descriptions. This revealed limited flexibility in translating between graphs and symbolic representations, as required by the curriculum. It was also found that learners did not consider all points within the feasible region as possible solutions, focusing only on boundary points. The average performance for the linear programming question in 2012 was 31.5%. The statements below were included in the Grade 12 final examination paper (Paper 1) in 2013.

A company manufactures both short-sleeved shirts and long-sleeved-shirts. The constraints below govern the production of the shirts per day.

- **No more than 80 short-sleeved shirts can be produced per day.**
- **A minimum of 50 long-sleeved shirts must be produced per day**
- **It is necessary to produce a maximum of five long sleeved shirts for each short sleeved shirt produced**.
- **Each short-sleeved shirt has 5 buttons and each long-sleeved shirt has 4buttons**
- **At most 800 buttons are available for the production per week**

The report revealed that most candidates were unable to formulate the constraint for the statement: "It is necessary to produce a maximum of five long-sleeved shirts for each short-sleeved shirt produced." They failed to recognise that this condition represents a proportional relationship between the two variables. As a result, most candidates answered the question incorrectly. Instead of writing $y \leq 5x$ OR $\frac{y}{5} \leq x$, they came with incorrect statements, for example, $x^3 5y$. Another challenge was the incorrect use of inequality symbols when formulating constraints. Overall, this question was poorly answered by candidates (DBE, 2013). It is therefore noteworthy that linear programming was removed from the FET Mathematics curriculum in South Africa after this period, which may have brought some relief to both teachers and learners who experienced difficulties with the topic.

As in mathematical modelling, a clear understanding of the given situation is essential. Failure to comprehend the context in linear programming often leads to the incorrect formulation of inequalities and, consequently, an inaccurate feasible region. In turn, an incorrect interpretation of the problem situation results in a flawed mathematical model. It is therefore essential that learners read and interpret problem scenarios with understanding before attempting to construct mathematical representations.

2.8. Examples of questions that apply mathematical modelling

1. An oversized pick-axe with a handle of 13m is an exhibit in Kasel, Germany (see Fig. 2.12). It is said the Hercules (9 m tall), the statue symbol of Kassel (see Fig. 2.13), threw the big pickaxe from his place of residence, found in the mountain Wilhelmshöhe over Kassel, down to the Fulda River.

   Calculate the possible height of the giant that would enable him to handle this pickaxe? Would it fit to the Kassel Hercules himself?

Figure 2.12: An oversized pick-axe

Source: Blum 2015

Figure 2.13: Statue of Hercules

Source: Blum 2015

Solution

Taking a normal person to be 1,8m tall and the handle of normal pick-axe to be 1m, proportionality can be employed to determine the tallness of this huge human being called, Hercules.

Height of normal person is 1,8 metres

Handle of normal pick-axe is 1 metre

$\times \frac{13}{1}$ $\times \frac{13}{1}$

$\therefore$ Height of the giant $= 1{,}8m \times \frac{13}{1} = 23{,}4m$

Let the height of the giant be x

Length of an oversized pick axe is 13 metre

The suitable giant to handle this oversized pick-axe must have a height of 23,4m or more. Therefore, the giant Hercules would not be able to handle this pick-axe since he is only 9 metres tall.

2. During documentation, an exhibition of modern art which happens once in five years in Kassel, Germany, goods cost more in downtown Kassel than in the shopping mall, dez. For example, the price of a Hercules T-shirt was €15.99 at downtown (town centre), while it was €12.99 in dez, which is not too far. **Do you think is beneficial to drive to dez just to purchase this Tee-shirt there?** (Blum 2015).

Solution

This question is an example of a real-life problem. Before deciding where to buy a T-shirt, one must consider the cost of travelling to the downtown area and compare the total cost (travel expenses plus the price of the T-shirt) with the price of purchasing the same T-shirt locally. The combined cost of travel and purchase should not exceed the local price. In addition, time is an important factor to consider. The time spent travelling to town may be used more productively for other activities. For example, for a student, this time could be used to complete an assignment or work on a thesis. The problem above can be solved by following the steps outlined below.

Step 1: A model of the situation was constructed (see Fig. 2.14).

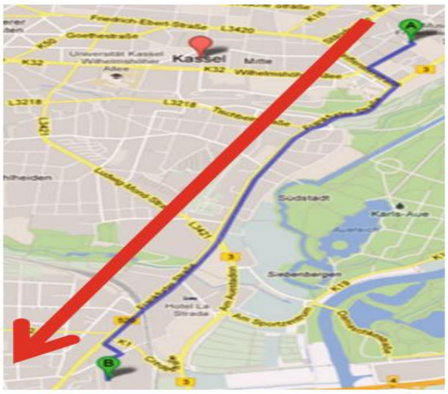


Figure 2.14: Cognitive map of the situation

Source: Blum (2015)

Step 2: The mental was simplified and structured by assuming that the distance would be travelled by car that consumes 10 litres of gas for every 100km, cost of gas is €1,599 per litre and that the distance to be travelled from the town centre to the dez was 5km.

Step 3: An appropriate model was constructed by mathematising these thoughts and relationships:

Cost of T-shirt at downtown = €15, 99.

Cost of T-shirt at dez = €12.99 + 2.*dab* where *d*: distance, *a*: consumption and *b*: gas price.

Step 4: Working out mathematically indicated that that the cost to dez is:

$$= €12,99 + 2 \times 5 \times \frac{10}{100} \times €1,599$$
$$= €12,99 + €1,60$$
$$= €14,59$$

The cost of the T-shirt at the mall is cheaper than at town centre.

Step 5: When this situation is interpreted mathematically in the pragmatic world, one can deduce that it is cheaper by 1.40 € to go and buy the T- shirt at the shopping mall.

Step 6: The results were validated by asking the following questions: Is it economical to travel a distance of 10 km just to save €1.40? Think about the time you have used for driving and see that time was going to be used to look at more exhibitions at Kassel. What about the risk of exposing yourself to accidents? The next step was to refine the representation and evaluate it against the simple mathematical solutions.

Step 7: Last step was to write down the whole solution

2.9. Learning theories and epistemological frameworks suitable for mathematical modelling study

This section discusses learning theories and cognitive frameworks that can be applied in the teaching and learning of Mathematics, including mathematical modelling. The theories and frameworks considered—constructivism, APOS theory, and the concept of image and definition—are discussed due to their shared emphasis on cognition and meaning-making. Collectively, they highlight how learners actively construct their own mathematical knowledge through mental processes and experience.

2.9.1 Constructivism

Ertmer and Newby (2013) describe constructivism as a relatively contemporary learning theory, grounded in psychological and epistemological perspectives of the twentieth century, particularly in the work of Piaget. Constructivism is based on the idea that learning involves making sense of prior experiences. While it shares similarities with cognitivism in that both view learning as a mental process, Jonassen (1991, as cited in Ertmer and Newby, 2013) distinguishes the two by arguing that cognitivists view the mind as an information processor, whereas constructivists believe that the mind actively interprets and filters environmental input to construct its own reality. From this perspective, learning is not the passive acquisition of knowledge but the active construction of meaning (Ertmer and Newby, 2013).

Piaget and Vygotsky devoted much of their scholarly work to understanding how individuals develop, construct, and share knowledge about the world and one another (Bruner, 1997). Constructivism has largely been shaped by Piaget's work, who is regarded as an early proponent of cognitive constructivism (Bozkurt, 2017). In contrast, social constructivism is associated with Vygotsky's theory of social interaction and learning. Both Piaget and Vygotsky acknowledge the influence of social factors in the development of reasoning and logical thinking (Powell & Kalina, 2009).

Lawless (2019) explains constructivism as a learning theory grounded in the idea that learners are active participants in the learning process. According to this view, learners construct meaning by connecting new information to their prior knowledge and lived experiences. Advocates of constructivism argue that the most effective learning occurs when learners actively engage with concepts and reflect on their experiences. Although knowledge can be transmitted, understanding cannot be directly imposed; rather, it must be constructed internally by the learner. Context and experience therefore play a crucial role in how new information is interpreted, evaluated, and integrated, often leading learners to revise their existing understandings (Lawless, 2019).

Similarly, Bada (2015) describes constructivism as an epistemological approach in which knowledge acquisition is viewed as a cognitive construction process. It holds that learners generate knowledge by integrating new information with existing cognitive structures. Constructivists further argue that learning is influenced by context, learner expectations, and attitudes. Although constructivism originated in psychology to explain individual knowledge construction, it is also widely applied in education to emphasise how learners actively construct meaning from their environment and prior experiences (Bada, 2015).

From a constructivist standpoint, learners are expected to be active agents in the construction of knowledge. Their role includes interpreting information, reconstructing ideas, and making decisions based on existing cognitive structures. Conceptual development and creativity are enhanced through the exchange of diverse perspectives among learners (Pods & All, 2010). At the initial stage of this study, collaborative learning was intended to facilitate knowledge construction through the sharing of ideas, thereby aligning the study with a social constructivist framework. However, due to the COVID-19 pandemic and associated restrictions during data collection, the study design was modified from group-based activities to individual tasks. Mcleod (2019) further notes that constructivism can be categorised into three main types, each of which is elaborated upon by different scholars in the sections that follow.

(a) Social constructivism

Social constructivism is highly relevant to mathematical modelling. Tasarib (2025) notes that collaborative work is a common feature of modelling activities, making social constructivism a suitable framework for understanding mathematical modelling as an effective teaching strategy. Group work and classroom discussions are therefore central to this approach, as learners can share diverse perspectives, reasoning strategies, and solutions when engaging in collaborative tasks.

As McLeod (2019) explains, social constructivism holds that knowledge is constructed through the integration of various interactive processes and is shaped by the interconnectedness of people, culture, and society. In this view, cognitive development first occurs at the social level between individuals and is later internalised by the learner. Similarly, Pritchard (2009) traces the roots of constructivist thinking to Piaget, who described the child as a "lone scientist" actively exploring and interpreting the world independently. This perspective highlights individual knowledge construction through interaction with the environment.

Pritchard (2009) further explains that social constructivism extends this view by emphasising the importance of interaction between the learner and others. Language plays a central role in this process, as learners use dialogue and discussion to express, challenge, and refine ideas. Such interaction often occurs with a more knowledgeable other, although it may also take place between peers of equal ability. Importantly, a more knowledgeable person is not necessarily someone older or in a formal position of authority, such as a teacher or parent, but rather anyone with greater understanding of a particular concept in each context.

Prior knowledge plays a significant role in the construction of new understanding. To participate meaningfully in discussions, learners draw on both prior and newly acquired knowledge. It is through this integration that new ideas are developed and refined during dialogue. Although learning is often associated with formal schooling, it occurs in a variety of contexts wherever meaningful interaction takes place. In classroom settings, the teacher typically assumes the role of the more knowledgeable other and is responsible for facilitating and sustaining productive dialogue (Pritchard, 2009).

Pritchard (2009) also notes that learners working collaboratively in pairs or small groups are engaging in socially constructed learning. In contrast, learning environments that rely solely on silent, individual work limit opportunities for interaction and the co-construction of knowledge. While individual work may occasionally be appropriate, it should not dominate classroom practice, as it can restrict opportunities for socially mediated learning and the development of shared understanding.

(b) Radical constructivism

Radical constructivism is largely associated with the work of Ernst von Glasersfeld. According to this perspective, all understanding or knowledge is actively constructed rather than passively received through the senses. Knowledge is therefore not something that can be transmitted directly into the mind; instead, it is constructed by individuals based on their personal experiences and interpretations (McLeod, 2019).

Rodney, Klobuchar, and Geelan (2001) further explain that radical constructivism, as developed by von Glasersfeld, is grounded in two key principles:

- Individuals actively construct knowledge rather than passively receiving it.
- The primary purpose of cognition is to organise and coordinate one's experiences of the world.

(c) Cognitive constructivism

This form of constructivism emphasises that knowledge is actively constructed by learners, and that the process of construction is influenced by their existing cognitive structures. In this sense, the acquisition of knowledge is closely related to the learner's stage of cognitive development. McLeod (2019) notes that this strand of constructivism is largely derived from the work of Jean Piaget. Similarly, Ackerman (2001) argues that Piaget's theory provides a strong foundation for understanding how children think and behave at different stages of development, as well as their evolving interests and abilities.

According to Piaget, children view the world differently from adults, and these perspectives are coherent, logical, and meaningful within their own developmental stage. Their thinking is reasonable and appropriate to their current needs and experiences, but it continues to evolve through interaction with people and the environment. This perspective has important implications for education, which can be summarised in three key ideas.

First, teaching is inherently indirect. Learners interpret instruction in relation to their prior knowledge and experiences rather than simply absorbing information as presented. Second, the transmission model of communication, which views learning as a one-way transfer of knowledge, is inadequate. Ackerman (2001) explains that Piaget rejected the idea that knowledge can be transmitted, stored, and later reproduced unchanged; instead, knowledge is constructed through active engagement with people, objects, and the world. Third, learning theories that ignore contextual and developmental influences provide an incomplete account of learning, as they overlook the roles of environment, context, and individual differences in shaping understanding.

Gaikwad (2020) further notes that cognitive constructivism is closely linked to information-processing theory, which views learners as active agents who construct meaning based on prior experiences. The preceding discussion highlights the crucial role of prior knowledge in the construction of new understanding. Given the complexity and diversity of constructivist perspectives, Hoover (1969, as cited in Amineh & Asl, 2015) proposed general principles that cut across these views. He identified two core ideas that capture the essence of constructivism:

1. Learners construct new knowledge by building on their existing knowledge.
2. Learning is an active process in which learners interpret and reorganise their understanding in response to new experiences.

In conclusion, although constructivism is often categorised into different strands, all versions share a common underlying principle: knowledge is actively constructed by learners through engagement and interaction. It is not simply acquired through the passive transmission of information from a teacher or other authority figure.

2.9.2 Action, Process, Object and Schema (APOS)

Tsafe (2024) explains that APOS is an acronym for Action, Process, Object, and Schema. APOS theory has played a significant role in the development of mathematics education and instructional practices in schools. As an extension of constructivist learning theory, it promotes a more focused and comprehensive learner-centred approach to mathematics instruction. The theory posits that learning begins with an action in the learner's mental framework, which is later internalised and develops into a process. At this stage, learners begin to reflect on and mentally reconstruct the activity, which is essential for meaningful understanding. At the object level, learners perceive the previously constructed process as a complete mathematical entity. Finally, at the schema level, the learner integrates related actions, processes, and objects into a coherent cognitive structure, forming a comprehensive mental representation of the concept.

Tsafe (2024) further outlines the four sequential stages through which learners develop mathematical understanding:

- Action: This refers to a learner's mental execution or manipulation of a mathematical idea in response to a problem or external stimulus.
- Process: At this stage, learners begin to make sense of mathematical concepts through internal mental operations. These processes include interiorisation, encapsulation, coordination, reversal, de-encapsulation, thematisation, and generalisation. Interiorisation refers to the repeated mental performance of a mathematical action until it becomes internalised.
- Object: An object is formed when a learner views a mathematical process as a complete entity that can be reflected upon. At this stage, the learner recognises that a mathematical concept (such as an equation, expression, function, or number) can be treated as a single object derived from underlying processes. This represents an object conception of the concept.

- Schema: A schema is a coherent collection of actions, processes, and objects that are interconnected within a learner's cognitive structure. Existing schemas are used to interpret and solve new and complex mathematical problems, while new experiences continuously modify and extend these schemas.

According to Dubinsky (2010), APOS theory and its application in teaching are based on two key assumptions:

- Assumption on mathematical knowledge: An individual's mathematical knowledge is reflected in their tendency to respond to perceived mathematical problem situations by constructing or reconstructing cognitive structures and using them in a social context to make sense of those situations.
- Hypothesis on learning: Mathematical concepts are not understood directly; instead, learners construct meaning using existing cognitive structures. Learning is facilitated when learners possess appropriate mental structures aligned with the mathematical concept being taught. In the absence of such structures, meaningful understanding becomes difficult or unlikely.

2.9.3 Concept image and Concept definition

Tall and Vinner (1981) argue that, now a mathematical concept is introduced, individuals already possess a complex cognitive structure in their minds that gives rise to different mental images associated with that concept. Many mathematical ideas are encountered informally in various contexts before they are formally defined.

According to Tall and Vinner (1981), a *concept image* refers to the totality of all mental images, associated representations, and cognitive processes that a learner uses to understand a concept. It represents the complete cognitive structure linked to that concept. In contrast, a *concept definition* refers to the formal or personal definition of a concept. While it may resemble the accepted mathematical definition, it is essentially the individual's own formulation of what the concept means, which may or may not fully align with the formal definition used by mathematicians.

Tall and Vinner (1981) further explain that concept development occurs as individuals encounter new ideas, during which existing mental images may lead to cognitive conflict. For example, a learner who holds the concept image that "subtraction always makes a number smaller" may have trouble when learning about integers, particularly when subtracting negative numbers. This conflict arises because the existing concept image is no longer consistent with the extended mathematical understanding required.

This situation can be illustrated as follows:

$$\begin{array}{ll} \text{Before}: 10-3=7 & 6-5=1 \\ \text{Later}: 4-(-2)=6 & -3-(-7)=4 \end{array}$$

When subtracting whole numbers, learners often observe that the result is smaller than the number being subtracted from. This reinforces the idea that "subtraction reduces a number." However, this intuitive understanding becomes problematic when negative numbers are introduced. In such cases, subtraction may lead to an increase rather than a decrease, thereby contradicting the learner's existing mental model.

To address such cognitive conflicts, Tall and Vinner (1981) argue that all mental elements associated with a concept—whether explicit or implicit—form part of what they call the *concept image*. They further explain that concept images do not develop in a strictly logical or linear manner. Instead, they evolve through sensory input that activates certain cognitive structures while inhibiting others. As a result, different aspects of a concept image may be triggered by different stimuli, sometimes in ways that are not internally consistent. The term *evoked concept image* is used to describe the part of the concept image that is activated at a given moment. Because different experiences may trigger different evoked images, conflicting understandings can arise, and if these are activated simultaneously, cognitive confusion may result.

Vinner and Dreyfus (1989) note that formal definitions are commonly provided for mathematical concepts, especially in school and university settings. However, learners do not always rely on these formal definitions when determining whether a given example fits a concept. Instead, decisions are often guided by their concept images. They define concept image as the totality of mental representations associated with a concept, including images, diagrams, graphs, symbols, and perceived properties. Learners' concept images are shaped by their exposure to examples and non-examples, which means that their understanding of the range of instances covered by a definition may differ from the formal mathematical scope.

In contrast, a *concept definition*, according to Tall and Vinner (1981), refers to the verbal formulation used to describe a concept. This may be a formal definition accepted within mathematics or a personal definition constructed by an individual. Importantly, individuals may refine or modify their concept definitions over time. The combination of a person's concept image and their personal concept definition leads to what is referred to as a *concept definition image*. In some cases, this structure may be incomplete or lack coherence, yet it remains part of the broader concept image. Rösken and Rolka (2007) similarly describe concept definition as a personal interpretation of a mathematical idea, aligned but not always identical to formal mathematical definitions. They also confirm the findings of Tall and Vinner (1981) regarding the relationship between concept image and concept definition.

Furthermore, Rösken and Rolka (2007) found that learners are exposed to diverse forms of information during the learning of Mathematics. The way in which learners integrate new mathematical ideas depends on their beliefs, values, and prior experiences. They therefore argue that concept formation can also be understood through epistemological beliefs, suggesting that philosophical perspectives on knowledge can be used to analyse how concepts develop. Similarly, Cottrill (2003) maintains that this framework allows learners to construct understanding from cognitive foundations rather than relying solely on formal mathematical structures.

Concept image and concept definition are particularly relevant to the present study, which investigates learners' mathematical modelling competencies. In solving real-world problems, learners are required to interpret scenarios, construct mathematical models, and evaluate solutions. Within this context, concept image refers to learners' personal and often intuitive understanding of real-world situations and the mathematical ideas used to represent them. Data such as written work, diagrams, and explanations can be analysed to gain insight into learners' mental representations. In addition, learners' varied concept images—whether visual, symbolic, or procedural—can be identified and categorised across different stages of the modelling cycle to support a deeper analysis of their mathematical thinking.

2.9.4 Multiple representations

A typical classroom comprises heterogeneous learners who differ in the ways they think, explore, and make sense of mathematical ideas. It is therefore essential for teachers to employ multiple representations to accommodate diverse learning needs. While some learners may easily interpret a concept through a particular representation, others may struggle with the same form. For this reason, teachers should support learners in expressing the same mathematical idea in different ways, enabling each learner to engage with the representation that best supports their understanding.

Friedlander and Tabach (2001) argue that multiple representations—such as numerical, verbal, graphical, and algebraic forms—can make the learning of algebra more meaningful and effective. However, each representation carries its own strengths and limitations, which both teachers and learners need to recognise.

Verbal representation is commonly used in presenting problems and communicating final solutions. It also supports reasoning, the identification of general patterns, and the connection between Mathematics and real-life contexts as well as other disciplines. Despite these advantages, Friedlander and Tabach (2001) caution that verbal representations may also be ambiguous or misleading. In addition, language differences, lack of universality, and individual stylistic variations can create barriers to effective mathematical communication.

Numerical representation, according to Friedlander and Tabach (2001), is often familiar to learners at the early stages of algebra learning. It provides a foundation for understanding mathematical problems and exploring specific cases, often serving as a bridge to more abstract representations. However, its major limitation lies in its inability to support generalisation. Numerical approaches may fail to present a holistic view of a problem, potentially leading to incomplete solutions or overlooked aspects. As a result, its effectiveness as a problem-solving tool may sometimes be limited.

Graphical representation is generally preferred by learners with visual learning tendencies. Nevertheless, graphs may be affected by issues such as scaling inaccuracies and often represent only portions of a function's domain or range. Mainali (2021) emphasises that the use of multiple representations enhances the teaching and learning of Mathematics. The most used forms include algebraic, verbal, numerical, and graphical representations. Teachers should therefore encourage learners to translate between these representations to deepen understanding and develop proficiency in Mathematics.

Tripathi (2008) illustrates the importance of multiple representations through a metaphorical story of five blindfolded individuals who were asked to describe an elephant. Each person touched a different part of the animal—such as the tail, leg, trunk, ear, and stomach—and therefore produced different and seemingly contradictory descriptions. However, when the blindfolds were removed, they realised that each description was valid but incomplete. Only by integrating all perspectives could they form a complete understanding of the elephant.

This narrative demonstrates how a single mathematical representation often captures only one aspect of a concept. Relying on one representation alone is therefore limiting, as it restricts understanding to a partial view. A comprehensive understanding emerges only when different representations are integrated, each contributing unique insights that enrich conceptual meaning. Ainsworth et al. (1997) further identify three ways in which multiple representations support learning: (a) they complement each other by highlighting different aspects of a concept and providing richer information than a single representation; (b) they constrain each other, thereby reducing interpretive errors; and (c) they promote deeper engagement when learners are required to translate and connect representations.

Similarly, Bossé et al. (2020) emphasise that mathematical representations and the ability to translate between them are essential competencies for learners. Previous research on representational translation shows that learners' ability to move between forms reflects their understanding of underlying mathematical concepts. Perception, application, and conversion among representations therefore indicate the depth of conceptual understanding.

In mathematical modelling, the use of multiple representations—such as graphs, tables, verbal descriptions, symbols, and physical models—is crucial, as it enhances problem-solving, strengthens conceptual understanding, and accommodates diverse learning preferences. After interpreting a real-world problem, learners may use different representations to model the situation. The ability to translate from one representation to another, for example from verbal to algebraic form, is an important indicator of conceptual understanding and mathematical proficiency.

2.10. Importance of language proficiency in implementing mathematical modelling

Daroczy et al. (2015) note that word problems are a standard component of Mathematics curricula across all grade levels. Unlike traditional mathematical expressions that rely on symbolic notation, word problems present mathematical relationships through short narrative contexts. In some cases, these narratives explicitly encode numerical relationships between quantities. However, despite their widespread use, word problems are often experienced as highly challenging by learners from early schooling through to adulthood. One key factor contributing to this difficulty is linguistic complexity.

Language plays a significant role in learners' performance in Mathematics, particularly in multilingual contexts where the language of instruction differs from learners' home language. When learners are required to learn and be assessed in English, yet English is not their first language, they may struggle to comprehend assessment tasks fully, resulting in reduced performance. In South Africa, for example, Mathematics assessment papers are typically set in English, which creates comprehension challenges for many learners. Research indicates that both the medium of instruction and learners' home language have a significant impact on achievement in Mathematics (Mabena et al., 2021).

In mathematical modelling, the initial phase of constructing a situation model is cognitively demanding and often constitutes a major barrier to success. Govender and Machingura (2023) found that many learners struggle to understand and make sense of modelling problems presented to them. A key difficulty is the inability to progress from the initial comprehension stage to subsequent stages of modelling due to limited understanding of the problem situation. This suggests that many learners experience challenges in translating real-world contexts into precise mathematical representations.

Similarly, Göksen-Zayim et al. (2019) found that learners encounter significant challenges when solving mathematical modelling tasks, largely because these tasks are typically presented in contextualised textual form. Across all stages of the modelling process, language proficiency plays a critical role. In the initial stage, referred to as the "conception stage," learners must read, interpret, and understand the problem accurately. In the second stage, they must translate their mental representations into formal mathematical models, a process involving mathematisation and problem solving. Finally, in the interpretation stage, language is again essential as learners express mathematical results in meaningful contextual terms.

In their study, Göksen-Zayim et al. (2019) administered a modelling task consisting of two parts: one contextualised real-world problem and another decontextualised mathematical task. Analysis of learners' written responses, task-based interviews, and classroom discussions revealed that language constituted a major barrier, particularly during the conceptualisation phase. Learners across all grade levels frequently reread the task but still struggled to construct accurate representations of the situation. In many cases, the transition from textual description to mathematical model was incomplete. The interviews further highlighted the linguistic difficulties learners experienced and the varied strategies they used in attempting to engage with the tasks.

Zerafa (2016), in a study involving Grade 3 learners in Malta, examined the role of language in word problem solving. Participants were given word problems in both English and Maltese, their home language. The findings showed that learners perceived word problems as more difficult when presented in English compared to Maltese. In addition, learners demonstrated better comprehension, improved recall, and more efficient problem-solving strategies when tasks were presented in their native language. They also completed tasks more quickly, as they did not need to translate or reinterpret the language of the problem.

Overall, the literature demonstrates that language plays a crucial role in mathematical modelling competency. It is essential for understanding problem contexts, translating real-world situations into mathematical representations, and interpreting and communicating solutions. In other words, proficiency in the language of instruction is fundamental for successfully navigating the stages of mathematical modelling. Inadequate language proficiency can hinder learners' ability to construct accurate representations and solve complex problems effectively.

In the South African context, most learners in the Further Education and Training (FET) phase learn Mathematics through English as the language of learning and teaching. It is therefore essential that learners develop sufficient proficiency in English to support their engagement with mathematical modelling tasks. This requirement also applies to learners who study

Mathematics through Afrikaans, as proficiency in the language of instruction is critical for accurately interpreting and solving modelling problems.

2.11. Beneficial effects and drawbacks of implementing mathematical modelling

The use of the mathematical modelling approach may appear highly beneficial; however, it is not without challenges that may hinder its effective implementation in schools. This section explores both the advantages and limitations of using mathematical modelling as an instructional approach in Mathematics education. In many cases, the introduction of any new instructional approach is accompanied by challenges such as a lack of trained personnel and insufficient resources. The discussion begins by examining the rationale for using mathematical modelling as an instructional method. This is followed by an exploration of the potential impacts of employing mathematical modelling as a teaching and learning strategy in Mathematics education.

2.11.1 Advantages of mathematical modelling

Blum (1993) notes that over the past three decades there has been a growing international shift towards the integration of mathematical modelling in Mathematics instruction. This approach is now being implemented across educational levels, from school to university. He further identifies several reasons for prioritising modelling in Mathematics teaching, which are rooted in the broader aims and objectives of Mathematics education. These include pragmatic, formative, cultural, and psychological arguments.

(a) **Pragmatic arguments**

Modelling supports learners in understanding and engaging with real-life situations and contextualised problems, thereby bridging the gap between school Mathematics and everyday experiences.

(b) **Formative arguments**

Through modelling, learners develop general competencies such as problem-solving skills, critical thinking, and positive dispositions towards exploring unfamiliar situations. These skills are essential for adaptability and lifelong learning.

(c) **Cultural arguments**

Blum (1993) argues that modelling reflects an important aspect of human intelligence, history, and real-world practice. Mathematics should therefore be taught not only as a body of knowledge but also as a means of reflecting on the discipline as part of human culture and scientific development.

(d) **Psychological arguments**

Modelling strengthens the learning of mathematical content by providing meaningful and relevant examples. Such contexts can enhance conceptual understanding, improve retention of knowledge, and foster more positive attitudes towards Mathematics.

Several studies have confirmed the benefits of mathematical modelling in teaching and learning. As noted in Chapter One, research suggests that modelling as a didactic approach can improve learners' achievement in Mathematics (Sokolowski, 2015). The use of real-life and contextualised problems helps learners recognise the relevance and applicability of Mathematics in everyday situations. In addition, modelling has been shown to reduce anxiety and fear associated with Mathematics while increasing learner motivation and engagement (Arseven, 2015).

A study conducted by Nguyen (2016) in Vietnam, involving 180 students, examined learners' performance on modelling tasks using tests that included contextual problems. The findings revealed that 49% of learners produced incorrect or invalid models or incorrect solutions, while 17% did not attempt the problems or were unable to formulate any mathematical model. A key difficulty identified was learners' inability to translate real-world situations into mathematical representations.

Nguyen (2016) emphasises that the final product of the modelling process is a mathematical model, which must be constructed using appropriate mathematical tools and expressed in symbolic form. He further argues that mathematical modelling promotes active learner participation, as it requires learners to construct knowledge rather than passively receive information. This implies that teachers must shift from traditional instruction to facilitative roles that guide learners in interpreting and applying Mathematics to real-world contexts.

The study also found that collaborative learning, particularly working in small groups, can support learners in engaging more effectively with modelling tasks. In addition, teacher scaffolding—such as providing hints and structured questioning—was identified as essential in supporting learners through the modelling process. Scaffolding should involve a gradual progression from simpler to more complex questions to guide conceptual development.

Nguyen (2016) concludes that sustained engagement with modelling activities is necessary, rather than one-off implementation. He suggests that mathematical modelling should become an ongoing practice in the classroom to improve its effectiveness. Ultimately, he anticipates that the findings of his research could contribute to strengthening the teaching and learning of modelling, with the possibility that mathematical modelling and problem-solving may become central competencies assessed in Vietnam in the future.

2.11.2 Challenges of implementing mathematical modelling

According to Bawa and Zubairu (2015), the use of mathematical modelling in classroom instruction is a demanding and time-consuming process. They further note that assessment poses a significant challenge, largely because learners may generate multiple valid solution strategies for the same problem. As a result, teachers are required to anticipate and understand a wide range of possible learner responses. This complexity makes it difficult to assess modelling tasks fairly and consistently. Consequently, the intended learning outcomes of modelling cannot always be evaluated with the same level of objectivity typically associated with traditional Mathematics assessment. Educators who lack experience in mathematical modelling often struggle with this perceived lack of standardisation (Ramos-Rodríguez et al., 2021). In addition, time constraints remain a recurring concern in the implementation of modelling activities.

The issue of time is also highlighted by Gould (2013) in her doctoral study, which investigated teachers' perceptions and misconceptions about mathematical modelling. Participants in her study reported that mathematical modelling activities are time-intensive and difficult to implement within the constraints of regular classroom schedules. Contributing factors included an already congested curriculum and the extensive preparation required to design and facilitate effective modelling lessons. These challenges collectively limit the practical integration of modelling into everyday teaching practice.

Similarly, Kaiser (2020) observes that although mathematical modelling plays a valuable role in supporting the teaching and learning of Mathematics, its actual use in schools remains limited and below expected levels. Teachers, in many cases, experience more challenges than benefits when attempting to implement modelling-based instruction. Kaiser (2020) categorises these challenges into four broad areas, as outlined below.

a) Impediments or hindrances based on standpoint of instruction

The implementation of mathematical modelling in classroom instruction is demanding and requires considerable time. Blum and Niss (1991) argue that the typical time allocated to Mathematics lessons—usually between 45 minutes and one hour per period—is insufficient for the effective use of modelling approaches. In the context of South African secondary schools, Mathematics is allocated approximately 4 hours and 30 minutes per week in the current curriculum (DBE, 2011). This translates to about 54 minutes per day. Given this limited time allocation, it becomes difficult to implement mathematical modelling strategies in a meaningful and sustained manner within regular classroom periods.

b) Pupil-related obstacles

According to Kaiser (2020), mathematical modelling lessons are often unpredictable, which discourages rote learning or memorisation strategies. The author further notes that learners frequently experience difficulties even when attempting a single step within the modelling process. These challenges are compounded when learners are required to navigate all stages of the modelling cycle. As a result, many learners tend to prefer more structured problem types in which they can simply apply a known formula to obtain a solution.

c) Teacher-related obstacles

Educators often encounter difficulties when implementing mathematical modelling in the classroom. Kaiser (2020) notes that these challenges include the considerable time required to prepare, update, and adapt modelling tasks for instructional use. Teacher-related obstacles also extend to the demands of classroom practice, which become more complex and less predictable when modelling approaches are used. In addition, teachers are required to develop new skills and competencies to effectively facilitate this approach to teaching and learning. Blum and Niss (1991) further observe that some teachers lack confidence in their ability to teach mathematical modelling effectively. They also highlight that difficulties arise when modelling tasks are derived from contexts outside a teacher's area of specialisation. For example, a

Mathematics teacher without a background in Physical Sciences may struggle to support learners in solving modelling problems based on Physical Sciences contexts. This highlights the need for interdisciplinary understanding and adequate professional development to support effective implementation of mathematical modelling.

d) Material-related obstacles

Kaiser (2020) notes that educators currently have insufficient modelling examples available to support effective classroom implementation. As a result, many teachers eventually exhaust the limited materials at their disposal. Ferri and Blum (2009) observe that although mathematical modelling has been made compulsory in Mathematics curricula in countries such as Germany and is assessed as part of learners' achievement goals, it cannot be assumed that learners are taught by teachers who possess adequate expertise in modelling. They further argue that many teachers were not sufficiently trained in mathematical modelling during their pre-service preparation at colleges and universities. This challenge is compounded by the fact that modelling is often vaguely specified in curricula, leaving future teachers without clear guidance on how to interpret and implement it effectively.

Teachers are expected to be subject experts who can support learners effectively. Ferri and Blum (2010) emphasise that Mathematics teachers should not only be knowledgeable in their subject content but should also have specialised competence in mathematical modelling. This would enable them to guide learners appropriately and design learning environments that encourage active engagement in modelling processes. Traditionally, the role of the teacher was viewed as that of a transmitter of knowledge; however, contemporary perspectives position the teacher as a facilitator of learning (Bligh, 1998, as cited in Bye, 2017). The shift away from teacher-centred instruction was intended to reduce passive learning and promote the development of learners' skills, competencies, and personal growth. Despite this shift, the teacher continues to play a central and influential role in the classroom.

It remains essential for teachers to have a strong grasp of the content they teach, given their critical role in shaping learners' understanding. Ma'rufi et al. (2017) argue that effective Mathematics teaching requires not only strong subject-matter knowledge but also the ability to present content in ways that promote meaningful understanding. However, content knowledge and pedagogical delivery alone are not sufficient. Teachers also need a deep understanding of learners, curricula, educational goals, and instructional materials. Teaching mathematical modelling, in particular, is a complex task. Scholars such as Hernández et al. (2016) and Cheng (2013) confirm that many teachers—especially those new to modelling—experience significant difficulties when implementing it.

Hernández et al. (2016) illustrate these challenges by identifying several guiding questions that teachers may struggle with when preparing modelling lessons. These include how to formulate contextual questions, what additional information learners may require, how such information should be obtained, what assumptions should be made during model development, how to support learners in generating assumptions, which problem-solving strategies learners are likely to use, how to manage group and whole-class interactions, when learners are likely to experience difficulties, how to intervene without taking over the process, and how learners can evaluate their models and solutions.

Cheng (2013) further demonstrates the practical difficulties teachers face when they lack sufficient knowledge and experience in mathematical modelling. He describes a case in Singapore where an inexperienced teacher designed a modelling task based on his own limited understanding. Due to insufficient conceptual grounding in modelling, the teacher frequently intervened during the lesson, effectively guiding learners toward predetermined solutions rather than allowing them to engage in independent modelling. Similarly, Ramos-Rodríguez et al. (2021) identify the formulation of real-world problems as another major challenge faced by teachers implementing modelling approaches.

It would have been valuable for the present investigation to explore teachers' knowledge and competence in mathematical modelling, including whether they were exposed to it during their university training. However, such an exploration falls outside the primary focus of this study, which is centred on learners' modelling competencies. Nevertheless, existing research provides important insights. For example, Ciltas and Isik (2013) examined the modelling skills of 35 prospective primary school Mathematics teachers enrolled in a modelling course. A pre-test revealed that participants experienced significant difficulties across various stages of the modelling process, with a mean score of 25.35%.

After a four-week intervention on mathematical modelling, a post-test showed substantial improvement, with the average score increasing to 72.85%. Despite this improvement, participants still struggled particularly with interpreting their solutions. Nevertheless, interviews indicated that the prospective teachers valued the inclusion of mathematical modelling in the curriculum due to its connection with real-life situations.

Similarly, Kaygısız and Şenel (2023) investigated the modelling competencies of Grade 4 learners using a real-life problem involving the planting of olive trees in a rectangular area. Learners were required to determine spacing and the number of trees that could be planted. Data were collected through written work, video recordings, observations, and field notes. The findings showed that while learners were able to formulate initial models, many struggled to develop accurate representations suitable for real-world problem solving. Time constraints also affected their performance, leading some learners to include unnecessary diagrams and tables. Overall, learners demonstrated partial competency, successfully completing the early stages of modelling but experiencing difficulties in interpreting and validating their solutions.

Despite the challenges outlined in this section, the positive contributions and necessity of mathematical modelling, as discussed in Section 2.6.2, remain evident. It is therefore essential to explore strategies that can mitigate or overcome the challenges associated with its implementation in Mathematics education.

2.11.3 Measures to eliminate challenges of implementing mathematical modelling

Bawa and Zubairu (2015) suggest that challenges associated with assessing mathematical modelling can be reduced by using multiple assessors as well as well-designed rubrics with clearly defined criteria and weighted components. They further note that teachers may also adapt rubrics sourced from online professional networks to suit specific assessment contexts. In addition, they argue that the modelling cycle itself can be used as a framework for developing appropriate assessment criteria, thereby improving fairness and consistency in evaluation.

Despite these assessment challenges, Bawa and Zubairu (2015) emphasise several important reasons for including mathematical modelling in school curricula. Firstly, learners need to develop the ability to apply Mathematics in practical contexts from an early stage, which prepares them for higher education, workplace demands, and everyday problem-solving. Early exposure to modelling can therefore strengthen learners' overall understanding of Mathematics. Secondly, mathematical modelling enables learners to develop a more holistic understanding of Mathematics as an interconnected discipline. Lastly, modelling supports the development of imagination and reasoning in relation to real-world phenomena that can be represented using mathematical functions, including linear relationships.

Although challenges exist in implementing mathematical modelling, various strategies can be employed to address these difficulties. Asempapa and Sturgill (2019) propose four key instructional strategies for effective implementation. Firstly, teachers should develop a deep understanding of the modelling process and be aware of the different approaches learners may use. This requires careful observation of learners as they define problems, construct solutions, and interpret their models. Secondly, teachers should engage closely with learners by listening to their discussions and encouraging dialogue, debate, and justification of ideas, thereby promoting critical reflection on the validity and relevance of models. Thirdly, teachers should create a learning environment that allows learners to work independently with minimal intervention, enabling them to take ownership of the modelling process. Lastly, classrooms should be structured in ways that promote positive attitudes towards modelling, using authentic tasks drawn from real-life situations.

Passarella (2021) argues that the successful implementation of mathematical modelling can be enhanced through improved pre-service teacher training, as well as ongoing professional development for in-service teachers. Such training should include exposure to realistic problem-solving activities and pedagogical approaches such as problem posing. In addition, introducing learners to mathematical modelling at an early stage of schooling is beneficial. Supporting this view, Wei et al. (2022) found that early exposure to modelling promotes the development of learners' mathematical representation, reasoning, communication, and discussion skills. These issues are further elaborated in Section 2.4.

2.11.4 Benefits of implementing mathematical modelling in the formative years of education

Mathematical modelling should be introduced during the early phases of schooling, as many learners experience Mathematics as an abstract subject. In this view, Mathematics is often perceived as a collection of rules, procedures, and concepts that are disconnected from real-world applications. There is therefore a need to help learners recognise the relevance and usefulness of Mathematics in everyday life, which can be effectively achieved through early exposure to mathematical modelling. Several studies support this view, including Wei et al. (2022) and the Common Core State Standards Initiative (CCSSI, 2010).

As noted in Section 2.3.4, early exposure to mathematical modelling is also one of the ways of reducing implementation challenges in later grades. Wei et al. (2022) conducted a study involving Grade 3 learners and their teachers, in which a six-month intervention programme was introduced. The activities were largely social and collaborative in nature, requiring learners to work together to solve problems. Through these interactions, learners engaged in discussions, developed hypotheses, debated ideas, and generated solutions. This collaborative engagement contributed significantly to the development of their mathematical reasoning, representation, and explanatory skills.

Similarly, the CCSSI (2010), a United States initiative aimed at developing common standards for Grades R–12 in Mathematics and literacy to prepare learners for higher education and the workplace, emphasises that mathematical modelling should begin in the lower grades. It highlights that mathematically proficient learners should be able to apply previously learned concepts to solve problems arising from real-life situations, workplace contexts, and community-based challenges. At the primary school level, this may involve representing simple addition problems mathematically. In the middle grades, learners may use concepts such as ratio and proportion to plan school events or analyse community-related issues. By high school, learners are expected to apply geometric properties and spatial reasoning to solve complex and often open-ended problems requiring strategic thinking.

Stohlmann and Albarracín (2016) further argue that although mathematical modelling has traditionally been emphasised at the high school level, greater attention should be given to its development in primary education if learners are to become proficient modellers. They emphasise that modelling competence develops gradually over time and therefore must be nurtured from an early age. Early exposure to mathematical modelling not only enhances learners' appreciation of Mathematics but also helps them to view it as a meaningful and applicable tool for understanding their daily experiences. This early engagement can ultimately foster more positive attitudes towards Mathematics from a young age.

2.12 Relevance of critical thinking, communication, collaboration and creativity to mathematical modelling in education

In Chapter One, creativity, critical thinking, communication, and collaboration were identified as essential competencies required of learners in the 21st century. Since this study focuses on learners' competencies in mathematical modelling, it is important to provide a detailed discussion of these skills and their relevance to modelling processes. The current South African Mathematics curriculum places strong emphasis on mathematical modelling, with real-life problem contexts integrated across all phases of the curriculum. In addition, the curriculum outlines clear aims for Mathematics education in South Africa, which reflect these competencies.

Three key aims of the curriculum were considered in this regard, all of which align with the 4Cs. In summary, the curriculum seeks to develop learners who can:

• Draw conclusions through the application of critical thinking and creativity.
• Work effectively both independently and collaboratively in group settings.
• Communicate mathematical ideas clearly using symbols, visual representations, and appropriate language across different contexts (DBE, 2011a).

A closer examination of these aims reveals a strong alignment with the 4Cs—creativity, critical thinking, collaboration, and communication. This indicates that the current Mathematics curriculum is designed to equip learners with competencies that are essential for success in the 21st century. The relevance of these skills to mathematical modelling is further explored in the following paragraphs.

2.12.1 Creativity and its importance in acquisition of Mathematics plus mathematical modelling

Creativity plays an essential role in education. At school level, the concept of creativity is most associated with subjects such as Visual Arts and English Language. In the South African curriculum, for example, English Paper 3 includes a Creative Writing component in which learners are expected to produce compositions and letters using imagination and original ideas. Similarly, one of the key objectives of the Visual Arts curriculum is to encourage learners to investigate, create, and realise innovative ideas in response to both structured and open-ended tasks by drawing on personal experience and knowledge of historical and contemporary visual studies (DBE, 2011b). In both subjects, creativity is explicitly foregrounded, which contributes to the common perception that creativity is mainly confined to artistic disciplines.

However, Cropley (2020) argues that creativity is not limited to the arts but is also evident in fields such as technology, business, medicine, manufacturing, defence, administration, and education. Creativity may result in tangible products such as books, artworks, music, buildings, and machines, as well as intangible outcomes such as ideas, services, processes, and systems of operation. When such outputs are original, novel, or effective in achieving a desired purpose, creativity is considered to be present. It may also manifest in different forms of human activity, ranging from abstract expressions such as conveying emotion or aesthetic appreciation to adopting new ways of interpreting or understanding phenomena.

The concept of creativity is often used in three related ways: to describe processes (e.g., innovative thinking), personal characteristics (e.g., a creative individual), and outcomes (e.g., an original product). In this sense, creativity can be understood both as a process that leads to new products and as a product emerging from the interaction between the individual and the process. This is captured in the well-known "4Ps" model—person, process, product, and press—where "press" refers to environmental factors that either support or constrain creativity (Cropley, 2020). According to the Merriam-Webster Dictionary, creativity refers to the ability to produce new things or generate original ideas. This definition is particularly relevant to Mathematics, where learners are often required to develop their own strategies to solve problems, especially in non-routine or unfamiliar contexts.

2.12.2 Definition of mathematical creativity

Nadjafikhaha, Yaftian and Bakhshalizadeh (2011) argue that defining mathematical creativity is a complex task, as it is difficult to clearly delineate its structure and characteristics. Although there is no universally accepted definition, various scholars have proposed different interpretations. According to Poincaré (1948, as cited in Sriraman, 2008), mathematical creativity can be understood as "discernment and choice." Sriraman (2008) interprets "choice" in this context as the ability of learners of Mathematics to distinguish between problems or ideas that lead to meaningful mathematical progress and those that do not generate new insights. However, this definition has been criticised for neglecting the aspects of originality and innovation. In this sense, reducing mathematical creativity to selection between valuable and non-valuable elements is likened to equating artistic sculpture with merely removing unwanted material.

Laycock (1970, as cited in Nadjafikhaha et al., 2011) defines mathematical creativity as the ability to analyse a problem from multiple perspectives, identify patterns, generate examples and variations, and select appropriate strategies for solving unfamiliar mathematical problems. Similarly, Chamberlin and Moon (2005) describe mathematical creativity as the ability to produce original and novel solutions to real or hypothetical problems through mathematical modelling. Davis (2018) further highlights that there is currently a “creativity gap” in many schools, as most creative engagement occurs outside formal classroom settings. She argues that creativity should not be viewed as an optional enrichment activity but rather as a measurable and essential skill that enhances learning, strengthens motivation, deepens understanding, and increases learner enjoyment. For creativity to flourish, intrinsic motivation is essential.

Although Mathematics has a long-established history as a discipline, creativity remains an essential requirement in the Mathematics classroom. However, based on my experience as a Mathematics teacher and subject advisor supporting Grade 10 to 12 teachers, I have observed that limited opportunities for creativity are created in many classrooms. The demanding and densely packed curriculum in Grades 10 and 11 often leaves little room for creative exploration, as teachers focus primarily on completing the prescribed syllabus. In Grade 12, there is additional pressure to complete the curriculum early, often by July or August, to allow sufficient time for revision, which largely consists of practising past examination papers. Furthermore, accountability pressures from the Department of Basic Education, particularly in underperforming schools, tend to encourage procedural teaching approaches focused on algorithmic drilling. As a result, both teacher and learner creativity is often suppressed. Teachers rarely adopt instructional strategies that promote creativity, while learners are encouraged to follow fixed procedures. This leads to a classroom environment where learners frequently adopt a single method for solving mathematical problems, thereby limiting creative expression.

Mann (2006, as cited in Grégoire, 2016) argues that the goal of Mathematics instruction should extend beyond obtaining correct answers and should instead focus on developing innovative and creative thinkers. Given that Mathematics is embedded in many areas of scientific inquiry, it is necessary to integrate creativity into its teaching and learning. Developing mathematical talent therefore involves more than transmitting procedural knowledge; it requires providing learners with opportunities to engage with open-ended problems that do not have predetermined solutions or fixed answer sets (Grégoire, 2016). However, Grégoire (2016) notes that mathematical creativity has historically not received the same attention as creativity in the arts, possibly due to the misconception that Mathematics is purely logical and rule-bound, leaving little room for innovation. This perception has contributed to an overemphasis on reproduction of procedures rather than exploration and discovery.

Creativity implies that an outcome is original and not merely a replication of existing ideas. The degree of novelty may vary across tasks, ranging from entirely new solutions to modified or adapted versions of existing ideas. Importantly, creative ideas must also be useful, meaningful, and appropriate to the intended purpose. While creative thinking allows for innovation, it is still constrained by factors such as language, logic, available resources, and conceptual frameworks (Grégoire, 2016). To foster creativity in Mathematics, learners should be exposed to ill-structured and open-ended tasks over sustained periods and be given the freedom to explore multiple solution strategies. This enables them to develop alternative perspectives and deepen their conceptual understanding through problem formulation and re-interpretation.

Muscat (2015) similarly argues that Mathematics teaching should not be confined to textbooks, classroom routines, or narrowly defined curriculum outcomes. Instead, it should provide learners with opportunities to apply mathematical ideas in meaningful, real-life contexts. From the early grades, teachers should be supported in adopting innovative pedagogical approaches that move beyond traditional methods dominated by factual recall and procedural exercises. Such approaches enable learners to become active problem-solvers and independent thinkers. When learners engage with authentic, real-world mathematical problems, they are better able to develop critical thinking skills and transfer knowledge across different contexts. In this way, Mathematics education can nurture learners who are critical, confident, creative, and willing to take intellectual risks. These learners are more capable of investigating problems, formulating questions, and designing their own mathematical inquiries based on lived experiences. Exposure to such approaches strengthens their problem-solving abilities and enhances their capacity to apply mathematical knowledge in diverse real-life situations (Muscat, 2015).

The discussion above highlights the importance of integrating real-life contexts in Mathematics teaching. Such integration enables learners to make meaningful connections between classroom Mathematics and everyday experiences. Muscat (2015) further notes that creatively exploring Mathematics is a two-way process, allowing teachers to assess learners' understanding of key concepts while simultaneously developing their skills. She identifies several benefits of creative exploration in Mathematics, including promoting learner responsibility, encouraging inquiry and problem-solving, strengthening real-life connections, fostering collaboration and communication, stimulating imagination and innovation, supporting the use of technology, accommodating diverse learners, enhancing reasoning, integrating subject areas, and improving conceptual understanding and retention.

Many of these benefits align closely with the principles of mathematical modelling, particularly in relation to collaborative learning, real-world application of Mathematics, the promotion of innovation, and the development of problem-solving and exploratory skills.

2.11.3 Critical thinking and its usefulness to mathematical modelling

Ozkahraman (2011, as cited in Simbolon et al., 2017) defines critical thinking as a process of searching for, acquiring, judging, reviewing, and comprehending information in order to broaden perspectives through metacognitive engagement and the ability to apply knowledge creatively while taking calculated risks. Similarly, Rowe et al. (2015) describe critical thinking as the ability to draw appropriate conclusions based on evidence, reasoning, and intellectual independence. Ennis (1991) defines it as reasonable reflective thinking aimed at deciding what to believe or do. He further includes creative dimensions within critical thinking, such as formulating hypotheses, making assumptions, considering alternative perspectives, posing questions, and exploring possible solutions. This definition highlights reflection, rationality, and decision-making as central components of critical thought.

Ruggiero (2012, as cited in Murawski, 2014) characterises critical thinkers as individuals who move beyond conventional patterns of thinking to more advanced and analytical levels of reasoning. Such individuals tend to generate a wider range of ideas than average thinkers and refine these ideas through systematic questioning and evaluation. They can examine problems from multiple perspectives, explore alternative approaches, and generate diverse solutions before selecting the most appropriate course of action. In addition, critical thinkers are more willing to take intellectual risks and explore unconventional ideas. Their decisions are guided by evidence rather than emotion, and they are aware of their own limitations. They also consistently evaluate the logic and feasibility of their reasoning in order to improve their thinking processes.

In Mathematics, problem-solving requires learners to read and interpret problems carefully, especially when encountering unfamiliar tasks. The emphasis is not only on obtaining the correct answer but also on demonstrating the reasoning process, including the steps, justification, and logical arguments used. This requires learners to think systematically and logically. Critical thinking was therefore included in this study because participants were required to solve word problems that demanded reasoning, interpretation, and informed decision-making regarding appropriate methods and formulae.

Raudenbush (2016) emphasises that critical thinking is a key factor in Mathematics because it distinguishes learners who merely perform procedures from those who truly understand mathematical concepts. Learners who rely on memorised formulas and routine procedures often lack conceptual understanding and focus only on reproducing learned steps without questioning their validity. In contrast, learners who think critically can explain why procedures work and justify the strategies they use to arrive at solutions.

Educators' Voice (2015) argues that children are naturally curious and are born with foundational scientific skills such as observing and questioning their environment to make sense of the world around them. However, this natural curiosity is often not sufficiently nurtured within formal schooling systems that tend to apply a uniform, one-size-fits-all curriculum. Early childhood educators therefore have a responsibility to cultivate learners' curiosity and sense of wonder to develop their problem-solving, investigative, and critical thinking abilities, which are essential for participation in the 21st-century world of work. This includes preparing learners for a labour market that requires critical thinking, collaboration, investigation, and effective communication of ideas.

According to Educators' Voice (2015), modern workplaces increasingly demand individuals who can think creatively, approach problems from multiple perspectives, and develop innovative solutions. In response, educational practices must shift away from traditional teacher-centred approaches in which learners are passive recipients of information and instead adopt more active, learner-centred methodologies. Inquiry-based learning is one such approach that promotes active engagement. In this model, learners are encouraged to ask meaningful questions, investigate problems, and evaluate possible solutions through experimentation and reflection. The teacher's role is to facilitate learning by guiding inquiry, supporting analysis, and encouraging learners to justify and defend their ideas. Rather than providing fixed answers, teachers pose open-ended questions that promote deeper thinking and sustained exploration. Active listening and responsiveness to learners' ideas are essential in supporting this process (Educators' Voice, 2015).

Educators' Voice (2015) further suggests that gamification can be used to enhance mathematical thinking. Gamification does not simply involve playing games for enjoyment, but rather the strategic use of game design principles and mechanics to promote engagement, motivation, and problem-solving. As Kapp (2012, as cited in Educators' Voice, 2015) explains, gamification involves the purposeful application of game elements and interactive design to engage learners and support learning. When effectively implemented, it can increase learner motivation, sustain engagement, and enhance understanding, particularly among younger learners. Through such interactive approaches, learners can develop stronger critical and mathematical thinking skills.

Simbolon et al. (2017) conducted a study investigating the impact of a problem-solving instructional strategy supported by Macromedia Flash on learners' critical thinking skills in solving linear equations with two variables. They emphasise that critical thinking is essential in Mathematics, yet many learners still demonstrate low levels of this skill due to traditional teaching approaches. In many classrooms, learners are required to follow procedural steps without opportunities to explore or justify their thinking, which limits their ability to reason independently. The study therefore highlights the importance of instructional approaches that actively develop critical thinking in Mathematics.

The findings of Simbolon et al. (2017) revealed that problem-solving strategies supported by Macromedia Flash were more effective than traditional lecture-based instruction. The use of interactive multimedia enhanced learners' ability to understand abstract concepts by making them more visual and accessible, while also increasing engagement and participation. The study concluded that instructional innovation can significantly improve learners' critical thinking skills and overall performance in Mathematics.

However, in many classrooms, teaching still emphasises memorisation of procedures rather than conceptual understanding. This approach limits learners' ability to solve unfamiliar or non-routine problems. To develop critical thinking, learners should be encouraged to analyse, interpret, organise, and evaluate information when solving problems. In this way, they are better prepared to tackle complex mathematical tasks. It is therefore important for teachers to move away from rote learning approaches.

Furthermore, the structure of some Mathematics textbooks may unintentionally reinforce procedural learning, as content is often organised strictly by topic. This can lead learners to associate specific methods with specific topics without developing deeper understanding of underlying concepts. For example, in topics such as Differential Calculus, learners may easily identify procedures to follow without engaging in critical reasoning. This highlights the need for teaching approaches that prioritise conceptual understanding and critical thinking over procedural repetition.

Example

The length of a rectangle is x metres and its breadth is $(16-x)$ metres. Determine the dimensions of the rectangle for it to have a maximum area.

Solution

$$\begin{aligned}\text{Area} &= \text{length} \times \text{breadth}\\ &= x(16-x)\\ &= 16x - x^2\end{aligned}$$

They will then differentiate the area and put the equation equal to zero.

$$\text{Area} = 16x - x^2$$

$$\frac{d\text{Area}}{dx} = \frac{d}{dx}(16x - x^2) = 16 - 2x$$

$$\begin{aligned}0 &= 16 - 2x\\ 2x &= 16\\ x &= 8\end{aligned}$$

$\text{Length} = x = 8 \text{ metres}$

$\text{Width} = 16 - x$

$= 16 - 8$

$= 8 \text{ metres}$

$\therefore$ The length is 8 metres, and the width is 8 meters.

When learners are still working on this topic, they may perform well because the teacher has drilled them on the procedures required to solve optimisation problems. However, when they are assessed later in examinations or in tasks that integrate multiple topics, they may struggle to recognise that the same underlying principles apply. As a result, they may fail to identify the appropriate strategy used during classroom instruction. In many cases, teachers focus on teaching learners procedural rules for solving specific types of problems without adequately explaining the underlying reasoning for each step. Consequently, learners are often encouraged to follow fixed routines rather than develop conceptual understanding. Typically, in solving optimisation problems, the following steps are emphasised.

:

2.12.4 Procedures for Resolving Optimisation Problems

- Use symbols, variables, and sketches, when applicable, translate the problem and come up with an equation representing the situation. In the above example, the equation is given by $\text{Area} = 16x - x^2$
- Determine the first derivative of the equation made simpler and equate it to zero. The above example yielded $16 - 2x = 0$
- Solve the value(s) give the maximum or minimum quantity and verify if they yield the desired solutions.

To support learners in developing genuine understanding of mathematical problems rather than relying on memorised procedures, textbooks should be structured in a way that promotes conceptual thinking. For example, at the end of each topic, textbooks should include mixed and integrated questions drawn from different topics. This would require learners to carefully analyse each problem and determine the appropriate strategy to use, rather than simply recalling and reproducing steps taught in class.

Amalia et al. (2019) conducted a study on the importance of critical thinking, focusing on learners' critical thinking skills when solving trigonometric ratio problems using a mathematical modelling approach. As part of their study, they developed a worksheet containing modelling tasks, a critical thinking assessment instrument, and a lesson plan on trigonometry based on modelling. A six-step modelling cycle was used, consisting of: (1) identifying and defining the problem, (2) formulating assumptions and identifying key variables, (3) mathematizing the situation, (4) analysing and evaluating the mathematical representation and its solution, (5) iterating or refining the model where necessary, and (6) implementing the model and communicating the results.

The study assessed critical thinking using four indicators: interpretation, analysis, inference, and evaluation. The results showed that analysis scored lower than interpretation and evaluation. This was attributed to challenges experienced during the assumption stage of the modelling process, which was identified as one of the most difficult phases for learners. A comparison of performance revealed that learners who were able to make appropriate assumptions performed better in the analysis component. Furthermore, strong analytical skills positively influenced inference skills, as learners who struggled to analyse problems accurately also had trouble in drawing valid conclusions.

Desmita (2005, as cited in Amalia et al., 2019) describes critical thinkers as individuals who reflect deeply on problems, delay judgement until sufficient information is available, and carefully analyse issues before making decisions. Such learners also understand the reasoning behind their actions. This underscores the importance of deliberately developing critical thinking skills in Mathematics education.

Similarly, Su et al. (2016) propose that critical thinking can be strengthened by emphasising reasoning, logic, and validity in problem-solving. Learners should be guided to move systematically from assumptions to conclusions in a logical manner. They should also be encouraged to generate multiple possible solutions, evaluate their validity, and justify their reasoning. In doing so, learners develop into reflective critical thinkers who are aware of how they use their mathematical knowledge. This reflective awareness is known as metacognition. Spencer (2018) further argues that metacognitive skills are essential for developing critical thinking and fostering lifelong learning, as they enable learners to become self-directed and capable of independently seeking knowledge in a rapidly changing world.

Since this study involved learners solving mathematical problems embedded in real-life contexts, it was important to observe the strategies they used in solving the tasks. It became evident that some learners were unsuccessful because they did not apply critical thinking or metacognitive strategies to fully understand the problems. They struggled with comprehension of the questions. For instance, in Question 3 of the test, learners were required to determine the dimensions of a rectangular garden that could be fenced on three sides using a 72-metre fence, given that the fourth side was already bordered by a rock wall. Some learners failed to identify key information in the question, such as the fact that only three sides required fencing. Consequently, they incorrectly calculated as though all four sides needed fencing.

In this study, evidence of critical thinking was observed when learners were able to fully interpret the problem, formulate appropriate assumptions, identify relationships between variables, and construct correct mathematical models (equations or expressions) representing the real-life situation. These learners proceeded to solve the models using appropriate mathematical procedures and obtained correct solutions. However, many participants failed to interpret their final solutions in relation to the real-world context and did not validate their answers. It can therefore be concluded that while some elements of critical thinking were demonstrated, it was not applied consistently across all stages of the modelling process.

2.13. Communication and its importance to mathematical modelling

The DBE (2011a) Mathematics curriculum document defines Mathematics as "…a language that uses notations and symbols to describe relationships in numbers, geometry, and graphs…" (DBE, 2011a:8). From this definition, Mathematics can be understood as a system of communication that relies on symbolic, visual, and linguistic representations. As a language, it must be communicated through both spoken and written forms, including words, symbols, graphs, and geometric representations. Communication therefore plays a central role in Mathematics teaching and learning and must be deliberately developed.

Mathematical communication enables learners to organise and express their mathematical thinking, both orally and in written form, as they engage in problem-solving. Once learners develop mathematical understanding, they should be able to communicate their reasoning clearly to others. In turn, the act of explaining mathematical ideas contributes to deeper conceptual understanding (Qohar, 2011).

One of the key aims of South Africa's Mathematics curriculum is to develop learners who can communicate effectively in various forms using symbolic, visual, and linguistic representations. In addition, the curriculum is grounded in an active, inquiry-based learning approach that promotes learning through doing rather than rote memorisation of procedures (DBE, 2011a). These two principles—active learning and mathematical communication—are closely interconnected, as meaningful participation in learning requires learners to articulate their thinking clearly. This implies a shift from traditional teacher-centred approaches, where the teacher is the sole source of knowledge, to learner-centred approaches where learners actively construct knowledge and communicate their ideas. In such classrooms, communication becomes unavoidable as learners explain, justify, and refine their thinking.

In recent years, there has been renewed emphasis on communication in Mathematics education, particularly through open-ended instruction and problem-solving approaches. These approaches recognise the importance of language in the teaching and learning of Mathematics. Historically, language received limited attention because Mathematics was often viewed as a subject centred on following fixed procedures, requiring minimal conceptual engagement. Consequently, creativity and communication were not prioritised. However, this view is gradually shifting towards process-oriented approaches that emphasise understanding, reasoning, and learner independence, all of which are supported through classroom communication (Halawati & Laelaseri, 2022).

Halawati and Laelaseri (2022) further argue that the adoption of active learning approaches, grounded in constructivist principles, requires a strong focus on mathematical communication. In active learning classrooms, learners are encouraged to discuss ideas, share reasoning, and engage collaboratively with mathematical tasks. Mathematical problems are often presented in multiple representations, including symbols, numbers, graphs, and verbal descriptions. Word problems require learners to interpret written language, translate it into mathematical expressions or equations, and then solve them. Therefore, learners who struggle to comprehend the language of a problem are unlikely to formulate correct mathematical models, resulting in unsuccessful problem-solving.

Mathematical modelling tasks typically take the form of word problems grounded in real-life situations and begin with a guiding question such as: "What is the maximum area that can be enclosed by a fence of a given length?" Because these tasks are presented in verbal form, learners must read carefully, interpret meaning, and translate the information into mathematical representations. Stohlmann and Albarracin (2016) emphasise that effective verbal and written communication is essential during mathematical modelling activities. They further note that learners should collaborate, exchange ideas, and communicate their reasoning throughout the modelling process. Upon completion, learners are expected to clearly explain the mathematical concepts used, the modelling steps followed, and the reasoning behind their solutions. This makes both oral and written communication integral to modelling.

Similarly, Vale and Barbosa (2017) argue that communication is fundamental to learning Mathematics and should therefore be intentionally developed in classrooms. Teachers can support this by allocating time for discussion and encouraging learners to share their mathematical thinking. They further contend that mathematical language enables learners to articulate their reasoning, refine their ideas, and communicate strategies accurately to themselves and others. The quality of communication in the classroom is influenced by the nature of tasks, the types of questions asked, and the extent of teacher-facilitated discussion. Therefore, teachers should design learning environments that promote multiple forms of communication to support mathematical reasoning and understanding.

2.13.1 Collaboration and its importance to mathematical modelling

Developing learners who can work effectively both individually and collaboratively is one of the goals of South Africa’s current Mathematics curriculum (DBE, 2011a). This implies that some activities are designed for individual work, while others are intended to be completed in groups to promote collaboration and shared learning. From my own experience as a learner, mathematical tasks were primarily completed individually. Group work was mostly reserved for subjects such as Physical Sciences, particularly during practical experiments. This was largely influenced by the limited availability of laboratory apparatus, as theoretical work was still done individually. Learning predominantly as individuals tended to foster a competitive environment, with limited willingness to assist peers. In contrast, contemporary educational approaches have shifted significantly towards collaborative learning. Current developments in education increasingly emphasise group-based learning as a strategy for enhancing understanding and promoting social interaction. In line with this shift, the South African Mathematics curriculum now strongly supports collaborative learning as part of effective teaching and learning practice (DBE, 2011a).

2.13.2 Collaborative and cooperative learning

Different scholars define collaborative and cooperative learning in varied ways, and in some cases the terms are used interchangeably. However, Panitz (1996) distinguishes between the two. He defines collaboration as a philosophy of interaction among individuals, characterised by shared responsibility for actions, mutual respect, and recognition of peers' skills and contributions. In contrast, cooperation is viewed as a structured form of interaction aimed at achieving a specific product through group effort.

Although collaboration and cooperation are conceptually different, they are often used interchangeably because both involve learners working in groups. Kozar (2010) also makes a clear distinction between the two. She explains that cooperative learning focuses on group members working together to produce a final product, whereas collaborative learning emphasises shared engagement in the entire knowledge-building process. In cooperative learning, group members may complete tasks individually and then combine their outputs, while in collaborative learning there is continuous interaction, discussion, negotiation, and shared decision-making throughout the task.

Research indicates that learners often benefit more from group-based learning than from working individually. Sofroniou and Poutos (2016), for example, investigated engineering students enrolled in a Mathematics module to determine whether group work could enhance critical and analytical thinking, deepen conceptual understanding, and improve academic performance. Their findings showed that group-based learning improved learners' understanding of content and led to better examination performance.

Page (2017) describes cooperative learning as a learner-centred instructional approach in which learners work together to develop shared understanding of a concept. In this approach, learners contribute their knowledge, skills, and ideas to construct meaning collectively, while the teacher's role shifts from information transmitter to facilitator. Each learner is assigned responsibilities and remains accountable for contributing to the group's success. In addition to academic learning, learners also develop teamwork skills and an understanding of interdependence within groups.

In this study, the terms cooperative and collaborative learning are used synonymously, as both refer to learning in group contexts. This discussion was included to highlight the value of group work in learning. Due to the COVID-19 pandemic during data collection, learners completed the test individually rather than in groups. Javed (2013) defines cooperative learning as the use of small groups in which learners work together to enhance their own learning as well as that of their peers. His study in Pakistan found that learners taught through cooperative learning performed better in Mathematics than those taught through traditional methods, leading to the recommendation that group-based instructional approaches should be widely adopted.

Mandušić and Blašković (2015) argue that contemporary teaching approaches require teachers and learners to engage collaboratively in exploring and solving problems, generating ideas, and producing solutions. This process involves discussion, explanation, and evaluation of ideas. They further note that collaborative learning includes various forms such as cooperative learning, team learning, and community-based learning. The common feature across these approaches is group interaction. In collaborative learning, learners are jointly responsible for both the process and outcome of learning, with the teacher acting as a facilitator who guides rather than directs learning.

Johnson et al. (2014) similarly explain that learning environments may be structured as cooperative, competitive, or individualistic. Cooperative learning is characterised by positive interdependence, where learners work together to maximise both individual and group learning. In contrast, competitive learning involves learners working against one another, while individualistic learning requires learners to work independently without consideration of others' progress. Vega and Hederich (2015) add that these different structures shape learner interaction and influence learning outcomes in distinct ways. In cooperative settings, learners share responsibility, support one another, and are collectively accountable for success.

Barczi (2013) found that cooperative learning enhances problem-solving skills in Mathematics and encourages creative thinking. As learners become more familiar with group-based learning, they require less teacher support and demonstrate increased independence in solving problems. Similarly, Vega and Hederich (2015) found that cooperative learning improved learners' performance in Mathematics and language, particularly because high-achieving learners supported their peers, resulting in improved understanding and confidence among low-achieving learners.

Koçak et al. (2009) further emphasise that group work promotes active participation, self-confidence, and communication skills, while also reducing learner dependence on the teacher. Learners develop respect for others' opinions, learn to collaborate, form social connections, overcome fear of making mistakes, and improve their ability to express ideas.

Sofroniou and Poutos (2016) also demonstrated that learners who participated in group work performed better in challenging Mathematical topics such as integration compared to those taught through traditional methods. Participants reported improved understanding, increased confidence, and the development of multiple strategies for solving problems. Observations further indicated that group interaction enhanced analytical and critical thinking skills.

Overall, although scholars differ in their conceptualisation of cooperative and collaborative learning, both approaches fundamentally involve group-based learning. In this study, learners engaged in group-related problem solving involving real-life contexts, making it important to consider the benefits of such approaches. Šerić and Praničević (2018) highlight two key advantages of group work: first, it equips learners with collaboration skills needed beyond school; and second, it prepares them for workplace environments where teamwork, shared responsibility, and coordinated problem-solving are essential. This suggests that group work not only enhances academic learning but also prepares learners for effective participation in professional contexts.

2.14 Problem-solving

Since the existence of a problem is the essential catalyst for any problem-solving process, this section begins by defining the concept of a problem as described by various scholars. Voskoglou (2021), drawing on Schoenfeld (1983), Green et al. (2005), and Martinez (2007), notes that a problem can be understood as a barrier that must be overcome in order to achieve a desired goal. A key feature of a problem is that the solver does not immediately know the method or procedure required to reach a solution. Similarly, Mayer (2003), as cited in Rahman (2019), defines a problem as a situation in which a goal must be achieved, but no routine or readily available procedure exists for accomplishing it.

Rahman (2019) further argues that problem-solving is one of the essential 21st-century skills. He describes 21st-century skills as a broad set of knowledge, competencies, workplace behaviours, and personal attributes required for success in contemporary society, particularly in education and employment. Problem-solving is therefore regarded as a critical skill that learners need both within and beyond the classroom.

In the context of STEM education, Iwuanyanwu (2020) emphasises that one of the key purposes of teacher education is to prepare learners to become effective problem-solvers who can respond to challenges encountered in real-world and workplace contexts. He further argues that the nature of problems presented in STEM subjects should intentionally support the development of 21st-century skills. However, despite the importance of these competencies, insufficient attention is often given to explicitly teaching the skills required to achieve them. As a result, traditional teaching approaches—where textbooks often mediate instruction between teachers and learners—continue to dominate STEM education in many contexts.

Amalia et al. (2017) highlight that contemporary global developments require individuals to develop skills that differ significantly from those emphasised in the past. These include critical thinking, digital literacy, problem-solving, independence, and the ability to think autonomously while collaborating with others. The modern world is characterised by complex challenges such as pandemics (e.g., COVID-19), HIV and AIDS, unemployment, teenage pregnancy, climate change, and poverty. Addressing these challenges requires individuals who can identify problems, analyse their causes, generate possible solutions, evaluate alternatives, and implement appropriate interventions. Consequently, problem-solving is an indispensable skill that must be deliberately developed through schooling.

To develop strong problem-solving abilities, learners must be exposed to such skills from an early stage. Within the school curriculum, Mathematics is one of the key subjects through which problem-solving is explicitly emphasised. In several countries, including South Africa, Australia, and Singapore, problem-solving is a central component of the Mathematics curriculum. For example, the Australian Mathematics curriculum is structured around four proficiency strands—understanding, fluency, problem-solving, and reasoning—adapted from the framework proposed by Kilpatrick et al. (2001) (Atweh & Goos, 2011). In Singapore, problem-solving is the core focus of the Mathematics curriculum. It emphasises the application of mathematical ideas in diverse contexts, including non-routine, open-ended, and real-world problems. Problem-solving is therefore embedded as a pedagogical approach across all school levels (Ministry of Education Singapore, 2006).

Similarly, South Africa's Curriculum and Assessment Policy Statement (CAPS) requires that problem-solving and mathematical modelling be integral to the teaching and learning of Mathematics. This is reflected in two specific aims of the curriculum, which state that learners should:

• develop cognitive and problem-solving skills through tasks that require reasoning ("when" and "why") as well as procedures ("how"); and
• engage in mathematical modelling that incorporates real-life problems where appropriate (DBE, 2011a).

The problem-solving approach referred to in the first aim does not refer to rote or routine procedural application, but rather to a problem-centred approach, which is discussed further in subsequent sections. This is particularly relevant to the present study, which investigates learners' mathematical modelling competencies developed through engagement with real-life problem situations. Learners are expected to use both reasoning and mathematical knowledge to address problems that cannot be solved by simply recalling taught procedures.

In this sense, learners are required to move beyond routine problem-solving, where known steps are applied to familiar questions, towards a problem-centred approach that demands interpretation, reasoning, and selection of appropriate mathematical concepts. To clarify this distinction, the study further contrasts routine procedural problem-solving with more open-ended, problem-centred mathematical modelling tasks.

2.14.1 How routine problem-solving differs from problem-centred approaches

Killen (2006) distinguishes between routine problem-solving and the problem-centred approach. In routine problem-solving, learners are first taught specific procedures and are then expected to apply these procedures to solve mathematical problems, such as word problems in Mathematics. In contrast, the problem-centred approach is a more exploratory and learner-driven method of teaching through problem-solving. It involves engaging learners with real-life or realistic problems that enable them to construct knowledge both within and beyond Mathematics.

A key difference between these approaches lies in the role of the teacher during learning. In the problem-centred approach, learners are not instructed on specific methods or procedures to follow. After being presented with a problem, they work in pairs or groups while the teacher observes their progress. If assistance is needed, the teacher does not directly provide solutions or prescribe steps but instead facilitates thinking without directing learners toward a predetermined method. Learners are therefore encouraged to explore multiple strategies and approaches. In contrast, in routine problem-solving, the teacher plays a more directive role by explicitly teaching procedures from the outset. Learners are then expected to practise and master these procedures through repetition until they become proficient (Killen, 2006). This approach is characteristic of traditional teaching methods, where success is often measured by the learner's ability to reproduce demonstrated procedures accurately.

From this perspective, routine problem-solving can be understood as the application of previously taught methods to standard mathematical exercises or tasks. Learners are first taught the relevant concepts and techniques and are subsequently required to apply them to similar problems. During practice sessions, the teacher may guide learners by demonstrating examples and assisting them in following established procedures. In this sense, learners are regarded as successful if they can accurately replicate the steps demonstrated by the teacher.

By contrast, in the problem-centred approach, the teacher organises learners into pairs or groups and presents them with complex, often unfamiliar problems. Learners are then expected to explore and construct their own solutions. The teacher's role is limited to observation and facilitation, without intervening to correct or direct learners' thinking processes. Once solutions are developed, learners present and justify their reasoning, explaining the strategies used and the conclusions reached. This approach allows learners to generate their own methods rather than relying on prescribed procedures.

Avenant (1990) describes the problem-centred approach as a teaching method in which learners are actively engaged in resolving problem situations presented by the teacher. It is an instructional approach that encourages meaningful learning by placing learners in rich problem contexts where they are required to collect information, make inferences, formulate hypotheses, experiment, and engage in cognitive processes such as comparison, classification, and reasoning. Through this process, learners construct new knowledge. The approach is therefore designed to promote inquiry and exploration as learners work towards solving mathematical challenges.

Biccard and Wessels (2012) argue for a shift away from traditional teaching approaches towards more contemporary methods that prepare learners for twenty-first century demands. They caution that rote learning of decontextualised rules and an overemphasis on procedures do not adequately support the development of higher-order thinking skills. Instead, they emphasise that learners should develop conceptual understanding, which enables flexible application of knowledge in new and unfamiliar contexts.

Furthermore, Biccard and Wessels (2012) assert that learners develop deeper understanding of Mathematics when they engage with contextual problems. They distinguish traditional problem-solving, often limited to routine word problems, from modern problem-solving approaches. In the traditional approach, learners are first taught procedures and then given similar problems to solve. In the modern approach, however, learners learn Mathematics through problem-solving itself, where problems serve as a means of developing mathematical concepts. In mathematical modelling contexts, learners use their own reasoning and prior knowledge to develop solutions, while the teacher supports learning by helping them make connections between concepts. This process strengthens and extends learners' understanding.
In this regard, Biccard and Wessels (2012) conclude that mathematical modelling empowers learners to learn through problem-solving and is therefore essential for effective Mathematics teaching and learning. Based on this understanding, it can be deduced that the problem-centred approach is well aligned with the development of mathematical modelling competencies, as it allows learners the flexibility to choose appropriate methods rather than relying solely on teacher-directed procedures.

2.14.2 The importance of the correct selection and sequencing of the problems

Bolat and Arslan (2024) describe routine problems as tasks with which learners are familiar and which can be solved systematically by applying previously learned procedures in a sequential manner. Because such problems are predictable, they enable learners to reinforce their mathematical operational skills through the gradual application of recently acquired concepts. In contrast, non-routine problems do not follow a familiar or previously learned procedure. They lack a clear or standard method of solution and therefore require higher-order thinking skills, creativity, and flexible problem-solving strategies.

Foster (2018) similarly explains that routine problems, often referred to as exercises, form a fundamental component of school Mathematics. They allow learners to practise and replicate specific methods that have already been demonstrated by the teacher. In such cases, learners are usually aware of what is expected before they begin the task. Foster further argues that mastery of routine problems is essential for developing fluency in Mathematics and provides a foundation for engaging with more complex problem types.

Building on this, Foster (2014) emphasises that without sufficient fluency in core facts, procedures, and concepts, as well as a broad repertoire of mathematical reasoning skills, learners are likely to struggle with unfamiliar or non-routine problems. He therefore argues that learners must deliberately develop fluency and reasoning skills if they are to become competent problem solvers.

From the above discussion, it can be inferred that learners should initially engage with routine problems to develop basic skills and procedural fluency before progressing to non-routine problems that require reasoning, creativity, and strategic thinking. This progression enables learners to apply established methods while gradually developing the capacity to tackle problems where the solution path is not immediately evident. Consequently, teachers play a critical role in ensuring that learners develop a strong foundation of mathematical skills before being exposed to more complex and unfamiliar problem situations.

Both routine and non-routine problems are important in the context of mathematical modelling. As previously noted, mathematical modelling and non-routine problem-solving both involve open-ended situations that require critical thinking and multi-step reasoning to address unfamiliar problems. However, routine problems also play a supportive role in mathematical modelling by building learners' confidence and strengthening their procedural fluency before they engage with complex modelling tasks. Therefore, learners need to develop a solid grounding in mathematical procedures as a foundation for effective engagement in mathematical modelling activities.

2.14.3 Importance and criticism of alternative problem-solving strategies

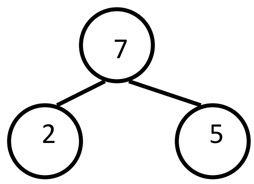
Coleman (2019) raises the question of why learners need to be taught multiple strategies for solving mathematical problems. She notes that some argue that mastering standard algorithms—such as simple computations like $2 + 2 = 4$ or $6 \times 8 = 48$—is sufficient for learners to obtain correct answers. However, she contends that while knowledge of procedures is important, learners also need to understand the principles and processes underlying mathematical operations to apply them effectively in different contexts.

Coleman (2019) further argues that learners require a range of strategies for solving problems, much like a carpenter who would not go to a worksite with only a hammer, but rather with a variety of tools suited to different tasks. In the same way, learners need multiple “tools” or strategies to approach mathematical problems flexibly and effectively. To illustrate this point, she demonstrates different methods for adding 998 and 337, showing how the same problem can be solved using more than one strategy:

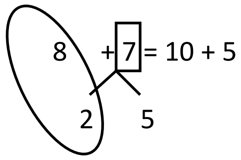
First approach	**Alternative approach**
998	998 + 337 = 1000 + 335
+ 337	=1335
1335	

In the first approach, learners follow a step-by-step procedural method. First, they add 8 and 7 in the units column to obtain 15, write down 5, and carry 1 to the tens column. Next, they add 9 and 3, together with the carried 1, to obtain 13; they write down 3 and carry 1 to the hundreds column. Finally, they add 9 and 3, along with the carried 1, to again obtain 13, which is then written in the final answer. Although this method leads to the correct result, Coleman (2019) argues that it often relies on procedural language that does not necessarily promote conceptual understanding. She further notes that such a method can be challenging, particularly when learners are required to perform the calculation mentally.

An alternative method reduces this complexity by using the concept of number bonds. This approach requires learners to understand that numbers can be decomposed into smaller parts. For example, 7 can be expressed as 2 + 5, illustrating how numbers can be broken down and recombined to simplify computation.



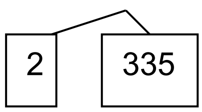
Coleman (2019) explains that learners can use such number components to reformulate a calculation into a simpler form. Learners who understand the principles of addition and can decompose and recombine numbers into more manageable parts can recognise that 7 + 8 can be rewritten as 10 + 5. This transformation simplifies the calculation, making it easier to solve mentally.



8 + 7 = 15

Coleman (2019) explains that the addition of 998 and 337 can appear complex, but it becomes much easier once learners understand number bonds and how to apply them in calculations. The first step is to recognise that 998 is close to 1000, requiring only 2 to reach it. By adjusting the numbers accordingly, the calculation can be simplified to 1000 + 335. This transformation makes the computation easier, even for young learners, as 1000 + 335 can be quickly calculated to give 1335, which is equivalent to 998 + 337. Coleman (2019) further notes that this method is not only faster but also reduces the likelihood of errors.

998 + 337 = 1000 + 335



1000 + 335 = 1335

It can be observed that both approaches—the standard algorithm and the number bonds strategy—yield the same correct answer. Coleman (2019) argues that educators should equip learners with multiple strategies that enable them to solve mathematical problems efficiently, even without reliance on calculators or written working.

Similarly, Bingölbali (2011) investigated teachers' willingness to use and accept multiple solution strategies, as well as how they evaluate different approaches to problem-solving. The study focused on four main aspects: (1) teachers' receptiveness to alternative solution methods; (2) their beliefs and orientations towards Mathematics; (3) their challenges in assessing open-ended questions; and (4) their overall difficulties with Mathematics.

With regard to teachers' receptiveness to alternative solutions, the findings revealed a notable inconsistency between curriculum intentions and classroom practice. Although the Turkish Mathematics curriculum encourages teachers to explore multiple solution strategies in teaching, many teachers were not inclined to accept alternative methods. For example, 67% of teachers indicated that they would accept only answer A for Item 1 (presented alongside alternatives A, B, and C), even though more than one approach produced a mathematically valid solution. Only 15% of teachers indicated that they would accept both A and B as correct responses.

A.	**B.**	**C.**
32	32	32
$\times$ 25	$\times$ 25	$\times$ 25
160	10	50
+ 640	150	150
800	40	+ 600
	+ 600	800
	800	

Solution A reflects a traditional algorithmic approach, which, as noted by Bingolbali (2011), is typically preferred by teachers in classroom practice. In contrast, Solution B draws on the distributive property, expressed as $32 \times 25 = (30 + 2)(20 + 5) = 600 + 150 + 40 + 10$. Solution C, however, is constructed using the area model of multiplication, where calculations such as $(30 \times 20)$and $(30 \times 5)$are prioritised before $(20 \times 2)$and (2×5). Despite these methodological differences, all three approaches yield correct results.

Although it is widely recommended that both teachers and learners engage with multiple strategies for solving mathematical problems, there is limited evidence regarding teachers' preparedness to solve problems using diverse methods and to evaluate such approaches effectively (Bingolbali, 2011). His study revealed that many teachers have trouble in assessing learners' varied solution methods and tend not to value the use of multiple strategies.

Similarly, Lynch and Star (2013) explored teachers' perspectives on multiple solution strategies in middle and high school mathematics. Their study was motivated by the observation that, while primary school learners have significantly benefited from exposure to multiple strategies, there is comparatively less emphasis on this practice in secondary education. Drawing on Silver et al. (2005), Lynch and Star (2013) conceptualised multiple strategies as learners' ability to approach and solve mathematical problems using different procedures or techniques. They further noted that several countries, including Australia and Singapore, advocate for the use of alternative solution methods in mathematics instruction.

Research on teachers' beliefs highlights the complex relationship between instructional practices and mathematics pedagogy. According to Grossman and Stodolsky (1995), as cited in Lynch and Star (2013), teachers' responses to educational reform are shaped by their understanding of the mathematics curriculum, which is often perceived as dense, continuous, and demanding. This perception compels teachers to maintain a rapid pace to ensure curriculum coverage. Consequently, secondary school teachers tend to interpret policy reforms through the lens of curriculum demands, learner diversity, and sequencing challenges, which differ markedly from those encountered in primary education.

A substantial body of research indicates that learners benefit from engaging with multiple strategies. Studies by Star and Rittle-Johnson (2008), Rittle-Johnson and Star (2007), Gentner and Namy (1999), and Silver et al. (2005), as cited in Lynch and Star (2013), demonstrate that exposure to diverse methods enhances learners' understanding. Furthermore, Bostic and Jacobbe (2010) and Ridlon (2009) found that learners show increased motivation and engagement when solving problems using multiple approaches. Mathematical problem-solving, therefore, naturally encourages the exploration of alternative strategies.

However, this approach has not been without criticism. Evans and Swan (2014) describe a more traditional "Triple X" teaching model—exposition, examples, and exercises—where a single method is introduced and reinforced through practice. In such contexts, alternative strategies are neither encouraged nor explored, as teachers expect learners to apply a predetermined procedure.

In contrast, classrooms that promote multiple strategies present a more complex instructional environment. Teachers must navigate diverse and sometimes unexpected solution methods, which may introduce unforeseen challenges. Learners may also encounter difficulties when required to generate and compare multiple approaches. Supporting this view, Silver et al. (2005), as cited in Quigley and Swan (2014), found that some interventions aimed at encouraging multiple solutions have yielded limited success. Similarly, Leikin and Levav-

Waynberg (2007), as cited in Evans and Swan (2014), observed that both teachers and learners may lack motivation to engage with multiple solution methods.

Despite strong support for multiple strategies, some researchers argue that their effectiveness is more pronounced in primary education than in secondary schooling. Lynch and Star (2013) suggest that time constraints and the extensive content demand of secondary curricula limit opportunities to explore multiple methods. Additionally, some teachers believe that teaching a single, efficient strategy is sufficient, particularly given the complexity of high school mathematics.

Lynch and Star (2013) also identified several challenges associated with implementing multiple strategies: the risk of learner confusion; time constraints and increased preparation demands; affective and motivational issues; learner resistance; entrenched beliefs that mathematics problems have only one correct method; limited teacher knowledge of alternative approaches; inadequate resources; and increased difficulty in assessing diverse solutions.

Wilkie (2016), however, found that learners generally perceive the use of multiple strategies as engaging and intellectually stimulating, even when tackling complex tasks. Despite this, some teachers remain reluctant to adopt such approaches, fearing that high-achieving learners may become disengaged while lower-achieving learners may experience confusion.

Overall, while the use of multiple strategies in mathematics has attracted criticism, it remains a valuable pedagogical approach. Open-ended and non-routine problems—such as those found in mathematical modelling—often allow for more than one valid solution. Encouraging learners to explore alternative methods can therefore enhance their problem-solving capacity and increase the likelihood of discovering diverse and meaningful solutions.

2.15. Metacognition

Metacognition was introduced in Section 2.4.2 in relation to critical thinking and is particularly important in the context of mathematical problem-solving. Lin (2001) defines metacognition as the capacity to recognise, analyse, and monitor one's own ideas, assumptions, and actions. Similarly, Woolfolk (2010) describes metacognition as knowledge about one's own thinking processes, encompassing the regulation of mental activities such as comprehension, reasoning, and problem-solving. Given its growing significance in education, metacognition is an essential component of this study. Research consistently shows that metacognitive skills play a crucial role in learners' success in mathematics. For instance, Schneider and Artelt (2010) found a strong relationship between metacognition and mathematics performance, demonstrating that increased metacognitive knowledge leads to improved achievement. Likewise, Güner and Erbay (2021) showed that learners with well-developed metacognitive skills are better able to apply appropriate techniques, use correct mathematical notation, and engage in logical reasoning, while those with limited metacognitive abilities struggle to interpret problems, select suitable strategies, and arrive at correct solutions. Lin (2001) further noted that learners benefit significantly from engaging in metacognitive activities such as monitoring, revising, self-assessment, and self-explanation, with these strategies being particularly beneficial for lower-achieving learners.

In contemporary learning environments, learners are often exposed to numerous distractions, including televisions, laptops, and mobile devices. While these technologies can support learning, they are frequently used for non-educational purposes, which may hinder concentration and academic engagement. This suggests that many learners have not yet developed effective self-regulation skills. Zimmerman (2002) identifies several key processes that are often lacking, including goal setting, strategic learning, time management, self-evaluation, help-seeking, and motivational beliefs such as self-efficacy and task interest. These challenges have contributed to increased interest in cognitive and metacognitive research. Social cognitive researchers, as noted by Schunk (1989) in Zimmerman (2002), have examined how social factors, including teacher modelling and instruction, influence learners' development of self-regulation. Zimmerman (2002) also demonstrated that learners who set specific goals and monitor their progress tend to achieve higher levels of performance.

Woolfolk (2010) identifies three types of metacognitive knowledge: declarative, procedural, and self-regulatory knowledge. Declarative knowledge refers to an individual's awareness of their own abilities, the factors affecting learning, and the resources required to complete a task—essentially, knowing what to do. Procedural knowledge involves understanding how to apply various strategies, while self-regulatory knowledge enables learners to determine when and why to use specific strategies. Together, these forms of knowledge support effective learning and problem-solving.

Metacognitive regulation involves three key processes: planning, monitoring, and evaluation (Woolfolk, 2010). Planning includes deciding how much time to allocate to a task, selecting appropriate strategies, organising resources, and determining the sequence of steps. Monitoring requires learners to reflect on their understanding by asking questions such as "Am I making progress?" or "Does this make sense?" Evaluation involves assessing both the processes and outcomes of learning, including whether alternative strategies are needed. While some actions may become routine and require minimal metacognitive effort, metacognition is particularly valuable when tasks are challenging but manageable. Quigley et al. (2019) further emphasise that self-regulated learning comprises cognition, metacognition, and motivation. Cognition relates to mental processes involved in learning, metacognition to the regulation of those processes, and motivation to the willingness to engage in learning tasks.

Spencer (2018) explains that metacognition begins with learners assessing the task and forming a clear understanding of what is required. This involves integrating prior knowledge with new information. Learners must then evaluate their strengths and weaknesses, plan appropriate strategies, implement and monitor these strategies, and ultimately reflect on their effectiveness. Effective problem-solvers can adjust their approaches when necessary, whereas less effective learners tend to persist with ineffective strategies. This cyclical process may vary in duration and does not always follow a strict sequence.

The importance of metacognition in this study is evident, particularly as it involves solving mathematical problems through modelling. Learners must demonstrate the ability to understand the problem, plan appropriate methods, apply suitable strategies, monitor their progress, and reflect on their solutions. Consequently, it is essential for teachers to foster metacognitive skills in the classroom. Spencer (2018) suggests several strategies to promote metacognition, including the use of a gradual release approach, integration of self-assessment, encouraging visualisation and evaluation, incorporating project management skills, and creating a classroom environment that allows for mistakes and risk-taking.

Lin (2001) emphasises the importance of designing instructional activities that explicitly support metacognitive development. Techniques such as modelling and prompting are particularly effective. In modelling, teachers demonstrate cognitive and metacognitive processes, while prompting involves guiding learners through questions and cues. For example, King (1992), as cited in Lin (2001), used prompt cards to encourage learners to generate metacognitive questions, which enhanced their understanding. Similarly, Chi et al. (1994), as cited in Lin (2001), found that learners who engaged in self-explanation through prompting demonstrated greater conceptual understanding.

Further evidence of the importance of metacognition is provided by Tzohar-Rozen and Kramarski (2014), who found that learners' success in mathematics depends not only on cognitive ability but also on their capacity for self-regulation. Aurah, Koloi-Keaikitse et al. (2011) similarly reported that learners with higher levels of metacognition are more effective problem-solvers. These findings underscore the need for teachers to actively support the development of metacognitive skills.

Metacognition is particularly relevant to this study because its core components—planning, monitoring, and evaluation—are essential for solving the mathematical tasks involved. During the planning stage, learners select strategies based on prior knowledge, such as understanding how to calculate the area and perimeter of a rectangle. As Spencer (2018) notes, successful task completion requires learners to connect prior knowledge with new learning. Given that mathematical modelling is a multi-step process, learners must continuously monitor their progress, check for errors, and ensure that their solutions are contextually appropriate. Finally, evaluation enables learners to assess the effectiveness of their methods and select the most appropriate strategies.

For example, one task in the study required learners to determine the dimensions of a garden that would yield a maximum area. This involves identifying the turning point of a quadratic function. Learners could employ various methods, including using the vertex formula, completing the square, or determining the midpoint of the roots. At more advanced levels, such as Grade 12, learners may also use calculus, where the turning point is identified by setting the derivative equal to zero. These multiple approaches highlight the importance of metacognitive skills in selecting, applying, and evaluating appropriate problem-solving strategies.

2.16 Examples of countries that use mathematical modelling

This section examines the implementation of mathematical modelling in countries such as Singapore, Germany, and South Africa. It explores how mathematical modelling has been integrated into educational practices, as well as the successes achieved and the challenges encountered in each context.

2.16.1 Singapore's mathematical framework and modelling

According to Kaur (2014), the developments in Singapore's education system over the years have led to significant changes in Mathematics education. These developments encompassed curriculum reforms, teacher development, learner engagement, and improvements in the learning environment. Collectively, these factors have contributed to Singapore's strong performance in international assessments such as TIMSS and PISA. The current school mathematics curriculum in Singapore can be described as inclusive, as it is grounded in a framework that places strong emphasis on mathematical problem-solving.

Kaur (2014) notes that in 1997, the then Prime Minister of Singapore called for major reforms in the education system to better prepare learners for the demands of the new millennium. The goal was to develop learners who could think independently and generate their own solutions when faced with challenges. This vision was encapsulated in the initiative "Thinking Schools, Learning Nation" (TSLN). As part of this reform, three key initiatives were introduced: Information Technology, National Education, and Critical Thinking and Creativity. To support these initiatives, changes were implemented across four key areas: assessment, teaching, curriculum, and teacher development.

To further strengthen the curriculum, the Curriculum Development Division of Singapore's Ministry of Education established a Mathematics Syllabus Review Committee. This committee was tasked with evaluating the effectiveness of the existing mathematics curriculum, originally adopted in 1981, and ensuring its alignment with contemporary developments in mathematics education. One of the key outcomes of this review was the development of a curriculum framework that emphasises problem-solving as the central focus of mathematics teaching and learning.

Singapore was selected for this study due to its consistently high performance in international assessments such as TIMSS. By examining Singapore's mathematics education system, this study seeks to identify effective practices that contribute to learners' strong achievement. Singapore's emphasis on problem-solving and the use of mathematical modelling in instruction aligns closely with the focus of this research.

Evidence from TIMSS 2015 highlights Singapore's leading position in mathematics achievement at both Grade Four and Grade Eight levels. The top-performing countries in these assessments are predominantly from Asia, with Singapore ranking first in both grades, indicating consistently high learner performance. In contrast, no African country appears among the top ten performing nations. According to Mullis, Martin et al. (2016), out of 49 participating countries in the Grade Four assessment, South Africa ranked 48th with an average scale score of 376, significantly below the TIMSS centre point of 500, outperforming only Kuwait. In the Grade Eight assessment, which included 39 countries, South Africa ranked 38th, with Saudi Arabia in the last position. The performance gap between East Asian countries and the next highest-performing countries was notable—23 points at Grade Four and 48 points at Grade Eight level. Among African countries, Morocco ranked highest at Grade Four (47th), while Egypt performed best at Grade Eight (34th) (Mullis et al., 2016). These results highlight significant disparities in mathematics achievement across regions and underscore the need to learn from high-performing systems such as Singapore.

Table 2.2: Average scores for top ten counties in Grades Four and Eight

Position	Grade Four		Grade Eight	
	Countries	**Mean Scale Score**	**Countries**	**Mean Scale Score**
1	Singapore	618	Singapore	622
2	Hong Kong	615	Korea Rep	606
3	Korea Republic	608	Chinese Taipei	599
4	Chinese Taipei	597	Hong Kong	594
5	Japan	593	Japan	586
6	Northern Ireland	570	Russian Federation	538
7	Russian Federation	564	Kazakhastan	528
8	Norway	549	Canada	527
9	Ireland	547	Ireland	523
10	England	546	United States	518

Source: Mullis et al (2016)

Sze (2009) explains that Singapore's Mathematics curriculum is guided by the Mathematics Framework, also known as the Pentagon Framework. This framework is grounded in the fundamental principle that both the processes and products of learning are essential for understanding Mathematics. It emphasises the application of mathematical knowledge to problem-solving, requiring learners to integrate concepts and skills while developing competencies such as reasoning and communication. In addition, the framework promotes exploratory learning and encourages the development of positive attitudes towards Mathematics.

The Ministry of Education in Singapore has mandated the use of the Mathematics (Pentagon) Framework across all levels of schooling, from primary education through to A-level. Consequently, all mathematics teaching and learning activities are structured and guided by this framework (Ministry of Education Singapore, 2014). The principles underpinning the Mathematics Framework, as illustrated in Figure 2.12, highlight problem-solving as its central focus. This underscores the fact that Mathematics education in Singapore is fundamentally anchored in problem-solving.

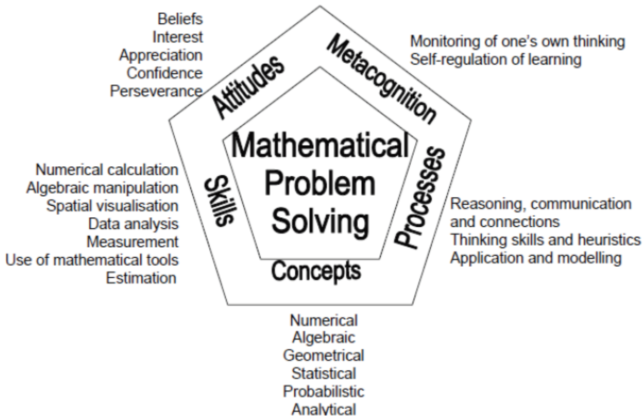


Figure 2.15: The Singapore Mathematics Framework

Source: Ministry of Education (2014)

Apart from problem-solving, the Singapore Mathematics Framework also gives attention to affective dimensions such as perseverance, interest, confidence, and appreciation of Mathematics. As illustrated in Figure 2.7, the development and success of mathematical problem-solving is supported by five interconnected components, namely metacognition, attitudes, processes, skills, and concepts (Ministry of Education Singapore, 2014). This indicates that effective problem-solving in Mathematics depends on the interaction of these five interdependent elements.

Concepts: The framework includes key mathematical concepts in algebra, number, geometry, statistics, probability, and analysis. To ensure mastery, learners must be exposed to a variety of meaningful learning contexts that promote conceptual understanding and application.
Processes: Mathematical processes refer to the essential thinking and doing actions required for successful engagement in Mathematics. These include reasoning, modelling, application, communication, connecting ideas, cognitive processes, and exploration.

Metacognition: Metacognition refers to learners' awareness and regulation of their own thinking, particularly in relation to selecting and using appropriate problem-solving strategies (Ministry of Education Singapore, 2014).

Skills: The curriculum emphasises essential mathematical skills such as algebraic manipulation, numerical computation, data analysis, measurement, three-dimensional visualisation, estimation, and the use of mathematical instruments. These skills are fundamental for effective engagement with mathematical tasks.

Attitudes: Learners' attitudes play a critical role in Mathematics achievement, as they influence engagement and persistence. These include beliefs about Mathematics and its usefulness, interest and enjoyment in learning Mathematics, appreciation of its beauty and power, confidence in using mathematical ideas, and perseverance in problem-solving. Positive attitudes are developed when learners recognise the relevance of Mathematics in everyday life. Lee and Ng (2015) describe mathematical modelling as a bridge between classroom Mathematics and real-world problem-solving. They argue that traditional Mathematics instruction has often focused primarily on conceptual understanding and procedural fluency, with limited emphasis on real-life application. This has contributed to the misconception that Mathematics consists only of formulas, theorems, and proofs.

Despite Singaporean learners' consistently strong performance in international assessments such as TIMSS and PISA, there has been growing recognition of the need to strengthen mathematical literacy and 21st-century competencies. One key response to this need has been the introduction of mathematical modelling into the curriculum. Proponents of modelling argue that it strengthens the connection between classroom learning and authentic real-world problem-solving (Lee & Ng, 2015).

Lee and Ng (2015) further note that mathematical modelling has increasingly been incorporated into curricula globally, as evidenced by the growing participation in the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) conferences. In Singapore, modelling was formally introduced into the mathematics curriculum framework in 2003. However, its implementation faced challenges, particularly due to limited teacher preparedness and insufficient understanding of modelling pedagogy. As a result, modelling activities were often treated as optional enrichment rather than integral classroom practice. Time constraints and assessment pressures within traditional examination systems further limited its adoption.

Ang (2010), as cited in Lee and Ng (2015), argues that without explicit inclusion in the national curriculum, it would have been difficult for teachers to recognise the value of modelling activities within the structured demands of teaching, learning, and assessment. Chan (2013), as cited in Lee and Ng (2015), further reports that many teachers were unfamiliar with the concept of mathematical modelling, often confusing it with bar modelling or strip diagrams commonly used for word problems.

Similarly, Ng (2013), as cited in Lee and Ng (2015), observes that teachers often struggle to distinguish between mathematical modelling and mathematical applications, as both involve real-world problem contexts. However, Stillman et al. (2008), as cited in Lee and Ng (2015), clarify that modelling begins with authentic real-world situations requiring learners to interpret, simplify, and mathematise problems to construct models. In contrast, applications typically involve selecting pre-determined mathematical procedures to fit a given real-world context. Chan et al. (2012) explain that traditional problem-solving in Singapore was previously teacher-centred, with learners repeatedly practising routine word problems demonstrated by teachers. However, curriculum reforms have shifted toward open-ended tasks such as investigations, which encourage learners to explore and discuss mathematical ideas through modelling activities.

In a study by Chan et al. (2012), primary school learners with no prior modelling experience were assessed on their modelling competencies, including assumption-making, reasoning, and task interpretation. The findings revealed that learners struggled to translate real-world problems into mathematical models and had difficulty making realistic assumptions. None of the learners completed all phases of the modelling cycle, although variation in approaches was observed. The authors concluded that these challenges were expected, given the learners' lack of prior experience, and aligned with findings from other studies showing that novice modellers often do not complete the full modelling cycle.

For both teachers and learners in Singapore, open-ended and non-linear modelling tasks present significant challenges. Teachers require strong pedagogical knowledge to guide learners through iterative modelling phases, particularly in supporting the process of mathematisation. Despite ongoing curriculum reforms, studies continue to highlight implementation difficulties. Ng (2010; 2013), as cited in Lee and Ng (2015), notes that insufficient teacher preparedness in terms of both mindset and pedagogical competence remains a key challenge.

In response, the Mathematical Modelling Outreach (MMO) initiative was established to support implementation. According to Lee and Ng (2015), MMO was designed with three main objectives: first, to collaborate with schools in strengthening learners' reasoning, thinking, and communication through modelling; second, to connect classroom Mathematics with real-world problem-solving by engaging learners in model construction; and third, to provide teachers with foundational training on modelling concepts, facilitation strategies, and task design.

Singapore's sustained success in international assessments such as TIMSS has led researchers to investigate contributing factors. These include a coherent national curriculum, strong teacher development systems, and a societal belief in the importance of Mathematics for economic development. Central to these factors is the emphasis on problem-solving within the Singapore Mathematics Framework. Clark (2009) describes this framework as a pentagon model with problem-solving at its centre, supported by five interconnected components. Textbooks are carefully designed to reflect this structure and to reinforce learners' understanding of these interrelated components. The curriculum also promotes reflection on thinking processes, communication, and problem-solving strategies. Clark (2009) further notes that recent curriculum reforms have strengthened attention to metacognition and communication during problem-solving.

Finally, Clark (2009) explains that model drawing is widely used in Singapore to support word problem-solving. From Grade 2 onwards, learners are taught to represent problems using bar models or rectangular diagrams. These visual representations help learners to: (1) visualise abstract relationships; (2) simplify complex information using partitionable structures; and (3) represent algebraic relationships before formal algebra is introduced.

2.16.2 Mathematical modelling in Germany

Among the countries that have incorporated mathematical modelling into their school mathematics curricula, Germany is a prominent example. According to Greefrath and Vorhölter (2016), awareness of mathematical modelling in Germany increased significantly during the 1980s. During this period, several modelling cycles were developed and discussed to explain the processes and aims of modelling, as well as the rationale for integrating modelling and applications into Mathematics education. They further note that the final quarter of the 20th century marked a shift towards a competency-oriented approach, with increased attention to empirical and practical research as well as international collaboration, following earlier influences of didactical reforms that promoted realistic and application-based instruction.

In 2003, Germany introduced national Educational Standards that made mathematical modelling a compulsory competency. As a result, modelling became one of the six general mathematical competencies expected of learners. Since then, considerable effort has been directed towards integrating modelling into classroom practice. In addition, the emergence of digital technologies has significantly transformed both the teaching and learning of mathematical modelling.

Recent research, both qualitative and quantitative, indicates that much attention in modelling education has been placed on learners, often overlooking the crucial role of teachers in effectively implementing modelling in classroom practice. However, this trend has begun to change, as increasing numbers of researchers in Germany now focus on teachers' modelling competencies. Ferri and Blum (2010) emphasize that mathematical modelling is now a compulsory element of the Mathematics curriculum in Germany, meaning all learners are required to engage in modelling activities. This makes it essential for teachers to have strong knowledge of modelling processes and effective pedagogical approaches for teaching them. Consequently, teacher education programmes at universities must be designed to address these competencies comprehensively. Despite modelling being a mandatory component of the curriculum and a key competency in national Educational Standards, many teachers still lack sufficient preparation, largely because modelling was not adequately addressed during their university training.

At the beginning of the 20th century, Mathematics education in Germany was influenced by a reform movement led by Johannes Kühnel (1869–1928). Greefrath and Vorhölter (2016) explain that Kühnel advocated for more realistic and interdisciplinary approaches to Mathematics teaching, arguing that traditional instruction was often abstract, artificial, and disconnected from real life. He criticised the use of unrelated and decontextualised problems and instead promoted tasks that were meaningful and engaging for learners. During this period, applications were considered central to learning Mathematics, not necessarily to prepare learners for everyday life, but to help them visualise and understand abstract concepts. Typical applied topics included averages, proportional reasoning, and decimal arithmetic. Kühnel's ideas were widely influential and remained prominent until the 1950s.

Between the 1960s and 1970s, Mathematics education in Germany underwent further development, with a stronger emphasis on formal and theoretical structures. Nevertheless, applications remained important, although in modified forms. Firstly, problems were increasingly framed with clearer mathematical structures, such as proportional relationships. Secondly, the scope of applications was broadened to include topics such as probability. Thirdly, teaching approaches and instructional methods were further refined and strengthened. Arseven (2015) notes that one of the central goals of Mathematics education in German secondary schools is the development of learners' mathematical modelling competencies. She argues that modelling makes Mathematics more meaningful and relevant to learners. However, she also observes that modelling has not yet received the level of attention it deserves. The gap between curriculum intentions and classroom practice is partly attributed to the fact that modelling is also challenging for teachers, as it requires engagement with real-world contexts and interdisciplinary knowledge. Arseven (2015) suggests that if prospective teachers are introduced to modelling theory during their tertiary education, they will be better equipped to implement it across different school levels. This highlights the importance of adequately preparing teachers to support learners effectively.

Despite its importance, Arseven (2015) reports that mathematical modelling is not yet fully implemented at the expected depth in German schools. Although it has been part of the curriculum since 2005 and is recognised as one of the core competencies in Mathematics education standards, its implementation is still largely concentrated in secondary education, with limited exposure at the primary level. This has contributed to gaps in both teacher and learner understanding of modelling.

Blum and Ferri (2009) further highlight a global trend towards increased emphasis on mathematical modelling in school curricula, with Germany serving as a key example where modelling is included among the six core competencies in national standards. However, they caution that classroom practice often falls short of these expectations. In many cases, word problems are used primarily as exercises in applying mathematical procedures rather than as authentic modelling tasks. Furthermore, such problems are often not sufficiently realistic or meaningful. They argue that the limited presence of modelling in classrooms is largely due to its complexity, which also challenges teachers, as it requires engagement with open-ended, real-world situations. As a result, modelling instruction tends to be less structured and less predictable than traditional teaching approaches.

Thomas et al. (2015) similarly emphasize the importance of strengthening mathematical modelling and applications in German classrooms. They argue that learners should understand the relevance of Mathematics to everyday life, the environment, and scientific disciplines. This shift in perspective has influenced changes in instructional approaches, as focusing solely on procedural application within textbook constraints is no longer considered sufficient. Instead, real-world contexts are increasingly used to demonstrate the usefulness of Mathematics, while also equipping learners with problem-solving skills applicable to daily life and professional contexts.

A notable initiative in this regard was implemented by the University of Hamburg, where a mathematical modelling programme was introduced in German schools in 2000 and funded by a private foundation. According to Thomas et al. (2015), the programme focused on prospective secondary school teachers and aimed to bridge the gap between university mathematics and school-level instruction. Learners aged 16 to 18 worked in groups under the supervision of trainee teachers, engaging in modelling tasks during class and after-school sessions.

Kaiser and Schwarz (2006) explain that the primary goal of this initiative was to strengthen the role of modelling in university mathematics education and to enhance teachers' pedagogical knowledge related to modelling. At the same time, learners were expected to develop competencies that would enable them to independently formulate and solve real-world problems by translating them into mathematical representations.

They further argue that exposing future teachers to modelling during their training is an effective way to support its implementation in schools. This exposure helps teachers become familiar with modelling tasks so that they can integrate them into classroom practice. The modelling problems used in the programme covered a wide range of real-life contexts, including risk management, health insurance, emergency logistics, fisheries science, cancer treatment planning, handwriting analysis, flight pricing, internet infrastructure, traffic flow during major events, and school timetable design (Kaiser & Schwarz, 2006).

According to Kaiser and Schwarz (2006), the selection of modelling tasks was guided by several criteria. Tasks needed to be accessible to learners, not overly complex in terms of understanding real-world contexts, and aligned with learners' mathematical abilities. The availability of experts who could provide authentic problem scenarios was also an important factor. Additional enrichment activities included company visits and lectures by applied mathematicians to broaden learners' understanding of real-world applications of Mathematics. Evaluation of the programme showed that learners could engage with complex and higher-order modelling tasks. Despite the time required, many learners remained actively engaged throughout the course and reported high levels of satisfaction. The results also indicated that average learners were able to solve challenging modelling problems successfully, suggesting that such tasks are not limited to high-achieving students. Furthermore, learners' solutions were of notably high quality compared to typical performance levels.

The study also revealed changes in learners' and prospective teachers' beliefs about Mathematics. Initially, many viewed Mathematics as a fixed, abstract subject focused primarily on computation. After participating in the programme, however, their perceptions shifted, with many recognising Mathematics as relevant to everyday life and essential across various fields. Learners also emphasised the importance of incorporating real-world problems into Mathematics instruction.

Prospective teachers reported that their teaching experience was enhanced and recommended that practical modelling training be included in university programmes. Learners also valued collaborative group work, noting that it helped them manage uncertainty and share problem-solving responsibilities more effectively. Overall, Kaiser and Schwarz (2006) conclude that modelling-oriented instruction can successfully transform Mathematics teaching, although it requires more time and presents greater complexity than traditional approaches.

Kaygısız and Şenel (2023) investigated the modelling competencies of Grade Four learners through a real-life problem-solving activity involving the spacing and planting of olive trees in a rectangular area. Learners were required to determine distances and calculate the number of trees that could be planted. Data collected through written work, video recordings, observations, and field notes showed that while learners were able to propose models, they struggled to construct accurate solutions for real-world contexts. They also experienced difficulties in time management and often produced unnecessary representations such as graphs and tables. Overall, learners demonstrated partial competence in interpreting, solving, and verifying their solutions.

Similarly, Maaß (2006) conducted a study with Grade Seven learners in Baden-Württemberg, Germany, focusing on modelling tasks such as estimating the surface area of a Porsche 911, analysing traffic congestion, interpreting mobile phone bills, and solving real-world estimation problems involving heat transfer and projectile motion. The findings indicated that successful modelling requires more than following procedural steps; it also involves the development of sub-competencies within modelling competence.

Lee and Ng (2015) report that Germany has also implemented "modelling weeks" in schools for more than a decade, during which learners work in groups under the guidance of teachers or university lecturers to solve authentic real-world problems.

2.16.3 Mathematical modelling in South Africa

Although mathematical modelling was formally introduced into the South African school Mathematics curriculum for Grades R–9 in 2003, research on the topic had already been undertaken by several scholars prior to its inclusion. Among these early contributors is Professor Michael de Villiers, whose work has had a significant influence on the development of mathematical modelling in South Africa. His research dates to the late 1980s, when he published *Teaching Modelling and Axiomatization* (1987). This was followed by several other influential studies, including *Relevant, contextualized teaching versus irrelevant, decontextualized teaching* (1992), *The role of technology in mathematical modelling* (1994), and *Mathematical Modelling and Proof* (2004). These works collectively highlight his sustained contribution to linking modelling, pedagogy, and mathematical reasoning.

Given that the present study investigates the competency levels of Grade Eleven learners, it is important to provide an overview of the development of mathematical modelling within the South African context. Prior to the introduction of the National Curriculum Statement (NCS) in 2003, South African Mathematics curricula placed limited emphasis on modelling. The NCS marked a significant shift by explicitly introducing mathematical modelling as part of the curriculum, a focus that has been retained and strengthened in the Curriculum and Assessment Policy Statement (CAPS) introduced in 2011.

The CAPS document identifies mathematical modelling as a key aim of Mathematics education in South Africa, highlighting its central role in the curriculum. It encourages the integration of real-world contexts across all content areas, emphasising that authentic problems should be used whenever possible. Fictional or artificial contexts are discouraged, while meaningful real-life issues related to social, cultural, health, political, scientific, and environmental themes are promoted (Department of Basic Education [DBE], 2011). This strong curriculum emphasis reflects a deliberate effort to make Mathematics more relevant and application oriented.

The growing prominence of mathematical modelling in South Africa was further demonstrated when the country hosted the 18th International Conference on the Teaching of Mathematical Modelling and Applications (ICTMA) in Bellville, Cape Town, from 23 to 28 July 2017. This marked the first time the conference was held in South Africa and attracted Mathematics education researchers from across the world to share experiences and research findings on modelling and its applications.

However, even before modelling became a formal curricular focus, elements of it were already present in earlier policy documents. The Revised National Curriculum Statement (RNCS) for Mathematics in Grades R–9 (Department of Education, 2002) required learners to construct mathematical models to describe, represent, and solve problem situations. These problems were expected to address environmental, health, economic, social, and cultural issues. Although this indicates that modelling was already being introduced, it was not yet emphasised to the same extent as in the NCS and CAPS curricula.

Several studies have investigated mathematical modelling in the South African classroom. Biccard (2010), for example, examined the development of modelling competencies among Grade Seven learners working in groups. The study involved twelve learners divided into three groups based on prior achievement levels in Mathematics: one group of high achievers and two groups of lower-achieving learners. None of the participants had previously been exposed to modelling-based instruction and were accustomed to traditional teaching methods.

The findings showed that all groups demonstrated some improvement in modelling competencies over time. The high-achieving group began with stronger initial competencies and showed steady progress, while the lower-achieving groups also demonstrated notable development, although they required more sustained exposure to modelling tasks. Biccard (2010) concluded that the development of modelling competencies is complex and interconnected, but that such competencies can improve progressively through structured group engagement.

Similarly, Wessels (2006) investigated how learners in Grades Four to Seven represent and organise data in open-ended modelling tasks. Conducted in a Pretoria primary school, the study required learners of varying ability levels to engage with two contextualised tasks designed to elicit spontaneous modelling responses. The researcher analysed learners' participation in key modelling processes, including problem formulation, data collection, analysis, and interpretation. Data were collected in the form of written work, verbal explanations, drawings, and diagrams. The study found that learners' use of multiple representations was indicative of effective engagement in modelling activities.

Jacobs and Durandt (2017) explored pre-service Mathematics teachers' perceptions of mathematical modelling following their first exposure to model-eliciting activities. The study involved fifty third-year pre-service teachers at the University of Johannesburg, who were preparing to teach Grades 10 to 12. The findings revealed generally positive attitudes toward modelling, with most participants expressing enjoyment and a willingness to develop their modelling skills further. However, nearly half of the participants lacked confidence in their ability to handle modelling tasks. The study also found differences in attitudes based on gender and achievement level, with female participants and those with lower Mathematics achievement (below 70%) displaying fewer positive attitudes toward modelling.

Govender and Machingura (2023) conducted a study on Grade 10 learners' mathematical modelling competencies in a Western Cape school. Participants were given word problems requiring the use of simultaneous equations, and data were collected through written tasks, observations, and interviews. Learners' competencies were categorised into three levels: incompetent, slightly competent, and competent. The results showed that most learners fell into the incompetent category. The study further revealed that learners struggled to formulate appropriate assumptions, translate real-world contexts into mathematical models, and interpret their solutions meaningfully.

Mosimege (2017), in a presentation at the 18th ICTMA conference held in South Africa, outlined key aspects of mathematical modelling in the national curriculum. These included the specific aims of the Mathematics curriculum, the role of modelling as a curriculum goal, context-based teaching and learning theories, and examples of socio-cultural modelling contexts such as cultural villages. He further highlighted the responsibilities of teachers, which include teaching relevant mathematical concepts, identifying applicable real-world contexts, ensuring that problems are drawn from realistic social, economic, health, environmental, political, and scientific settings, and integrating real-life problems into everyday classroom practice.

Taken together, these studies suggest that mathematical modelling is becoming increasingly prominent in South African Mathematics education. However, its successful implementation depends largely on teachers' readiness and competence. As Jacobs and Durandt (2017) also indicate, some pre-service teachers still lack sufficient confidence and competence to effectively facilitate modelling activities in the classroom.

2.16. Conclusion

This chapter provided an overview of the literature on mathematical modelling by first presenting various definitions of the concept. It further discussed the advantages and challenges associated with mathematical modelling in mathematics education. Key 21st-century skills such as creativity, critical thinking, communication, and collaboration were also highlighted as essential for learners to succeed in modelling activities. Since this study involves problem-solving, the discussion also focused in detail on problem-solving and metacognition as important skills that learners need to develop for effective engagement in mathematical modelling. In addition, the chapter presented examples of countries where mathematical modelling has been incorporated into the Mathematics curriculum, illustrating different approaches to its implementation. The literature reviewed indicates that the introduction of mathematical modelling into school curricula has not been without challenges, as various implementation difficulties have been identified across different educational contexts. The theoretical foundations of this study are discussed in the next chapter. Chapter 3 focuses on Constructivism and the five strands of mathematical proficiency as the guiding theoretical frameworks. It also examines relevant learning theories and outlines the mathematical competencies required for proficiency in mathematical modelling.

CHAPTER THREE: THEORETICAL FRAMEWORK

3.1 Introduction

In this chapter, different perspectives that help to explain the phenomenon under study are discussed. The overall structure and guiding vision of the study are also outlined. The chapter presents Constructivism and the strands in the learning of Mathematics as the theoretical frameworks underpinning the study. This is followed by a discussion of mathematical modelling processes, problem-solving models, and mathematical modelling competencies. Constructivism is adopted as the primary theoretical framework in this study. It is used to interpret and assess learners' mathematical competencies as demonstrated in their responses to test items based on real-life problem situations. In other words, Constructivism provides a lens through which learners' abilities are understood in relation to their engagement with the mathematical modelling approach. The framework also supports an examination of how learners construct and apply procedures during the modelling process when solving classroom-based Mathematics tasks. In addition, Kilpatrick's strands of mathematical proficiency are used as a complementary theoretical framework. This framework further supports the analysis of learners' mathematical development and competencies within the context of mathematical modelling.

3.2 Models of problem-solving

It is difficult to discuss mathematical modelling without also considering problem-solving. According to Voskoglou (2021), Mathematics is fundamentally a discipline in which the problem-solving process is systematically practised and analysed. Mathematical modelling can be viewed as a specialised form of problem-solving that focuses on finding solutions to real-life problems. In this sense, mathematical modelling may be regarded as a subset of problem-solving and can be represented as illustrated in Figure 3.1:

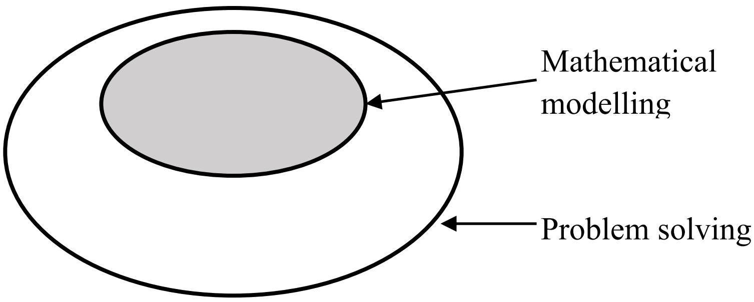


Figure 3.1: Illustration of problem solving and mathematical modelling

In this study, learners' competencies in mathematical modelling were investigated through tasks requiring them to solve mathematical problems involving area and perimeter to determine the dimensions of rectangular fields or camps that would yield the maximum area. Since these tasks were inherently problem based, it was necessary to discuss problem-solving, as already done in Chapter Two (Section 2.11). In this chapter, however, attention is given specifically to two problem-solving models, namely those proposed by Polya (1957) and Lester (2013). These models provide structured guidelines that help learners approach problem-solving systematically rather than haphazardly. They also outline step-by-step procedures that learners can follow when solving mathematical problems.

Aydoğdu and Ayaz (2008) identified several key objectives of teaching problem-solving in Mathematics, including improving learners' self-concept in relation to problem-solving, enhancing perseverance and willingness to attempt new problems, familiarising learners with problem-solving strategies, and encouraging a systematic approach to tackling problems. They further emphasise the importance of exposing learners to multiple solution strategies for a single problem, developing their ability to select appropriate strategies, improving accuracy in executing procedures, and ultimately increasing their success in obtaining correct solutions.

Saman and Chin (2017) argue that modern learners need to develop creative and innovative thinking skills, as thinking ability is closely linked to the capacity to apply Mathematics in real-life situations. They further note that problem-solving skills can only be strengthened when they are given deliberate and consistent attention. However, they observe that much of the problem-solving in schools still focuses on routine procedures presented in textbooks, rather than on authentic and varied problem situations. Since strategies differ depending on context, learners must also develop the ability to connect problem situations with critical thinking and collaborative learning. In this regard, the authors emphasise the importance of incorporating metacognitive strategies into problem-solving instruction, a concept that has already been discussed in detail in Chapter Two.

For learners to be successful in solving mathematical problems, it is essential that they present their work in a logical and systematic manner. This requires the use of structured approaches that guide reasoning and solution processes. Although several problem-solving models exist, this study focuses only on Polya's (1957) and Lester's (2013) models. While different models may consist of varying numbers of steps, both selected models are structured around four main stages.

3.2.1 Polya's model

This stage is used to assess learners' understanding of context-based problems and prepare them to translate such problems into mathematical form. The formulation of a mathematical model (devising a plan) represents the second stage in both Polya's model and the mathematical modelling process. In Polya's framework, the third stage is referred to as "carrying out the plan," whereas in mathematical modelling it is described as "solving the mathematical problem," during which appropriate mathematical skills and procedures are applied to obtain a solution. For the development of modelling competency, the final stage, "looking back" (Polya's Step 4), is particularly important, as it involves evaluating the solution in relation to the original real-world context. According to Ortiz (2016), Polya's model can be effectively used in classroom Mathematics instruction. The four stages of this approach include understanding the problem, devising a plan, carrying out the plan, and reflecting on the solution.

Understanding the problem: According to Ortiz (2016), at this stage learners attempt to fully comprehend the problem. Teachers play an important role by encouraging learners to engage actively with the task. Learners focus on identifying the unknowns, extracting relevant data, interpreting the given conditions, and introducing appropriate mathematical notation. In parallel, teachers prepare to assess the problem-solving process by identifying evidence of learners' understanding, checking whether their interpretations are correct and sufficient, and determining whether the given data and constraints have been appropriately analysed. Teachers may also provide guiding questions, hints, and relevant resources to support understanding, while simultaneously identifying misconceptions or gaps in learners' conceptual understanding.

Carrying out the plan: In this phase, learners implement the strategy developed in the previous step. They work through the solution systematically, checking each step for accuracy and consistency. Ortiz (2016) notes that learners should continuously verify their procedures to ensure correctness. Teachers support this process by asking probing questions that help learners justify each step and confirm its validity. They also encourage learners to revisit earlier steps when necessary to ensure that their solution remains logical, coherent, and aligned with the problem requirements.

Looking back: At this final stage, learners evaluate their solutions to determine whether they are reasonable and meaningful in relation to the original problem. Ortiz (2016) explains that learners reflect on whether alternative methods could have been used and whether the solution could have been anticipated more efficiently. They also consider whether their approach can be applied to other similar problems and compare their strategies with those of their peers to identify similarities and differences. Teachers, on their part, assess whether learners have verified their answers appropriately and encourage reflection on the usefulness and generalisability of different solution methods. They may also guide learners in making the problem more realistic, meaningful, or broadly applicable, while emphasising the value of using multiple approaches in problem-solving.

Ortiz (2016) further emphasises that Polya's model is highly effective in Mathematics teaching and learning. For example, when a teacher observes that a learner is struggling at the "devising a plan" stage, the teacher may provide targeted support at that point without immediately advancing the learner to later stages such as "carrying out the plan." Yuan (2013) also notes that Polya's problem-solving model has influenced both mathematicians and non-mathematicians, although its classroom implementation is still not as widespread as expected. In her work, Yuan (2013) applied Polya's model to help learners connect abstract mathematical ideas with concrete understanding, particularly in teaching concepts such as the lowest common multiple.

Every fifteen minutes, the M14D bus arrives (for example, at 8:00, 8:15, and 8:30 in the morning). Every twelve minutes, the M14A bus arrives (for example, at 8:00, 8:12, and 8:24 in the morning). Both buses' timetables run continuously throughout the day at the same pace. When are they supposed to meet up at the Union Square bus stop after 8:00 a.m. for the first time?

She reported that she encouraged learners to read the question carefully and with full understanding, which aligns with Step 1 of Polya's model. At this stage, the concept of multiples was not explicitly introduced; instead, learners were guided to think about two buses arriving at different intervals but eventually stopping at the same time. Yuan (2013) noted that in responding to the problem, learners listed the arrival times of each bus and determined the point at which both buses would stop together. This corresponds to Step 2 of the model. She further indicated that one of the completed charts produced by learners was as follows:

M14D	8:00	8:15	8:30	8:45	9:00	9:15	9:30
M14A	8:00	8:12	8:24	8:36	8:48	9:00	9:12

Learners may readily deduce from the chart that the two buses would arrive together at the Union Square bus terminal no later than 9:00. They can therefore identify that, after 8:00, the first simultaneous arrival occurs at 9:00. In Polya’s model, this corresponds to Stage 3, “carrying out the plan.”

Yuan (2013) observed that most learners did not experience difficulties in constructing the time chart and correctly determining the time at which the buses would arrive simultaneously. She further reflected on a previous teaching experience in which she introduced the topic through theoretical concepts and vocabulary related to determining the multiples of 15 and 12. Although learners were still able to obtain the correct answer (60) without difficulty, they struggled to transfer this understanding to real-life contexts such as the bus scheduling problem presented here.

Yuan (2013) therefore argued that teachers should begin with concrete, context-based experiences before moving to abstract concepts, enabling learners to construct the relationship between the two independently. After learners had arrived at the correct solution, she also asked them to reflect on the meaning of the original problem and reconsider its significance. This activity represents the fourth phase of Polya’s model, “looking back.” She noted that teachers often underemphasise this stage, yet it is crucial for bridging the gap between theoretical understanding and practical application.

Despite the strengths of Polya’s model, Ortiz (2016) highlights several challenges in its implementation. He argues that familiarity with the four-step process alone is insufficient to develop proficient problem-solvers, as effective problem solving requires flexibility rather than strict adherence to a linear procedure. In practice, learners may not always follow the steps sequentially; for example, they may begin working on a solution without fully understanding the problem. However, this should not discourage them from engaging with the task, as understanding may develop progressively during the solution process.

Ortiz (2016) further notes that while textbook problems are often neatly structured, real-life problems are typically complex, unstructured, and fragmented. In such cases, learners may begin attempting a solution while simultaneously developing their understanding of the problem. He also points out that intuition may sometimes guide learners toward a correct solution, but they still need to justify their reasoning using structured approaches such as Polya's model. Additionally, he cautions that the model's linear structure may limit creativity if applied rigidly.

Similarly, Wickramasinghe and Valles (2015) investigated the use of Polya's model in a statistics classroom to enhance learners' problem-solving skills. Prior to their intervention, they observed that learners often rushed into solving problems without adequately reading or understanding them. Many would simply select a formula and substitute values, a practice that rarely encouraged reflection or verification of answers. Over time, this led to errors that went unchecked, such as probability values exceeding 1, which is mathematically invalid. They therefore emphasised the importance of encouraging learners to verify and evaluate their solutions critically.

Their study, conducted with university students in New Mexico, involved dividing the class into two groups (Class 1 and Class 2). Both groups covered identical statistical topics, including probability, confidence intervals, and descriptive statistics, and completed the same assessments. However, Polya's problem-solving approach was implemented in Class 2 only.
Wickramasinghe and Valles (2015) describe the implementation of the model as follows:

Step 1: Understanding the problem

Learners were instructed to read each problem carefully, restate it in their own words, identify key concepts, and determine the unknowns. Emphasis was placed on comprehension, particularly for word problems.

Step 2: Devising a plan
Learners explored possible strategies such as identifying patterns, recalling similar problems, using diagrams, and formulating equations.

Step 3: Carrying out the plan
Learners executed the solution steps while continuously checking and verifying their calculations.

Step 4: Looking back
Learners evaluated the reasonableness of their solutions, interpreted their results, and assessed whether their methods were appropriate for the given context.

An example problem used in the study was given to both classes, and learners in Class 2 were expected to respond using Polya's structured approach.

Example:
By mistake, a manufacturer of DVD recording systems included 20 defective systems in a shipment of 100 going out to a small retailer. The retailer has decided to accept the shipment of DVD recorders only if none are found to be defective. Upon receipt of the shipment, the retailer examines only 20 of the systems. What is the probability that there will be 5 or less defectives? (Wickramasinghe and Valles 2015)

The results of Wickramasinghe and Valles's (2015) study indicated that the use of Polya's problem-solving technique in the classroom significantly improved learners' performance. Some learners in Class 1 produced probability values greater than one, reflecting a lack of verification of their solutions. In contrast, learners in Class 2 demonstrated greater accuracy, as they checked and validated their answers in line with the final step of Polya's model, "looking back."

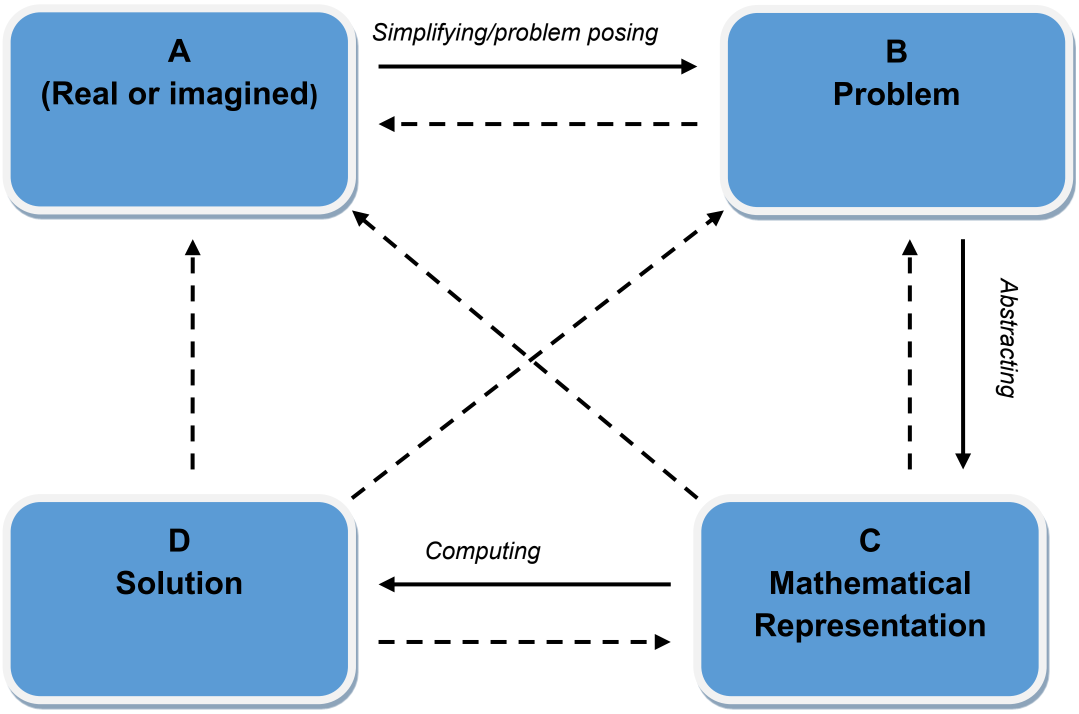
3.2.2 Lester's model

This model is used to guide learners in solving complex, non-routine word problems. It consists of four main components represented as Box A (Real or Imagined situation), Box B (Problem), Box C (Mathematical representation), and Box D (Solution). The solid arrows indicate the progression of activities from one stage to the next, where the outcome of each phase informs and influences the subsequent phase (see Figure 3.2).

The problem-solving model proposed by Lester (2013) comprises a series of phases that begin when an individual is confronted with a complex situation (Box A) and is assigned a specific task to complete, as indicated by the transition from Box A to Box B. At this initial stage, the problem solver begins by simplifying and reducing the complexity of the situation through the identification of relevant concepts and procedures that may be useful in addressing the problem. This requires making decisions about what information is essential and what can be temporarily set aside.

Simultaneously, the solver develops an understanding of the relationships among key concepts, which contributes to the construction of a realistic representation or mental model of the initial situation. This representation serves as a simplified version of the original context from which the problem is derived. Compared to the original scenario, this simplified model is easier to interpret, analyse, and use as a basis for further mathematical reasoning.

Figure 3.2: A model of complex mathematical activity   Source: Lester 2013



The next phase is the abstraction stage, represented by the solid arrow moving from Box B to Box C. At this point, formal mathematical concepts and notations are introduced for the first time. In this phase, the problem solver selects mathematical concepts that capture the essential features of the realistic model. This selection is often guided by an understanding of what representations will enable in subsequent solution processes.

A mathematical representation of the task and its context is then constructed using appropriate symbols and expressions to translate the real-world situation into mathematical form. Once this representation has been established, the original problem is reduced to a well-defined mathematical problem that is closely aligned with the constructed model. As illustrated in Box C, this stage results in a clear and structured mathematical formulation of the problem.

The transition from Box C to Box D represents the third phase of the process, which involves manipulating the mathematical representation and deriving mathematical results, as indicated by the computational arrow. At this stage, the problem solver applies mathematical knowledge, skills, and reasoning independently of the original context. For example, a problem may require the formulation and solution of a system of equations, where the solving process is governed by mathematical principles rather than contextual considerations.

According to Lester (2013), the final phase of the process involves moving between Boxes D, A, B, and C, and requires the solver to compare and interpret the results obtained in relation to the original situation, the problem formulation, and the mathematical representation, as indicated by the broken arrows. This comparison is not restricted to the end of the process; rather, it may occur continuously throughout all stages of problem solving.

Continuous monitoring and evaluation of one’s thinking is therefore essential for successfully engaging with complex mathematical tasks. More broadly, the ability to reflect on and assess one’s reasoning highlights the inherently complex nature of mathematical activity. The extent to which a solver engages in such comparisons with prior knowledge and experiences can be seen as a key indicator of task complexity and a distinguishing feature between routine and non-routine problem solving. In this sense, more complex tasks typically require a higher frequency of comparisons and evaluations, whereas routine computations involve fewer such reflective processes.

3.3 Constructivism and Mathematical Proficiencies as Theoretical Frameworks

This study employed constructivism and the strands of mathematical proficiency as the primary theoretical frameworks for investigating learners' competencies in mathematical modelling. In addition, APOS theory, together with the concepts of concept image and concept definition, as well as multiple representations, were used as analytical frameworks to guide the interpretation of learners' responses. This section has outlined the concept of constructivism and its implications for education. It has also incorporated the strands of mathematical proficiency developed by Kilpatrick et al. (2001) as a complementary theoretical framework to support the analysis of learners' mathematical understanding and modelling competencies.

3.3.1 Constructivism as a theoretical framework of this study

The primary philosophical approach guiding this study is constructivism. As indicated in Chapter Two, several definitions of constructivism have already been provided. To further consolidate this understanding, it is necessary to consider how different scholars conceptualise the term.

Anyanwu and Iwuamadi (2015) describe constructivism as an approach that promotes active participation, problem-solving, and collaboration, where learners construct knowledge based on prior experiences. In such classrooms, teachers act as facilitators who guide learning and encourage interactive dialogue. Similarly, Resnick and Glaser (2016) define constructivism as a philosophy of learning that enables individuals to develop new understanding through the interaction between existing knowledge, concepts, and prior beliefs. Shittu and Alex (2025) further emphasise constructivism as a learning theory that stresses active knowledge construction through experience and interaction, positioning it as a viable alternative to traditional instructional approaches.

Although constructivism initially emerged as a theory of learning, it has gradually expanded to influence broader domains, including teaching practices, curriculum development, and theories of personal and scientific knowledge. It has also informed several national curriculum frameworks and education policy documents. In essence, constructivism is no longer confined to explaining how learning occurs but has become a guiding framework for how teaching and curriculum design should be structured.

The pioneers of constructivism focused primarily on how learners construct new knowledge by building on prior experiences. In this regard, the shift toward constructivist thinking has also necessitated changes in pedagogical approaches, moving from teacher-centred instruction to learner-centred practices. Constructivist perspectives emphasise that learners actively construct knowledge when meaningfully engaged in learning activities. This shift is evident in curriculum reforms, including those in South Africa, where approaches such as mathematical modelling have been introduced in Mathematics education. Consequently, teachers are required to adapt their instructional strategies to align with such learner-centred, modelling-based approaches.

Although scholars define constructivism in different ways, these definitions largely converge on a common idea: knowledge is actively constructed by learners through the integration of new information with prior knowledge and experience. Yilmaz (2008) summarises the core assumptions of constructivist learning theory as follows:

1. Learning is a dynamic process.
2. Learning is a flexible activity.
3. Learning is context dependent.
4. Knowledge is constructed by the learner rather than being innate, passively received, or independently invented.
5. Knowledge is individually constructed and unique.
6. Knowledge is socially influenced.
7. Learning involves interpreting the world.
8. Learning is shaped by prior knowledge and experience.
9. Learning is supported by social interaction.
10. Effective learning requires meaningful, challenging, and open-ended problems.

A careful examination of these principles shows clear alignment with mathematical modelling approaches, particularly in relation to active learning, social construction of knowledge, reliance on prior knowledge, and engagement with open-ended problems. These similarities justify the use of constructivism as a relevant theoretical framework for this study.

The constructivist principles are further supported by Faulkenberry and Faulkenberry (2006), who argue that although constructivism exists in different forms, they all share a common emphasis on learner-centred learning. This implies that learners should be active participants in the learning process rather than passive recipients of information. During learning, learners draw on their prior knowledge, experiences, and ideas, all of which influence how they construct new understanding.

3.4. Seven Goals for Developing Constructivist Learning Environments

Honebein (1996), citing Cunningham, Duffy and Knuth (1993) as well as Knuth and Cunningham (1993), outlines seven instructional goals for constructivist teaching approaches. He then summarises these goals as follows:

To provide experience of the knowledge construction process: This goal emphasises that learners take responsibility for identifying what they want to learn within their field of study and for selecting appropriate learning strategies.

To provide knowledge of and respect for multiple perspectives: Real-life problems often have more than one valid solution or approach. Learners should therefore engage in activities that expose them to and allow them to explore diverse solutions.

To embed learning in realistic contexts: Classroom problems should be closely aligned with authentic real-world situations.

To encourage ownership and voice in the learning process: Learners should play an active role in determining what and how they learn, reflecting a learner-centred approach.

To embed learning in social experience: Since cognitive development is influenced by social interaction, learning should involve collaboration among learners and between learners and teachers.

To encourage multiple representations of knowledge: Relying solely on verbal or written communication limits learners' understanding. Therefore, alternative forms such as computers, photographs, and videos should be used to enrich learning experiences.

To foster awareness of the knowledge construction process: Learners should be able to justify and reflect on the strategies they use in solving problems, which is a key outcome of constructivist learning.

Several of these constructivist teaching goals align closely with mathematical modelling. Both emphasise learning in real-life contexts, learner-centred instruction, the existence of multiple solution strategies, learner autonomy in problem solving, collaborative learning environments, and the requirement for learners to justify their reasoning and solutions. These shared characteristics further demonstrate the strong connection between constructivism and mathematical modelling, thereby justifying constructivism as an appropriate theoretical framework for this study, which investigates learners' competencies in mathematical modelling.

3.4 Teaching and Learning Mathematics in Terms of Constructivists' View

Constructivists argue that knowledge cannot simply be transferred from one person to another; rather, it is actively constructed by the learner. This perspective implies that classroom teaching approaches must shift in line with constructivist philosophy. The traditional role of the teacher as a transmitter of knowledge must therefore be redefined. Reflecting on my own schooling experience, classroom learning was largely teacher-centred: learners sat quietly facing the teacher, who dominated the lesson through explanation. At the end of the lesson, learners were expected to reproduce what had been taught, and success was largely measured by memorisation and accurate recall.

In contrast, constructivism views learners as active participants in knowledge construction. Teachers are therefore expected to design and organise learning environments that promote active engagement, enabling learners to construct their own understanding of concepts rather than simply receiving information passively.

Bada and Olusegun (2015) identify two key goals of teaching Mathematics from a constructivist perspective. Firstly, learners should develop increasingly complex and abstract mathematical structures to solve meaningful problems. Secondly, learners should become self-motivated and independent in their mathematical thinking. In this view, learners perceive Mathematics as a tool for problem-solving and regard their own inquiry, exploration, and participation in discussion—not teacher explanation—as the primary source of knowledge. For such learners, understanding and reasoning about Mathematics becomes more important than merely completing procedural tasks. They see themselves as active constructors of mathematical knowledge rather than passive recipients.

In the present study, learners were given the freedom to apply their own methods when responding to test questions. They were required to draw on prior knowledge to solve unfamiliar, real-world problems.

Constructivists further emphasise that the development of learners' personal mathematical understanding is central to Mathematics education. This contrasts with traditional approaches, which prioritise established procedures and fixed algorithms. Although teachers in traditional classrooms may use concrete manipulatives to introduce concepts, these are often used as a bridge toward abstract symbolic procedures rather than as tools for sustained exploration. Constructivist instruction, on the other hand, encourages learners to develop and refine their own strategies for solving problems. Learners are not expected to replicate predefined methods but are encouraged to construct and justify their own approaches.

Lawless (2019) similarly notes that, within constructivist theory, the teacher's role shifts from instructor to facilitator or guide. She argues that learners learn best when they actively discover knowledge rather than being told what to do. Learning takes place through discussion, role-play, and collaborative problem-solving. Instead of standing in front of the class and delivering information, teachers move among learners, offering guidance, support, and scaffolding where necessary.

According to Har (2013), the constructivist teaching model enables learners to construct knowledge in ways that may represent objective understanding, support cognitive development, and emerge through social interaction. She further notes that constructivism does not prescribe a single fixed teaching method; rather, it is better understood in contrast to traditional behaviourist approaches. A comparison between constructivist and traditional instructional methods is therefore often used to highlight these differences (see Table 3).

Constructivism is a learner-centred approach that promotes active engagement with mathematical concepts. It prioritises discovery, problem-solving, and critical thinking over rote memorisation. Teachers act as facilitators who guide learners through problem-solving processes while encouraging reflection, discussion, and the application of mathematical ideas in meaningful contexts. At its core, constructivist Mathematics education emphasises engagement with real-world problem-solving situations. In contrast, traditional Mathematics instruction often focuses on the memorisation of algorithms and repetitive procedural exercises, sometimes without conceptual understanding (Murugesan et al., 2016).

Blum and Borromeo Ferri (2009, as cited in Tasarib et al., 2025) further emphasise that mathematical modelling requires learners to translate real-world problems into mathematical representations using appropriate symbols and terminology.

It is therefore evident that constructivism strongly supports mathematical modelling, as both approaches are learner-centred, promote active learning, and emphasise real-world problem-solving. The comparison between constructivist and traditional approaches is particularly relevant to this study, as it highlights the importance of using real-life problems in assessing learners' competencies in mathematical modelling.

Table 3.1: Comparison of Constructivist and Traditional Learning Approaches

Method	**Constructivist Learning**	**Traditional Learning**
Type of Learner	Learners are recognised as distinct individuals, and the process of learning is considered to be enhanced by their uniqueness.	Learners are similar in nature and characterised by their age, which the development of curricula and instructional strategies are based.
Accountability for Learning	Learning is owned by learners. In the learning process, it emphasises the active participation of learners.	The instructors own learning. Learners are receptive to knowledge and in a passive state while they are taught.
Academic Motivation	Real-world experiences that involve problem-solving and conceptual learning help learners become more motivated.	Reward systems enhance the behaviours of learners. Adhering to norms and achieving goals increases motivation.

Teacher's role	Teachers provide guidance, support and pose probing to their learners. Along with the learners, they have ongoing conversations.	Teachers issue instructions and anticipate that learners will follow them carefully to understand the learning material.
Interaction	Mutual learning occurs between teachers and learners. Through efficient communication, they can increase each other's knowledge and comprehension.	The process of learning is unbiased or impartial. Teachers impart knowledge to their learners through the completion of activities and assignments.
Collaboration	Learners work together to determine knowledge and reality. They must be careful to make sure they learn concepts and other material with factual knowledge.	To comprehend the material that is prescribed in the subject, learners must be focused and disciplined. Teachers provide learners with ideas and tips to help them reach their academic objectives.
Context	Learning is largely dependent on the context in which it occurs. Application and learning are directly related. By means of social interactions, learners take part in many activities.	Contextualisation of knowledge is removed. It is possible that learners may not receive the necessary skills to effectively comprehend the tasks. Learning takes place when results are quantified or measured.
Assessment	Assessment is a procedure that involves teachers and students interacting in both directions. Finding learning accomplishments of learning is related to the learning process.	It is conducted by teachers. It is used to determine how much the learners have grasped. Furthermore, it can determine how learners manage to move forward with the accomplishment of their academic objectives.

Knowledge	Experience causes modifications to knowledge, which is a dynamic concept.	Knowledge is inactive.
Working of Learners	Learners collaborate or work together in groups.	Learners work independently or on their own.

Source: Har (2012)

It can be observed that, in teaching and learning, constructivists advocate for instructional approaches in which teachers create learning environments that are closely connected to real-life contexts and allow for multiple possible solutions. Teaching should emphasise discovery learning, where learners actively construct meaning while teachers provide guidance, scaffolding, and feedback. In addition, teachers are expected to promote collaboration among learners while fostering a sense of ownership of learning through joint knowledge construction. Constructivist teaching approaches strongly emphasise cooperative learning, where learners work in mixed-ability groups and are collectively accountable for learning outcomes. Another key feature is problem-based learning, in which learners are presented with authentic, meaningful real-life problems that initiate inquiry. As learners work together to solve these problems, they are exposed to diverse perspectives, and through discussion and debate, their reasoning and critical thinking skills are developed.

Easen and Bolden (2005, as cited in Adams, 2006) argue that traditional views of learning focused primarily on observable behaviour. They caution that evaluating learning solely through context-bound performance ignores two key considerations: first, that learning occurs within the mind, and second, that behaviour is an unreliable indicator of cognitive processes. These critiques encouraged a shift in educational thinking, moving teachers beyond rigid performance measures towards a focus on learning processes and learner experiences. Consequently, teachers began to reconsider their instructional approaches from the learners' perspective.

Easen and Bolden (2005, as cited in Adams, 2006) further note that although test performance may reflect intellectual development, it may also simply indicate a learner's ability to interpret examination requirements. As a result, performance in tests can sometimes be improved without genuine conceptual understanding. In such cases, teaching may become focused on training learners to interpret exam questions correctly rather than developing deep procedural and conceptual understanding.

Adams (2006) similarly argues that debates on learning have enabled educators to move beyond narrow performance-based perspectives towards a focus on learning processes and learner development. These discussions encouraged teachers to critically reflect on their instructional practices from the viewpoint of learners. In contrast, behaviourist approaches viewed learners as passive recipients of knowledge, while teachers were seen as transmitters of information. Although behaviourism acknowledges that the environment influences learning and supports equal learning opportunities, it pays limited attention to cognitive processes. Despite extensive criticism, behaviourist approaches continue to appear in some classroom management literature (Porter, 2000, as cited in Adams, 2006).

In contrast, constructivism conceptualises learning as a process of mental construction in which new information is integrated with existing knowledge, skills, and experiences. Learning is enhanced when learners are actively engaged in constructing meaning. Pritchard and Woollard (2010) emphasise that meaningful learning is unlikely to occur when learners remain passive in the classroom. They therefore recommend that learners be encouraged to engage in purposeful talk related to the subject matter, while teachers establish clear expectations for appropriate interaction and focus.

They further identify five forms of social interaction that can enhance learning:

- Cognitive apprenticeship, where a more knowledgeable individual supports a learner;
- Collaborative learning, where learners work together towards a shared goal;
- Cooperative learning, where learners pursue individual goals while supporting others;
- Pastoral interaction, which addresses emotional and social development; and
- Instructional interaction, where structured guidance is provided to individual learners.

Although social interaction is central to constructivist classrooms, teachers must ensure that all learners' educational needs are addressed through well-designed activities. Peer support is often a key feature of such classrooms, where learners assist one another when difficulties arise. Brookes and Brookes (1993, as cited in Adams, 2006) propose several strategies to enhance constructivist learning, including promoting learner autonomy and initiative, using concrete materials and primary sources, incorporating learners' responses into lessons, assessing understanding continuously, encouraging discussion, asking open-ended questions, prompting justification of responses, and exposing learners to situations that encourage multiple viewpoints and debate.

These strategies align with the constructivist view that learning is an active process of personal knowledge construction. Bhattacharjee (2015) similarly argues that constructivist teaching encourages educators to support learners in constructing their own understanding, often through authentic tasks and real-life contexts. According to constructivist principles, learning is most effective and enduring when it is situated in meaningful, relevant environments connected to learners' lived experiences.

Sachdeva (2019) emphasises that learner autonomy is a central feature of learner-centred approaches, particularly those grounded in experiential and problem-based learning. She defines autonomy in Mathematics education as learners' ability to take responsibility for their own learning, engage in decision-making, participate in discussions, and initiate learning activities. Autonomous learners are expected to regulate their own learning processes and apply mathematical knowledge beyond the classroom to real-world contexts.

Thus, autonomy is a key requirement for competence in mathematical modelling, as modelling involves solving authentic real-life problems using mathematical reasoning. Wolf (2018), in *Modelling with Mathematics*, similarly argues for a shift away from traditional instructional approaches towards modelling-based teaching that promotes learner autonomy. She notes that learners accustomed to routine, procedural instruction often struggle with non-routine modelling tasks that require independent thinking. This challenge is compounded by the fact that many teachers still operate within traditional frameworks where only one correct method is emphasised. The introduction of mathematical modelling tasks can therefore help learners recognise that multiple solution strategies are possible. Wolf (2018) further argues that effective modelling instruction requires teachers to interpret learners' thinking and support them in justifying their solutions, positioning teachers as reflective practitioners and problem-solvers themselves.

In summary, the discussion above demonstrates a clear alignment between constructivist principles and mathematical modelling. Both emphasise real-world contexts, learner-centred instruction, active knowledge construction, collaboration, and multiple solution pathways. This strong conceptual relationship justifies the use of constructivism as an appropriate theoretical framework for this study.

3.5.1 Strands in learning Mathematics (mathematical proficiency)

The Merriam-Webster Dictionary defines "proficiency" as progress or advancement in knowledge or skill. In this study, Kilpatrick's five strands of mathematical proficiency were incorporated into the theoretical framework to explain how learners develop competence in Mathematics. Kilpatrick, Swafford, and Findell (2001) identify these strands as: (1) conceptual understanding, (2) procedural fluency, (3) strategic competence, (4) adaptive reasoning, and (5) productive disposition. Both social constructivism and the strands of mathematical proficiency were used as complementary frameworks to guide this study by shaping how learners' mathematical understanding is interpreted and by extending existing knowledge in the field.

Kilpatrick was commissioned by the National Academy of Sciences in the United States to lead a committee investigating how Mathematics is learned. The main aim of this work was to make recommendations for improving learners' achievement in Mathematics from preschool through to Grade 8. During this period, there were ongoing debates about the nature of mathematical learning. Some mathematicians argued that conceptual understanding should precede skill acquisition, while others maintained that skills develop before conceptual knowledge (Kilpatrick, Swafford, & Findell, 2001).

Kilpatrick (2009) explains that the committee introduced the term "mathematical proficiency" to describe the overall competence required in learning Mathematics. This was represented using a rope model consisting of five interwoven strands. The intertwined nature of the strands illustrates that mathematical learning does not follow a fixed sequence; rather, the strands develop simultaneously and reinforce one another. For a learner to become proficient in Mathematics, all five strands must be developed in an integrated manner. The five strands and their descriptions are presented below.

3.5.2 Conceptual understanding

Conceptual understanding, according to Kilpatrick, Swafford, and Findell (2001), refers to the comprehension of mathematical concepts, procedures, and relationships. Learners who develop this strand go beyond memorising isolated facts and procedures; they understand when and why a mathematical concept is applicable and how it connects to other ideas. By organising their knowledge into coherent and meaningful structures, they can link new concepts with prior knowledge, thereby facilitating deeper learning.

This strand also supports long-term retention of knowledge. When mathematical ideas are understood meaningfully, they are more easily recalled, reconstructed, and applied, even after some time has passed. In other words, learners are less likely to forget or misapply procedures they have conceptually understood. Furthermore, learners with strong conceptual understanding can monitor their own thinking and evaluate whether their solutions are reasonable and consistent.

Teachers often identify conceptual understanding through learners' ability to explain relationships between concepts and representations in their own words, although this is not always straightforward. This is because learners may demonstrate understanding internally before they are able to express it verbally. The ability to represent mathematical ideas in multiple ways and to explain how different representations serve specific purposes is widely regarded as an important indicator of conceptual understanding (Kilpatrick, Swafford, & Findell, 2001).

Andamon and Tan (2018) emphasise that one of the primary goals of Mathematics education is to enable learners to solve problems in a systematic and logical manner, such that similar problems can be addressed more effectively in the future. Mathematics plays a significant role in everyday life and is applied in fields such as computer science, physics, astronomy, and statistics. Through Mathematics, learners develop essential skills such as problem-solving, logical reasoning, and abstract thinking, which help them interpret and understand the world around them.

According to Andamon and Tan (2018), strong performance in Mathematics requires the development of conceptual knowledge. They define conceptual understanding in Mathematics as a deep and comprehensive grasp of fundamental concepts, including an understanding of why mathematical procedures and algorithms work. Thus, conceptual understanding involves learners actively engaging with mathematical ideas, making informed choices, and applying their knowledge through meaningful participation in learning activities.

3.5.3 Procedural fluency

According to Kilpatrick, Swafford, and Findell (2001), the strand of procedural fluency refers to the ability to carry out mathematical procedures accurately, efficiently, and appropriately. In other words, it involves knowing how to apply different procedures, when to use them, and being able to execute them correctly, precisely, and flexibly. Kilpatrick and colleagues (2001) further explain that, in the context of number work, procedural fluency is essential for supporting understanding of place value, the meaning of rational numbers, and the ability to compare and evaluate different computational methods. It includes a range of approaches such as written algorithms, mental computation strategies, and the use of tools like calculators, computers, or physical manipulatives such as beads and counters. They also emphasise that learners should develop speed and accuracy in performing basic operations with whole numbers (e.g., $6 + 7$, $17 - 9$, 8×4) without over-reliance on external aids. Such fluency is strengthened through consistent and purposeful practice, which helps learners build both confidence and efficiency in mathematical computation.

3.5.4 Strategic competence

According to Kilpatrick et al. (2001), this strand focuses on the ability to formulate, represent, and solve mathematical problems. They explain that it is closely related to the processes of problem formulation and problem solving commonly discussed in mathematics education literature. To solve problems effectively, learners need to be familiar with a range of strategies and be able to select the most appropriate approach for a given situation. This requires an understanding of how different methods can be applied to different types of problems. In order to represent a problem accurately, learners must first construct a mental model of the key elements of the situation. Once this internal representation is formed, they can translate it into a mathematical form that captures the essential features of the problem while filtering out irrelevant information.

3.5.5 Adaptive reasoning

According to Kilpatrick, Swafford, and Findell (2001), adaptive reasoning refers to the ability to think logically about the relationships between mathematical ideas and situations. It involves sound reasoning that is based on careful analysis, consideration of alternatives, and the ability to justify conclusions. In Mathematics, adaptive reasoning functions as a connecting structure that links different facts, concepts, and procedures, ensuring that they are used coherently and meaningfully. It also guides learners in checking whether their mathematical thinking is consistent and valid.

The authors further explain that adaptive reasoning requires deductive thinking, where conclusions are derived from established assumptions through logical steps. In this sense, a correct mathematical solution must be supported by a clear chain of reasoning. Learners are therefore encouraged not to rely solely on external validation from teachers or peers when uncertain about a solution, but instead to critically review their own work and assess whether their reasoning is valid and justified.

Research suggests that young learners' ability to reason logically is initially limited and develops gradually, often becoming more structured around the age of twelve. Studies indicate that when young children (approximately ages 4–5) explain their thinking, they tend to rely on prior knowledge and simple transformations of information. At this stage, they may also resist alternative explanations. However, their reasoning skills can be supported and strengthened using visual representations such as pictures and drawings. Kilpatrick et al. (2001) further note that learners are more likely to demonstrate reasoning when three conditions are present: a sufficient knowledge base, a clear and engaging task, and a familiar and comfortable context.

Adaptive reasoning is also evident when learners can justify their solutions independently. This includes providing clear reasons for the methods used to solve a problem. Although justification and proof are often associated with advanced levels of Mathematics, Kilpatrick et al. (2001) argue that learners should be introduced to reasoning from an early age. Even in the early grades, learners should be given regular opportunities to discuss their thinking, explain their strategies, and justify their answers. Teachers can support this by establishing classroom norms that require learners to explain and defend their mathematical ideas to peers. Such practices help learners develop the ability to challenge, justify, and articulate their reasoning, thereby strengthening both conceptual understanding and reasoning skills.

Furthermore, Kilpatrick et al. (2001) reiterate that adaptive reasoning develops when learners possess sufficient prior knowledge, are engaged in clear and motivating tasks, and work within familiar and supportive contexts. In related perspectives, Ostler (2011, as cited in Syukriani et al., 2017) distinguishes between strategic competence and adaptive reasoning by explaining that strategic competence refers to the ability to formulate appropriate mathematical models and select effective problem-solving strategies, whereas adaptive reasoning focuses on justifying and explaining the correctness of solutions within a given context. Similarly, Ozdemir (2012, as cited in Syukriani et al., 2017) associates strategic competence with learners' ability to regulate their learning behaviours and apply appropriate strategies to understand and solve mathematical problems.

3.5.6 Productive disposition

This strand refers to learners' belief that Mathematics is meaningful, useful, and worthwhile, combined with a sense of perseverance and personal efficacy (Kilpatrick et al., 2001). In other words, it describes a stage at which learners begin to recognise the value of Mathematics in their everyday lives and develop the belief that consistent effort in learning the subject can lead to success. At this stage, learners may start to view themselves as capable Mathematics learners who can perform well when they apply effort and persistence.

Kilpatrick et al. (2001) further argue that for learners to develop competence in the other four strands of mathematical proficiency, they must first believe that Mathematics is understandable and that it can be learned through sustained effort. The development of a productive disposition therefore depends on learners being exposed to learning experiences that help them see Mathematics as logical, accessible, and beneficial. It also requires learners to appreciate both the satisfaction of making sense of mathematical ideas and the value of perseverance in problem-solving.

As learners deepen their understanding of mathematical concepts, their ability to make sense of Mathematics also increases. In this way, productive disposition develops alongside conceptual growth. According to Kilpatrick et al. (2001), learners' attitudes toward Mathematics play a significant role in determining their overall success in the subject. Furthermore, Kilpatrick et al. (2001) explain that learners who develop a productive disposition tend to be confident in their mathematical abilities and increasingly believe that Mathematics is logical and understandable. Thus, mathematical proficiency involves more than conceptual understanding, procedural fluency, problem solving, and reasoning; it also includes a positive attitude toward Mathematics, which may vary among individuals. Mathematically proficient learners are those who believe that they can solve mathematical problems through effort and persistence, provided that the Mathematics involved is meaningful and makes sense to them.

3.5.7 The development and interlinking of mathematical strands

The development of mathematical proficiency cannot be achieved through a single practice or isolated effort. Rather, it is a gradual process that requires sustained engagement with concepts and procedures over time. Learners must repeatedly apply mathematical ideas, justify their reasoning, and make connections to previously learned concepts and techniques. Kilpatrick et al. (2001) emphasise that it is not sufficient for learners to merely practise routine exercises, such as simple fraction addition problems, after a procedure has been introduced. Instead, sufficient and varied practice is necessary to help learners understand algorithms deeply and develop the ability to explain and apply them across different contexts.

Awofala (2017) argues that strategic competence can be strengthened through continuous exposure to mathematical tasks that are embedded in real-life contexts. This competence is developed when learners are presented with problems that require them to understand the situation, plan a solution, and implement appropriate strategies to solve the problem mathematically. Kilpatrick et al. (2001) further explain that adaptive reasoning interacts with other strands of mathematical proficiency, particularly during problem-solving. Learners first use strategic competence to interpret and represent a problem and then apply exploratory strategies to develop possible solutions. Finally, adaptive reasoning becomes essential when learners evaluate the validity and correctness of their chosen strategies and solutions. In this process, conceptual understanding supports reasoning by enabling learners to use representations such as symbols and diagrams as a basis for logical justification. Although learners may initially arrive at solutions through computation, measurement, or graphical methods, adaptive reasoning is required when they begin to assess whether their solutions are appropriate and logically sound.

Kilpatrick et al. (2001) also note that conceptual understanding, procedural fluency, and strategic competence are interdependent and mutually reinforcing. Solving non-routine problems requires a strong grasp of the quantities and relationships involved in the task. At the same time, engagement with such problems strengthens learners' ability to handle routine procedures, while also deepening their understanding of key mathematical ideas such as unknowns, givens, conditions, and solutions. In this way, strategic competence is continuously involved in the development of procedural fluency, particularly during calculation and problem-solving processes.

Furthermore, knowledge acquired with understanding provides a foundation for the development of new mathematical ideas and supports learners in tackling unfamiliar problems. As learners develop conceptual understanding, they become better able to recognise connections between concepts and procedures and to construct logical explanations for mathematical relationships. This growing understanding builds confidence and enhances overall performance. For example, once learners understand place value and single-digit operations, they can extend these procedures to multi-digit calculations in number systems (Kilpatrick et al., 2001). This illustrates that conceptual understanding strengthens both procedural fluency and adaptive reasoning.

The importance of well-developed strands of mathematical proficiency is also demonstrated in empirical research. Awofala (2017), in a study conducted in Nigeria involving 400 secondary school learners (198 boys and 202 girls), assessed learners' mathematical proficiency across different dimensions. The results showed that 99.75% of learners demonstrated high levels of mathematical proficiency. Awofala (2017) attributed the strong performance observed in both internal and external examinations to the development of learners' mathematical proficiency. This suggests that all strands must be developed in an integrated manner, rather than focusing on only one or two, to improve overall achievement in Mathematics.

Similarly, Rittle-Johnson et al. (2001) highlight that understanding how knowledge develops and transforms is central to research in mathematics education. They identify conceptual understanding and procedural knowledge as two key forms of knowledge necessary for mathematical competence. Silver (1986, as cited in Rittle-Johnson et al., 2001) further emphasises that the integration of conceptual and procedural knowledge is essential for effective performance in Mathematics.

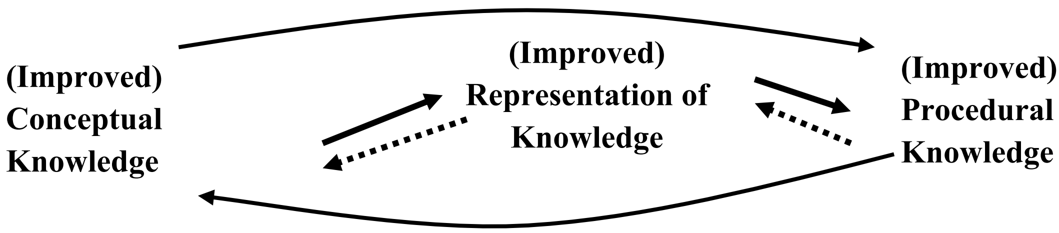


Figure 3.3: Relations between Conceptual and Procedural Knowledge
Source: Rittle-Johnson et al (2001)

Various scholars have presented differing views regarding the developmental relationship between procedural knowledge and conceptual understanding, despite the importance of both forms of knowledge. Much of the existing research has focused on determining which type of knowledge develops first. Debates have centred on the sequencing or prioritisation of one form of knowledge over the other. However, some studies suggest that conceptual and procedural knowledge are mutually reinforcing and develop in an iterative cycle. In this view, growth in one type of knowledge supports the development of the other, which in turn further strengthens the first, creating a continuous cycle of learning development, as illustrated in Figure 3.3 (Rittle-Johnson et al., 2001).

Chan et al. and Seto (2012) propose that learners' ability to develop mathematical models may indicate their capacity to construct mathematical concepts and express them through mathematisation. This process involves translating real-world situations into mathematical representations, thereby giving meaning to conceptual structures and enabling learners to interpret and solve authentic problems. Chan et al. (2012) further clarify that, although mathematisation is closely related to mathematical modelling, it is not synonymous with modelling. Rather, it forms part of the modelling process, in which learners work towards developing mathematical representations that adequately capture real-world contexts.

3.6 Solution plan as an intervention strategy for mathematical modelling

When examining the challenges learners experience in mathematical modelling, it becomes evident that most difficulties arise within the modelling process itself. Various scholars have identified specific areas where learners struggle. Some learners have trouble in understanding the problem context, others in formulating mathematical representations, while others struggle to validate or interpret their final solutions. These challenges are reflected in several empirical studies discussed below.

Marchisio, Roman, and Sacchet (2021) found that learners face multiple obstacles when solving mathematical modelling tasks. These include technical errors (such as incorrect input), misunderstanding of problem texts, failure to translate word problems into mathematical form, and difficulties in the solving phase, including incorrect formulation of strategies and computational errors.

Similarly, Schaap, Vos, and Goedhart (2011) identified several barriers within the modelling cycle. At the initial stage of understanding, learners often fail to comprehend the problem fully because they overlook important information in the problem statement. They may also make incorrect assumptions and struggle to identify relevant variables. At later stages, learners have trouble in establishing relationships between variables, often due to weak algebraic skills.

Edo, Putri, and Hartono (2013) also reported that learners struggle particularly with (a) formulating real-life situations mathematically and (b) evaluating the reasonableness and validity of their mathematical solutions. In the same vein, Jankvist and Niss (2020) found that many learners experience difficulties with mathematisation, particularly in translating word problems into appropriate mathematical expressions. As a result, they are unable to construct accurate representations of given situations.

Leong and Tan (2020) further observed that learners often struggle with the sequencing of the modelling process. They experience difficulties in making appropriate assumptions, which indicates that modelling tasks are cognitively demanding. Overall, learners frequently encounter challenges in making assumptions, performing calculations, and interpreting solutions meaningfully.

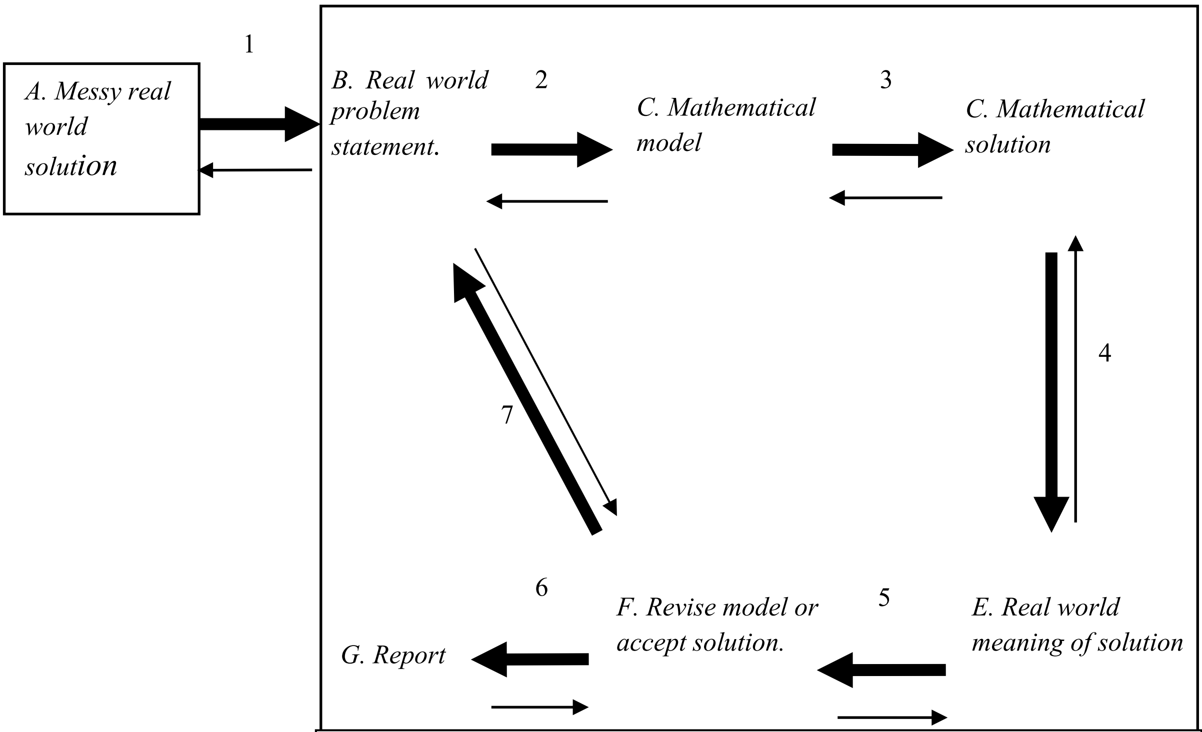
These challenges highlighted across studies need to be addressed to improve learners' performance in mathematical modelling. Chan et al. (2012) suggest that learners who can develop mathematical models demonstrate the ability to construct mathematical concepts and express them through mathematisation. This involves translating real-world situations into mathematical representations that give meaning to conceptual structures, thereby enabling learners to interpret and solve real-life problems.

Chan et al. (2012) further clarify that, although mathematisation is closely related to mathematical modelling, it is not equivalent to modelling itself. Rather, it forms part of the modelling process, as learners work towards developing appropriate mathematical representations of real-world contexts.

Research in mathematical modelling is not only concerned with learners' actions during problem solving, but also with how teachers support learners through the process. Stillman et al. (2007) emphasise this perspective by proposing a modified modelling cycle that incorporates both learners' cognitive processes and teachers' instructional support during modelling activities.

In the modelling cycle illustrated in Figure 3.4, the broader arrows represent transitions between phases, moving clockwise from the initial stage to indicate the overall progression of the modelling process. The cycle may end in a final report when a satisfactory solution is obtained, or it may return to earlier stages if the solution is deemed inappropriate or unacceptable. The numbered stages (1 to 7) represent the cognitive activities involved as learners move through different phases of modelling. The thinner arrows pointing in reverse directions highlight the non-linear nature of the process and indicate the presence of ongoing metacognitive activity.

Figure 3.9 further simplifies the solution process into four main stages of modelling, with the intention of making learners' difficulties easier to identify. This structure supports learners' self-diagnosis of challenges and allows for targeted intervention. When necessary, the "solution plan" provides adaptive support, enabling learners to overcome difficulties. If further support is required, the teacher is expected to provide additional prompts and appropriate feedback. The primary aim of the solution plan is therefore to identify learners' difficulties within the modelling process and to enhance the provision of responsive and adaptive teacher support.



1. Comprehending, structuring, simplifying, explaining context
2. Supposing, developing, mathematising
3. Using Mathematics to work out solutions
4. Interpretation of results of Mathematics
5. Evaluating, criticising, verification
    a. Communication and justification (if model is found adequate)
6. Reviewing the modelling procedure (if model ii judged

Figure 3.4: Modelling Process
Source: Stillman et al (2007)

In Chapter Two, I discussed several modelling cycles that differ in the number of stages they propose. For instance, Blomhøj and Højgaard (2003) present a six-step cycle, while Blum and Leiß (2007) propose a seven-step cycle (see Figures 2.1 and 2.4, respectively). However, Schukajlow et al. (2015) critique such multi-step models, arguing that an increased number of stages can create difficulties for learners, particularly those who are new to mathematical modelling, as they may struggle to distinguish and navigate between the different phases.

In response to this challenge, Schukajlow et al. (2015) propose a simplified four-step process referred to as a "solution plan", in which several stages are merged (as illustrated in Figure 3.7). Specifically, steps 2 and 3, as well as steps 5, 6, and 7, are combined to reduce complexity. To further support learners, the solution plan includes structured guidance at each stage, along with explicit instructions, examples, and demonstrations.

The four stages of the solution plan are as follows:

(a) **Apprehending the task**

At this stage, learners are required to read the problem carefully and form a mental image of the situation as a foundation for understanding the task.

(b) **Searching for mathematics**

Learners transition from visualising the problem to representing it mathematically. They identify relevant and irrelevant information, determine necessary variables, and make appropriate assumptions about the situation. Suitable formulas and procedures are then selected based on the problem context.

(c) **Using mathematics**

In this stage, learners carry out the required mathematical procedures, which may include solving equations or representing the situation using graphs and other mathematical tools.

(d) **Explaining results**

Finally, learners interpret their mathematical results in the context of the real-world situation. They validate their answers based on real-life reasoning and determine whether the solutions are reasonable and appropriate for the given context.

The effectiveness of scaffolding approaches such as the solution plan was also examined by Nuryadi and Hartono (2021). Their study showed that while the simplified modelling structure reduces the number of stages in the modelling process, it does not eliminate the essential principles of mathematical modelling.

Solution Plan

1. Understand Task
- Carefully read the wording in the word problem
- Form a mental picture of the situation clearly
- Produce a drawing or design

2. Searching Mathematics
- Find the data you require, and if needed, make assumptions
- Search for mathematical relationships (using a geometrical formula or an equation, for instance).

3. Using Mathematics
- Regarding such content, what is your knowledge? Utilise it to solve equations, reverse formulas, or create graphs, for example
- If it fails: Do you know any further mathematical approaches?

4. Explaining Results
- Finalise your results properly
- Link your outcome to the task and check if it is valid
- Write down your concluding response

Figure 3.4: A solution plan
Source: Schukajlow et al. (2015)

The term "scaffolding" was first introduced, according to Pea (2004), by David Wood, Jerome Bruner, and Gail Ross in a 1976 publication. They defined it as a form of support that enables learners to complete tasks or solve problems that would otherwise be beyond their independent capability, particularly without the assistance of a more knowledgeable adult or tutor. In this sense, scaffolding refers to the temporary support provided to learners, either individually or in groups, to help them meet learning challenges and gradually develop independence.

Van de Pol, Volman, and Beishuizen (2010) explain that the concept of scaffolding is derived from the construction industry, where a scaffold is a temporary structure used to support workers during the building or maintenance of tall structures. In education, the term is used metaphorically to describe temporary instructional support that enables learners to accomplish tasks they could not complete on their own.

Such support may take various forms, including modelling, guided questioning, prompts, explanations, or feedback that helps learners progress in their understanding. The following section outlines some key characteristics of scaffolding.

3.6.1 Features of scaffolding

According to Van de Pol et al. (2010), the three main characteristics of scaffolding are contingency, fading, and transfer of responsibility.

(i) **Contingency**: This refers to the support provided by the teacher in response to learners' immediate difficulties at a particular time. It is also described as responsiveness, meaning that the support is timely and adapted to learners' specific needs. It is also differentiated, as learners who experience greater difficulty receive more intensive support than those who are more proficient.

(ii) **Fading**: This involves the gradual withdrawal of support as learners begin to demonstrate improved understanding and increased competence. As shown in Figure 3.6, support is typically high at the beginning of a learning task, but it decreases progressively as learners become more independent.

(iii) **Transfer of responsibility**: This refers to the gradual shift of responsibility from the teacher to the learner. Initially, the teacher takes primary responsibility for guiding the learning process, including introducing concepts and explaining procedures. However, as scaffolding is reduced, learners assume greater responsibility for completing the task independently.

Figure 3.6 illustrates that at the beginning of a learning activity, the teacher provides substantial support and assumes a central role in guiding instruction. As relevant information is introduced and learners begin to understand the task requirements, responsibility gradually shifts from the teacher to the learners, who eventually take full control of the learning process.

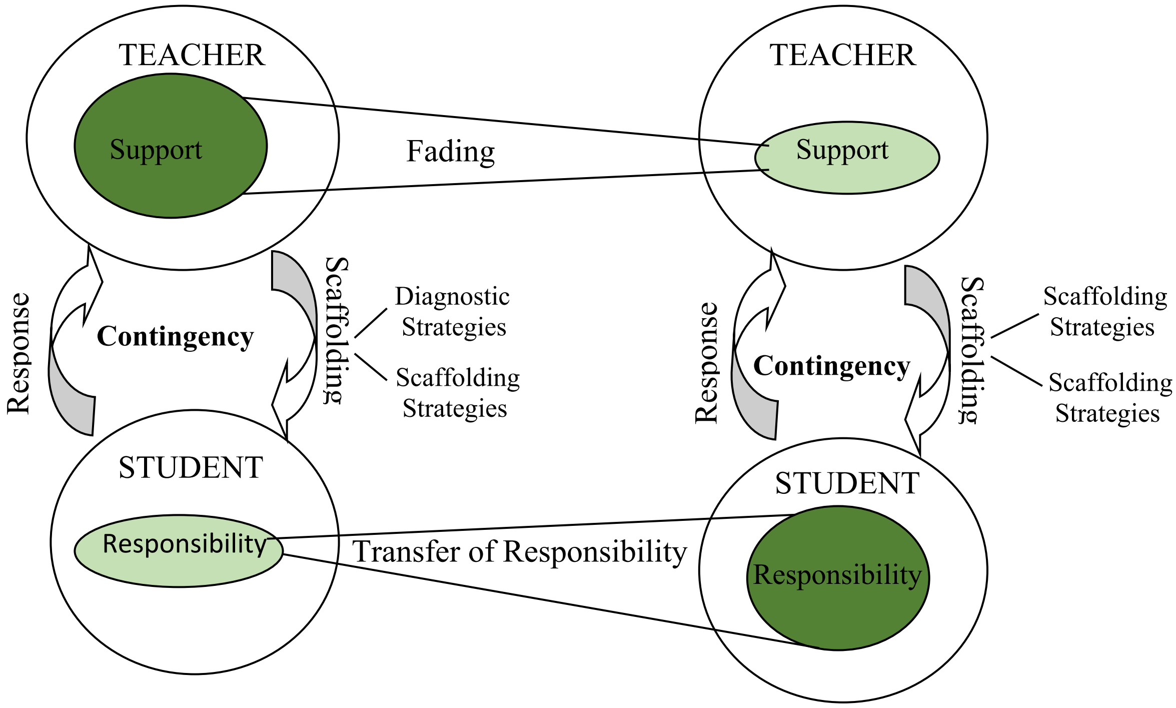


Figure 3.5: Characteristics of scaffolding

Source: Van de Pol et al. (2010)

In teaching and learning, scaffolding can be described as a form of instructional support that teachers provide to guide learners in solving problems and understanding concepts. This support is offered only when learners need it and is adjusted according to their level of understanding. As learners demonstrate increased competence and understanding of the concept being taught, the level of support is gradually reduced until they can work independently. The use of a solution plan simplifies the mathematical modelling process by reducing the number of steps learners are required to follow, thereby making the process more accessible and easier to navigate.

3.7 Conclusion

This chapter discussed four learning theories, namely behaviourism, cognitivism, constructivism, and connectivism. In addition, three epistemological frameworks were presented, namely APOS (Actions, Processes, Objects, and Schema), concept image and concept definition, and multiple representations. These theories and frameworks were used to explain how individuals acquire and develop knowledge. Furthermore, constructivism was discussed as the main theoretical framework, while the mathematical proficiency strands were introduced as an additional supporting framework. The solution plan was also presented as an intervention strategy aimed at improving learners’ competencies in mathematical modelling. The next chapter presents and discusses the research methodology and design used in this study.

CHAPTER FOUR: RESEARCH METHODOLOGY AND DESIGN

4.1 Introduction

This chapter presents a detailed account of the research methodology used in the study, aligned with its objectives. It also explains the philosophical foundations underpinning the research and highlights their importance in understanding the phenomenon under investigation. In addition, the researcher's philosophical stance is outlined, together with the rationale for the selected research approach and methods. The chapter further discusses the study population and sample, sampling techniques, issues of reliability and validity, as well as ethical considerations. Finally, the data collection instruments and procedures employed in the study are described.

4.2 Paradigm definition

According to the Merriam-Webster Dictionary, a paradigm is a conceptual and theoretical framework used within a scientific discipline to develop ideas, formulate laws, and draw conclusions, as well as to design tests that support them. Farooq (2017) describes a paradigm as a Greek-derived term meaning an example, model, or universally accepted perspective that serves as a standard. In this sense, a paradigm represents a structured set of concepts upon which theories and principles for knowledge generation are based. Similarly, Guba and Lincoln (1994) define a paradigm as a set of beliefs that shapes a worldview by explaining the nature of reality, the individual's place within it, and the possible relationships between individuals and the world.

From these perspectives, a paradigm can be understood as a model that explains how knowledge is constructed, interpreted, and investigated. Consequently, a research paradigm must be explicitly stated, as it provides the researcher with a clear orientation for investigating a phenomenon of interest. Rehman and Alharthi (2016) explain that a research paradigm encompasses the beliefs and assumptions regarding the nature of reality, what can be known about that reality, and how such knowledge can be acquired. Put differently, it represents a particular way of understanding and exploring the world. A research paradigm is therefore a theoretical framework consisting of assumptions related to ontology, epistemology, methodology, and methods.

The four components of a research paradigm are described as follows:

Ontology: Ontology concerns the researcher's beliefs about the nature of reality. It addresses questions about what reality is, how it exists, and what can be known about it. Ontological assumptions guide the researcher in determining the type of reality under investigation.

Epistemology: Epistemology refers to the nature and theory of knowledge and the processes through which knowledge is acquired, validated, and communicated. Rehman and Alharthi (2016) describe it as the philosophical study of knowledge and its justification. It also involves questions of objectivity, validity, causality, and generalisability, as noted by Patton (2002, in Rehman and Alharthi, 2016).

Methodology: Methodology refers to the overall strategy or plan that guides the research process. Ellen (1984, in Rehman and Alharthi, 2016) describes it as an informed approach to generating data, while Crotty (1998) views it as the blueprint that guides the selection of appropriate research strategies to achieve the desired outcomes.

Methods: Methods refer to the specific techniques and procedures used to collect and analyse data in relation to the research questions or hypotheses (Crotty, 1998).

Rehman and Alharthi (2016) further argue that it is important to understand research findings within the context of their underlying paradigms, as differing paradigmatic assumptions may lead to interpretations that are not easily transferable across frameworks. An awareness of a researcher's paradigmatic position therefore enhances the interpretation and evaluation of research. Conversely, researchers who are strongly rooted in a single paradigm and unfamiliar with alternative paradigms may struggle to critically engage with studies grounded in different assumptions.

In summary, a research paradigm can be viewed as a set of philosophical assumptions and beliefs that shape how a research problem is conceptualised, investigated, and interpreted. These assumptions are influenced by the researcher's disciplinary orientation and worldview. It is therefore essential for researchers to adopt an appropriate paradigm that provides a coherent framework for their study. The four components of a paradigm—ontology (reality), epistemology (knowledge), methodology (research strategy), and methods (data collection and analysis techniques)—can be systematically conceptualised as interrelated elements of the research process.

A model that explains how knowledge is understood	=	What does reality mean?	+	What is reality or knowledge, and how can I know it?	+	What process might we employ to learn?	+	What kind of tools may we employ to learn?

Paradigm = ontology + epistemology + methodology + methods

Figure 4.1: Mathematical illustration of a paradigm

Since the data were collected from learners' scripts and the analysis aimed to identify the difficulties learners encounter when using mathematical modelling to solve real-world problems, the study was qualitative in nature. It therefore relied on qualitative data. Consequently, an interpretivist paradigm was adopted. The interpretivist paradigm is discussed in detail in the next section.

4.3 Interpretive paradigm

Elster (2007) and Walsham (1995), as cited in Chowdhury (2014), describe interpretivism as an approach that focuses on the essential nature of individuals' lived experiences and their participation in cultural and social contexts. From this perspective, interpretivism refers to research approaches grounded in the view that human understanding of reality is constructed through social interaction. As a result, it generally does not rely on the methods of the natural sciences. Instead, interpretivist researchers seek to understand and explain human actions, behaviours, and social interactions within their specific contexts.

Interpretivism, also referred to as naturalistic or anti-positivist inquiry, is regarded as a more subjective approach to understanding social phenomena. Interpretivists hold that each situation is unique and can only be understood within its specific context. Social reality is viewed as a human construction shaped by shared meanings, language, and consciousness, rather than an objective, fixed truth. Accordingly, interpretivists reject the notion of an externally determined reality and instead argue that human behaviour is best understood within its social environment (Nel, 2019).

Nel (2019) further explains that interpretivism is concerned with how individuals express themselves, how they interpret experiences, and how meaning is assigned to events and phenomena. It emphasises the meanings that individuals, groups, and societies attach to their lived experiences. In interpretive research, patterns, themes, and concepts are expected to emerge inductively from the data. The researcher therefore seeks to understand real-life situations from the participants' perspectives, with the human mind viewed as actively constructing meaning. Interpretive inquiry is thus exploratory in nature and is best suited to qualitative research designs that aim to understand complex social phenomena in depth.

Photongsunan (2010) notes that interpretivist research differs significantly from positivist approaches. Whereas positivist research typically begins with hypotheses to be tested, interpretivist research is grounded in qualitative data from which explanations and meanings emerge. Furthermore, interpretivist studies are characterised using open-ended research questions, flexible designs, and relatively small samples. The use of small samples is intentional, as the focus is not on generalisation but on gaining rich, in-depth understanding of participants' meanings within a specific context.

According to Klein and Myers (1999), interpretivist research is rooted in the idea that knowledge is socially constructed through processes such as language, shared meanings, and collective understanding. They further argue that interpretivist inquiry recognises a close relationship between the researcher and the phenomenon being studied, including the contextual constraints that shape the research process. Methodologically, interpretivist research does not predefine variables or test hypotheses; instead, it aims to develop a deeper understanding of the social context and the reciprocal relationship between phenomena and their environments.

Several advantages are associated with the interpretive paradigm. First, its use of multiple perspectives enables researchers to understand phenomena holistically within their social contexts. Second, the use of interactive methods such as interviews allows researchers to access participants' beliefs, values, experiences, and emotions that may not be observable directly

(Pham, 2018). Kivunja and Kuyini (2017) further outline key characteristics of interpretivism in comparison with positivist and critical paradigms, which are summarised in Table 4.

Table 4.1: Characteristics of the Interpretive Paradigm

Component of a paradigm	**Interpretivist Paradigm**
Epistemology (how knowledge is acquired)	Knowledge is constructed through interactions with others.
Ontology (nature of reality)	Reality is subjective, diverse (multiple) and socially constructed.
Methodology (procedures used to acquire knowledge)	They use meaning oriented methodologies, such as s interviewing or participant observation
Methods (data to be gathered)	Qualitative data

4.4 Applicability or relevancy of the interpretive paradigm to this study

This study analysed learners' scripts from a test designed for the research, completed while they worked in groups. According to Nel (2019), interpretivism emphasises individuals' lived experiences, focusing on how they construct the social world and interact with others. From this perspective, meaning is continually constructed and refined through social interaction. The adoption of the interpretivist paradigm therefore informed the use of a qualitative research design, as interpretivism is closely aligned with qualitative inquiry. A defining feature of qualitative research is that data are collected in participants' natural settings and are described in words rather than numbers. In this study, the data captured what learners could and could not do in relation to mathematical modelling, rather than statistical measurements. In other words, the analysis focused on qualitative distinctions based on learners' demonstrated competencies and difficulties.

4.5 Research context

The current Mathematics Curriculum in South Africa identifies mathematical modelling as an essential component of the curriculum. This implies that the teaching and learning of Mathematics should incorporate modelling approaches in which real-life problems are integrated across all topics wherever appropriate. The problems presented to learners should be authentic and non-contrived, and should, where possible, address issues related to economic, political, health, and environmental contexts (DBE, 2011). The study was conducted in three schools within the Pongola Circuit. It investigated the mathematical modelling competencies of Grade 11 learners as they solved real-life problems involving area and perimeter. Data collection took place during the period in which strict COVID-19 regulations were in effect, including mandatory mask-wearing and maintaining a physical distance of at least one metre between learners. These restrictions necessitated that the test be administered either individually or in small groups in order to comply with safety protocols. In addition, the letters of consent sent to parents explicitly indicated that all COVID-19 regulations would be strictly adhered to throughout the data collection process. The researcher remained fully compliant with these ethical commitments and did not deviate from the assurances provided to participants and their parents.

4.6 Research Approach and Design

The study adopted a qualitative approach to address the research question. It investigated learners' competencies in mathematical modelling when solving real-life problems; therefore, a phenomenological research approach was deemed appropriate. Creswell (2009) explains that phenomenological research is used when a researcher aims to describe a phenomenon while also capturing the essence of participants' lived experiences of that phenomenon. In this study, I described in written form what Grade 11 learners did correctly and incorrectly when solving real-life problems using a mathematical modelling approach.

According to Nassaji (2015), descriptive research has become increasingly common in education, particularly within qualitative studies, due to the recognition that teaching and learning are complex processes. To understand this complexity, it is not sufficient to focus only on how learning occurs or the factors that influence it; rather, there is a need for an in-depth exploration of learners' behaviours, experiences, and interpretations. Nassaji (2015) further notes that although the terms "descriptive research" and "qualitative research" are sometimes used interchangeably, they are not identical. However, both approaches are well suited to classroom-based research conducted in natural settings. A key shared feature is the use of naturalistic data, where teaching and learning are studied in their authentic environments.

In light of these characteristics, the study can be described as both qualitative and descriptive in nature. This is because the data were collected in a regular classroom setting under natural conditions. Data were generated through a researcher-designed test administered to learners, and learners' scripts were subsequently analysed to determine their competencies in applying mathematical modelling to solve problems. In addition, learners' mathematical proficiency was analysed using Kilpatrick's five strands. As indicated in Chapter Three, both mathematical modelling competencies and mathematical proficiency served as the theoretical frameworks guiding this study.

4.6.1 Choosing qualitative research for this study

The qualitative design was adopted for several reasons. Firstly, the study was conducted in the learners' natural environments, namely their schools, which aligns with the principles of qualitative inquiry. Secondly, the data were derived from learners' scripts following a researcher-administered test, which were then systematically analysed to identify emerging patterns. Thirdly, the findings were presented in descriptive form, using words rather than statistical or numerical measures. Fourthly, the qualitative approach was considered appropriate because the study sought to gain an in-depth understanding of Grade 11 learners' mathematical modelling competencies. In addition, the sample size was not intended for statistical generalisation. Creswell (2013) notes that qualitative research does not prescribe a fixed sample size; instead, data collection continues until data saturation is reached, which occurs when no new information emerges from participants or data sources. Given the absence of a predetermined sample size in qualitative research, all 72 learners who volunteered to participate were included in the study. Overall, the qualitative approach was employed to explore and explain learners' competencies in mathematical modelling when solving real-life problems. The next section outlines the characteristics of qualitative inquiry.

4.6.2 Characteristics of a qualitative research design

To ensure the quality of a study focused on understanding the meanings individuals assign to social or human situations, Creswell (2013) argues that researchers should begin with clearly stated philosophical assumptions and the use of interpretive conceptual frameworks. Qualitative researchers typically employ an emergent design of inquiry, which involves collecting data in natural settings while being attentive to the people, places, and contexts under study. Data are analysed both inductively and, where appropriate, deductively to generate patterns, categories, or themes. The final report incorporates participants' perspectives, researcher reflexivity, rich description and interpretation of the phenomenon, as well as its implications for understanding or potential change.

The key characteristics of qualitative research outlined above are elaborated below:

Natural setting: In qualitative research, data are collected in the natural environments where participants experience the phenomenon under study. This contrasts with survey research, where data may be collected through distributed instruments. Instead, qualitative researchers engage directly with participants and observe their behaviour within authentic contexts. In this study, data were collected in learners' normal school environments, where participants completed the test under everyday classroom conditions.

Researcher as key instrument: In qualitative inquiry, the researcher is the primary instrument of data collection, often gathering data through document analysis, observation, and interviews. When tools are used, they are typically developed by the researcher and include open-ended tasks or questions. In this study, I developed the test instrument used for data collection and personally analysed learners' scripts to assess their mathematical modelling competencies.

Multiple methods: Qualitative research often draws on multiple sources of data, such as documents, observations, and interviews, to enhance depth and richness of understanding. Although this study primarily used a test as the data collection instrument, richness was ensured through the inclusion of participants from three different schools, allowing for variation in responses and contexts.

Inductive data analysis: Qualitative researchers build patterns, categories, and themes by moving from specific observations to broader interpretations. This involves organising data into meaningful units and refining themes through continuous engagement with the dataset. In some cases, deductive reasoning is also used to verify emerging themes against existing frameworks. In this study, learners' responses were analysed to identify patterns in their application of mathematical modelling processes.

Participants' meanings: Creswell (2013) emphasises that qualitative research prioritises participants' interpretations of a phenomenon rather than the researcher's assumptions. This involves capturing multiple perspectives and acknowledging diverse viewpoints. Accordingly, the themes in this study reflected learners' varied understandings and approaches to mathematical modelling tasks.

Emergent design: Qualitative research follows an evolving design, meaning that aspects such as research questions, sampling, and data collection strategies may be refined during the study as new insights emerge. The primary goal is to obtain the most meaningful and relevant data from participants using appropriate methods.

Reflexivity: Reflexivity requires researchers to acknowledge their background, assumptions, and potential influence on the research process. In this study, I maintained reflexivity by focusing on learners' actual responses without imposing my personal experiences of Mathematics classrooms. Attention was given to how learners performed each step of the mathematical modelling process, rather than subjective interpretation.

Holistic account: Qualitative research aims to present a comprehensive understanding of the phenomenon by integrating multiple perspectives and contextual factors. In this study, this was achieved by exploring learners' understandings of mathematical modelling, as well as the challenges and competencies demonstrated in solving real-life problems.

Overall, these characteristics guided the qualitative design and analysis used in this study.

4.7 Researcher's own experience

I am an experienced Mathematics educator with nineteen years of teaching experience in Grades 8 to 12, as well as sixteen years of service as a Mathematics Subject Advisor in the

Mpumalanga Province. In this role, I support teaching and learning by observing Mathematics lessons in classrooms and providing professional guidance to teachers. Over the years, I have developed key professional competencies, including instructional leadership, classroom observation skills, and the facilitation of continuous professional development.

As part of my responsibilities, I regularly conduct classroom visits to monitor teaching practices and support teachers in improving their instructional approaches. Through these observations, I have noted that many lessons remain predominantly teacher-centred, with teachers often positioned at the front of the classroom explaining procedures while learners follow prescribed steps to complete exercises. This practice highlights the limited use of learner-centred approaches in many classrooms.

Together with colleagues, I have facilitated workshops aimed at supporting teachers to adopt more interactive and learner-centred pedagogies that promote active learning. During these workshops and school-based support visits, teachers are guided on how to implement classroom arrangements that encourage collaboration, such as pair and group work, with the aim of improving learners' achievement in Mathematics. Kunsch et al. (2007) note that peer tutoring in Mathematics can significantly enhance learners' academic progress. While some improvement has been observed, with a number of teachers gradually adopting more innovative teaching strategies, a considerable proportion still rely on traditional methods.

I have also observed that classroom furniture is often arranged in a traditional format, with desks placed in rows and all learners facing the chalkboard, where the teacher typically stands. Given my familiarity with these conditions in schools within my jurisdiction, I considered it important to conduct this study outside my province to minimise bias and avoid the influence of prior knowledge of the schools under my supervision.

It is evident that in many Mathematics classrooms, learners are primarily engaged in routine exercises based on procedures demonstrated by teachers. Instruction often emphasises repetition and memorisation of steps, with success measured by learners' ability to reproduce taught methods accurately. This approach limits opportunities for creativity and problem-solving, as learners become overly dependent on teacher-directed strategies. In some cases, teaching is also strongly guided by textbook procedures and teacher guides, which further constrains learners' opportunities for innovation and independent thinking.

4.8 Research Site

Purposive sampling was used to select the schools because they were considered most suitable for providing relevant data to achieve the objectives of the study. Etikan, Musa and Alkassim (2016) describe purposive sampling as a technique in which the researcher deliberately selects participants based on specific characteristics they possess. Similarly, Creswell and Poth (2017) explain purposive sampling as a deliberate selection of settings, individuals, or materials that have attributes relevant to the research problem and are best positioned to provide meaningful responses to the phenomenon under investigation.

The selected schools were chosen based on their consistent performance in Mathematics in the Grade 12 end-of-year examinations over several years. In the preceding three years, each of these schools achieved an average pass rate above 60%. These high-performing schools were therefore considered appropriate for generating rich and relevant data on learners' competencies in mathematical modelling.

In addition, the schools were selected due to their geographical proximity to my place of residence. They are located near the border between KwaZulu-Natal and Mpumalanga Province, making them easily accessible. This accessibility allowed for repeated visits within a relatively short period and reduced travel-related constraints. For example, one of the schools is approximately 50 kilometres away.

Importantly, I deliberately avoided conducting the study in schools within Mpumalanga Province, where I serve as a Mathematics Subject Advisor, to minimise potential ethical and positional bias associated with researching within my own professional context. Selecting schools outside my immediate work environment also helped reduce the risk of undue influence and ensured greater objectivity. In contrast, choosing schools farther away would have resulted in increased transportation costs and extended travel time.

4.9. Population and sample

According to Banerjee and Chaudhury (2010), a population refers to the entire set of individuals or elements from which information is to be collected. They further explain that a statistical population may include not only people, but also other units such as measurements, outcomes, events, or biological indicators (e.g., weight, height, or haemoglobin levels), if it is clearly defined in terms of inclusion and exclusion criteria. In research, the population is determined by the research questions and objectives, and is often delimited by factors such as location, age, gender, or occupation. Therefore, it is essential that the population is clearly specified using appropriate inclusion and exclusion criteria.

Similarly, McMillan and Schumacher (2010) describe a population as a collection of cases—such as individuals, events, or objects—that meet specific criteria and to which the findings of a study are intended to be generalised. In this sense, a population represents the full set of elements from which research conclusions are drawn. These elements share identifiable characteristics that are relevant to the focus of the study. Shukla (2020) also defines population as the total group of units to which the results of a study can be applied. It includes all elements possessing the characteristics under investigation, and from which broader generalisations may be made.

In contrast, a sample refers to a subset of the population selected for the purpose of the study. McMillan and Schumacher (2010) define a sample as a portion of the population that shares the relevant characteristics of interest to the researcher. Mohsin (2016) similarly describes a sample as a smaller group of participants drawn from the population for research purposes, with individuals in the sample referred to as participants. Mugo (2002) adds that a sample is a clearly defined subset of a population whose characteristics are studied to make inferences about the whole population.

4.9.1 Population of the study

In relation to this study, the population comprised all Grade 11 learners from the three selected schools. As indicated in Section 4.9, these schools were purposively selected. If all Grade 11 learners in these schools had agreed to participate, a sampling procedure would have been used to select the study sample. However, the study was conducted during a period when COVID-19 regulations were still in effect in some institutions. As a result, it was not feasible to accommodate large numbers of participants in a single setting due to concerns about the potential spread of infection. In addition, the permission letters submitted to schools and parents clearly indicated that all COVID-19 safety protocols would be strictly observed to ensure the safety of learners. Ultimately, because only a limited number of learners volunteered to participate, all those who consented were included in the study.

4.9.2 Sample of the study

Initially, I intended to include 30 learners from each of the three schools, resulting in a total planned sample of 90 participants. To achieve this, stratified sampling would have been employed had a larger number of learners been willing to participate. Acharya, Prakash, Saxena and Nigam (2013) explain that stratified sampling involves dividing a population into subgroups (strata) that share common characteristics such as age, gender, income, or education level, after which a random sample is drawn from each stratum.

However, the number of learners who volunteered to participate in each school was fewer than 30. Consequently, all learners who consented to participate were included in the study. In effect, the sampling approach aligned more closely with volunteer sampling. Casteel and Bridier (2021) describe volunteer sampling as a technique in which participants are selected based on their willingness to take part in the study rather than through random selection.

In total, 75 learners participated in the study. This sample size was considered adequate for a qualitative inquiry. According to Alase (2017), sample sizes in phenomenological studies typically range between two and twenty-five participants, depending on the homogeneity of the group under investigation. The primary aim of interpretive phenomenological research is to develop a deeper understanding of the phenomenon under study rather than to achieve statistical generalisation.

Although the final sample exceeded the typical range suggested for phenomenological studies, it was retained because it provided richer and more diverse responses. Reducing the sample through further sampling procedures was not considered necessary, as the larger group enabled participants to be exposed to different approaches to problem-solving beyond routine procedures, such as reliance on Differential Calculus. In addition, the larger number of participants allowed for a broader representation of learners' competencies in mathematical modelling.

4.10 Data Collection Technique

Grade 11 learners' competencies in mathematical modelling were assessed using a brief test worth 25 marks. The test was administered to address the study's research questions and objectives. The same instrument was used across the three schools on different days. In the first two schools, the completed scripts were collected immediately after the test to prevent learners from sharing the content with the school that had not yet written. After all three schools had completed the test, copies of the instrument were provided to learners so that they could retain them for use in Grade 12 as an introductory resource for the application of Differential Calculus. Although the test was initially designed to be completed within 30 minutes, the duration was extended to 45 minutes after it was observed that learners at the first school were unable to complete it within the allocated time. Each learner completed the test individually and was assigned a code name by their teachers in place of their real names and school identifiers to ensure confidentiality and protect their identities. This study adopted a phenomenological research design to explore learners' competencies in mathematical modelling. Lester (1999) explains that phenomenological research focuses on how individuals experience and interpret phenomena, using qualitative approaches to gain in-depth understanding. This framework emphasises participants' perspectives and lived experiences. Although the test was initially intended to be administered in groups, the format was changed to individual administration due to COVID-19 safety protocols. All tests were conducted in the presence of the researcher, and the completed scripts were subsequently marked and analysed. The findings of this analysis are presented in the following chapter.

4.11 Trustworthiness of the study

Like any other researcher, I had to ensure that readers of this study develop trust and confidence in its findings and conclusions. It was therefore essential to ensure that the results were grounded in the data collected from participants and were not influenced by my personal assumptions or biases. Connelly (2016) defines trustworthiness in research as the degree of confidence in the data analysis, interpretation, and the procedures used to ensure the quality of a study. In this regard, a researcher must adhere to systematic procedures that demonstrate that the study is credible and worthy of scholarly consideration.

To ensure the trustworthiness of this study, credibility was enhanced by providing a transparent account of the research design, the mathematical modelling assessment, and the procedures used in analysing learners' responses. Transferability was strengthened by clearly describing the grade level, test design, and subject focus, allowing readers to judge the applicability of the findings in similar contexts. Dependability was ensured through consistency in administering the same test across the three schools, as well as through careful documentation of procedural adjustments, such as the extension of the allocated time. Confirmability was achieved by minimising researcher bias and grounding interpretations strictly in learners' actual performance.

All research processes and methods were clearly outlined. These details were provided in Chapter Four, where the research paradigm, population, sample, sampling techniques, and participant characteristics were described. The data collection procedures were also explained in this chapter, while the data analysis procedures were presented in Chapter Six. Yilmaz (2013) notes that quantitative and qualitative researchers differ in how they establish research quality; quantitative studies typically use concepts such as reliability, validity, and objectivity, whereas qualitative research is assessed using trustworthiness criteria such as credibility, transferability, dependability, and confirmability.

Since this study adopted a qualitative approach, these trustworthiness criteria were used to evaluate its rigour. According to Lincoln and Guba (1985), the integrity of qualitative research is established through credibility, transferability, dependability, and confirmability. Each of these components is discussed in detail below.

4.11.1 Credibility

Credibility, according to Lincoln and Guba (1985), refers to the extent to which research findings are trustworthy and reflect the authentic experiences or realities of participants. They propose several strategies for enhancing credibility, including prolonged engagement, persistent observation, negative case analysis, member checking, referential adequacy, peer debriefing, and triangulation. Each of these is explained below in relation to this study.

Prolonged engagement: This involves spending sufficient time in the field to build rapport with participants and gain a deep understanding of the research context. In this study, prolonged engagement was limited because I sought to minimise disruption to teaching and learning. I spent approximately 30 minutes at the first school and an additional 15 minutes at the other two schools. Extending the time further would have interfered with normal instructional activities.

Persistent observation: This strategy focuses on identifying and closely examining the most relevant aspects of the phenomenon under investigation. In this study, observation was not used as a primary data collection method; therefore, persistent observation in the conventional sense was not applicable. However, a form of extended engagement occurred during data analysis, where each learner's script was carefully and systematically examined to develop a deeper understanding of learners' competencies in mathematical modelling.

Peer debriefing: Peer debriefing involves engaging with impartial colleagues to critically review the research process and interpretations, thereby uncovering possible biases or assumptions (Lee et al., 2023). In this study, credibility was strengthened through peer debriefing during a virtual workshop held on 29–30 April 2021. During this process, peers and lecturers critically interrogated aspects of the study, leading to important revisions. For example, initially planned interviews intended to explore learners' attitudes towards mathematical modelling were removed as they were deemed misaligned with the study's focus. Similarly, a pre- and post-test design was replaced with a single-test approach, as the study was not aimed at measuring intervention effects. These critiques significantly improved the clarity and focus of the study, thereby enhancing its credibility.

Triangulation: The study employed data triangulation, which involves using multiple data sources to enhance understanding of a phenomenon. Lincoln and Guba (1985) note that in qualitative research, triangulation is primarily used to enrich and deepen understanding rather than merely to verify findings. In this study, triangulation was achieved through the use of data from three different schools. Further discussion of triangulation is provided under the criterion of confirmability.

Negative case analysis: This refers to the deliberate search for and examination of data that contradict emerging patterns or explanations. In this study, negative case analysis was not applicable because the focus was on assessing learners' competencies in mathematical modelling through a structured test. The outcomes were interpreted in terms of levels of competence rather than exploring divergent perspectives or narratives.

Referential adequacy: This strategy involves setting aside a portion of the data for later comparison with emerging findings to verify interpretations. In this study, three scripts from each school (a total of nine scripts) were withheld from initial analysis. The remaining scripts were analysed first to generate findings. Afterward, the reserved scripts were analysed and compared with the initial results to confirm and validate the consistency of the interpretations.

Member checking: Member checking involves returning data, interpretations, or findings to participants for verification to enhance accuracy and credibility. In this study, member checking was not conducted. This decision was made because triangulation across three schools was considered sufficient to ensure data quality, and because the data consisted of written test scripts rather than interview responses. Additionally, it was considered inappropriate to require learners to justify their answers retrospectively, as this could have caused discomfort or embarrassment. The use of multiple schools as data sources was deemed adequate for strengthening the credibility of the findings.

4.11.2 Transferability

Transferability, as defined by Lincoln and Guba (1985), refers to the extent to which the findings of a study can be applied or extended to other contexts or settings. Frey (2008) describes transferability as the degree to which qualitative results may be relevant to different participants and environments, while Anney (2014) further emphasises that it concerns the applicability of findings across varied contexts and populations. For a study to meet the criterion of transferability, the researcher must provide a rich and detailed description of the research context, processes, and participant selection, enabling readers to determine the extent to which the findings may be applicable in other settings. In this study, transferability was ensured by providing a comprehensive account of the research procedures, including participant selection, the population under investigation, sampling strategies, sample size, data collection methods, and detailed descriptions of data analysis and interpretation.

4.11.3 Dependability

Dependability in research, as defined by Lincoln and Guba (1985), refers to the consistency and stability of research findings over time. Frey (2008) describes dependability as the extent to which findings remain stable, while Korstjens and Moser (2018) emphasise consistency in the research process as a key indicator of dependability. To ensure dependability in qualitative research, it is important that the research process is clearly documented, systematically applied, and aligned with accepted analytical procedures, while also minimising researcher bias in data interpretation. This is often achieved through the development of an audit trail, which provides a detailed record of all research decisions, procedures, and data management processes. In this study, dependability was ensured using clearly documented procedures for sampling, data collection, and data analysis, as outlined in the relevant chapters. In addition, extracts from learners' scripts were systematically analysed to assess their competencies in mathematical modelling, providing a transparent basis for interpreting the findings. Ethical data handling procedures were also consistently applied throughout the study.

4.11.4 Confirmability

Confirmability, according to Lincoln and Guba (1985), refers to the extent to which research findings are shaped by the participants' responses and data rather than the researcher's biases, motivations, or assumptions. They identify several strategies for ensuring confirmability, including triangulation, audit trail, reflexivity, and confirmability audit, each of which is discussed below.

Confirmability audit: A confirmability audit involves an independent examination of the research process and findings by an external reviewer who is not directly involved in the study. Its primary purpose is to determine whether the conclusions, interpretations, and recommendations are supported by evidence from the data. This process allows an external party to critically evaluate the methodological decisions and verify the trustworthiness of the outcomes.

Audit trail: An audit trail provides a systematic and transparent record of all stages of the research process, from data collection through to analysis and reporting. In this study, an audit trail was maintained through detailed documentation of procedures, including data collection and data analysis processes. Extracts from learners' scripts were included to demonstrate how findings were derived, with each extract analysed and discussed in relation to the emerging results. In addition, separate analysis tables were developed for each school, where all observations and interpretations were systematically recorded.

Triangulation: Triangulation refers to the use of multiple data sources or approaches to enhance understanding and strengthen the credibility of findings. Patton (1999) notes that practical constraints such as time limitations, budgetary restrictions, and insufficient training may affect the extent of triangulation in a study. Nevertheless, studies that draw on multiple data sources or methods generally reduce the likelihood of bias compared to those relying on a single source. In qualitative research, triangulation often involves combining different forms of data to support cross-validation of findings.

Reflexivity: Lincoln and Guba (1985) describe reflexivity as a continuous, systematic reflection on how the researcher's position, assumptions, and actions may influence the research process and outcomes. They argue that researchers may approach the same phenomenon from different perspectives, leading to multiple valid interpretations. Reflexivity therefore requires the researcher to remain aware of potential bias throughout the research process and to document reflective observations.

In this study, reflexivity was enhanced using a structured rubric to assess each learner's script, which helped to minimise subjective interpretation. Three analysis tables (one per school) were used to systematically record findings from each script. The rubric guided the analysis by ensuring consistency across key criteria, including whether learners:

1. understood the problem by identifying relationships between variables and making appropriate assumptions;
2. formulated a correct mathematical model representing the situation;
3. solved the model correctly and applied appropriate mathematical procedures;
4. interpreted their solutions meaningfully; and
5. verified the validity of their results.

The use of this structured rubric and systematic recording helped ensure that interpretations were grounded in observable evidence rather than opinion. Consistency was maintained across all scripts; for example, when a learner failed to formulate an appropriate model, this was recorded uniformly across all cases. Importantly, the nature of the data (written scripts) and the structured analytical approach limited the influence of personal emotions or beliefs on interpretation. This would likely have been more challenging if interviews had been used, as interview data may be interpreted differently depending on the researcher's perspective.

4.12 Research Ethics

Data collection for this study commenced only after approval had been granted by the University of South Africa College of Education Ethics Review Committee. This approval ensured that the study complied with key ethical principles, including (i) informed consent, (ii) anonymity and confidentiality, (iii) beneficence (avoiding harm to participants), and (iv) respect for participants' privacy. Each of these ethical considerations is discussed in detail below.

4.12.1 Informed consent and privacy

In this study, I adhered to all formal procedures required for conducting research in school settings. To obtain permission to conduct the study at the three purposefully selected schools, I applied to the Head of the KwaZulu-Natal Department of Education (see Appendix C). Approval was granted (see Appendix D). In addition, I requested permission from the school principals and Grade 11 Mathematics teachers through formal written letters (see Appendices E and F). Sample responses from principals and teachers are included as Appendices G and H, respectively.

Since the study involved learners who were minors (below 18 years of age), parental consent was also required. Accordingly, letters were sent to parents requesting permission for their children to participate in the study. These letters outlined the purpose of the study, possible benefits, the researcher's and supervisor's contact details, and the institutional affiliation of the researcher. They also clearly stated that participation was voluntary, that learners could withdraw at any time without penalty, and that their children would not be exposed to any form of harm. Issues of confidentiality, anonymity, and the expected role of learners in the study were also clearly explained (see Appendix I).

In addition, assent was obtained directly from the learners. Assent letters explained the purpose of the study, the procedures to be followed, the data collection instruments, and the voluntary nature of participation, including the right to withdraw at any stage should they feel uncomfortable. Learners who agreed to participate signed an assent form, which was also countersigned by a witness aged 18 years or older (see Appendix J).

Ethical issues related to privacy were addressed throughout the consent process, particularly by clearly informing both parents and learners that participation could be withdrawn at any time. Although withdrawal was allowed, none of the learners chose to discontinue their participation. This may be attributed to several factors. Firstly, no personal or identifying information was collected, as learners completed the test using code names instead of their real names. Secondly, the purpose and procedures of the study were clearly explained to learners prior to participation. Thirdly, I visited the schools before data collection to introduce myself to the learners, which helped establish familiarity and reduce any perception of intrusion into their privacy.

4.12.2 Respect anonymity and confidentiality

Anonymity and confidentiality of participants in this study were ensured by concealing all identifying information during data collection, analysis, and reporting. Participants were instructed not to include any personal details on their scripts, such as their names or the names of their schools. Throughout the analysis process, all identifying information remained masked to protect participant identity. All research documents, including learners' test scripts and consent forms containing personal details, were securely stored in a locked and inaccessible cabinet, to which only the researcher had access. These measures ensured that the data remained protected throughout the study. In accordance with the University of South Africa College of Education Ethics Review Committee requirements, all collected data will be securely destroyed after the stipulated retention period.

4.12.3 Beneficence or kindness – Do not hurt

In relation to this study, I ensured that learners were protected from any possible harm or risks. The most significant risk during the period of data collection was the COVID-19 pandemic. To minimise the risk of infection, I adjusted the original data collection plan. Although learners were initially expected to work in groups of three, this was changed to individual work to maintain physical distancing requirements. In addition, all participants were required to wear face masks, and desks and chairs were sanitised before use. I did not take learners' temperatures in the classroom, as screening had already been conducted at the school entrances. However, each classroom was equipped with wall-mounted sanitiser dispensers, and learners were instructed to sanitise their hands before commencing the test.

Apart from COVID-19-related precautions, no other significant risks or potential harm were associated with the study. The nature of the study itself was non-invasive, as it was conducted within the normal classroom environment and involved only a written mathematical modelling test. Therefore, there were no foreseeable physical, psychological, or social risks to participants.

4.12.4 Conflict of interest

Since I was working in the Mpumalanga Province as a Subject Advisor for Mathematics, I chose to conduct this study in the KwaZulu-Natal Province to avoid any potential conflict of interest. I regard all schools that I support in my advisory role as part of my professional workplace. Based on the definition of conflict of interest provided in the Merriam-Webster Dictionary in the preceding section, conducting the study within these schools could have created a situation of competing interests. I also wished to avoid any perception that my position as a subject advisor might influence how participants or schools responded to the study, potentially resulting in preferential treatment that would not ordinarily be extended to individuals in different positions. Furthermore, conducting the study within my workplace context could have introduced bias, as my prior knowledge and experiences in these schools might have influenced my interpretations and shaped preconceived expectations about the findings. By selecting schools outside my professional jurisdiction, I aimed to ensure greater objectivity and reduce the risk of bias in the research process.

4.13 Conclusion

This chapter provided a comprehensive discussion of the research paradigm, including the suitability of the interpretive paradigm for the study, as well as the research design and methodology employed. It also outlined the research site, population, sample, and data collection procedures in detail. Furthermore, the chapter addressed the ethical considerations and the measures taken to ensure the trustworthiness and reliability of the study. The next chapter presents the data analysis and interpretation of the findings.

CHAPTER FIVE: DATA ANALYSIS AND INTERPRETATIONS

5.1 Introduction

The purpose of this study was to determine Grade 11 learners' competencies in mathematical modelling during problem-solving. To achieve this aim, learners completed a short individual test consisting of problems requiring them to determine maximum area, with a focus on scenarios involving the perimeter of a field or the length of fencing required. The test was developed by the researcher and included questions related to the area and perimeter of rectangular figures.

The findings presented in this chapter are organised around the main research question and its accompanying sub-questions.

Main research question:

What are the competencies of Grade Eleven learners in mathematical modelling when resolving real world problems?

Sub-questions:

- What is the understanding of the usage of mathematical modelling approach in solving real world problems by Grade 11 learners?
- What are the successes of Grade Eleven learners in solving non-routine realistic and practical problems using the mathematical modelling approach?
- What are learners' challenges in employing mathematical modelling to solve lifelike non-routine problems in Mathematics?
- Why do learners encounter barriers in formulating the real-life situation into a mathematical model?
- Are learners able to apply mathematical modelling procedures?

Constructivism informed the five stages of the mathematical modelling cycle used in this study, namely: understanding the problem, formulating the mathematical model, solving the problem, interpreting the results, and validating the solution and model. From a constructivist perspective, learners are expected to actively construct their own understanding rather than passively receive information.

In the first stage, understanding the problem, learners needed to actively engage with the context to make sense of the problem situation and develop an appropriate mathematical representation. This required them to make valid assumptions and establish relationships between variables. Such understanding could not be achieved through passive learning.

The second stage involved the formulation of a mathematical model, where learners were required to select appropriate mathematical symbols and construct an equation or expression that represented the real-world situation. Instead of being provided with specific formulas, learners had to determine suitable methods independently, thereby encouraging creativity and conceptual thinking.

The third stage required learners to actively solve the formulated mathematical problem, applying relevant mathematical procedures to obtain a solution. This was followed by the interpretation stage, where learners translated their mathematical results back into the original real-world context to make meaningful sense of their answers.

The final stage, validation, required learners to verify whether their solutions were reasonable and appropriate in relation to the problem context. Where results were found to be inappropriate, learners were expected to revise and refine their models accordingly. Across all stages of the modelling cycle, learners were required to actively engage with the tasks and draw on prior knowledge. For example, knowledge of functions was necessary to determine coordinates of maximum turning points in optimisation-type problems.

5.2 Analysis of Qualitative Data

I examined learners' written responses from the test used to generate data. The scripts were treated as documents and analysed using content analysis as the primary research technique. Wallen and Fraenkel (2001) define content analysis as the process of systematically examining textual or visual material contained in documents. Patton (2002) further describes it as a qualitative data reduction and sense-making process that involves identifying key patterns, consistencies, and meanings within large volumes of qualitative data. Similarly, Cavanagh (1997) views content analysis as a flexible technique for analysing textual data, while Gheyle et al. (2017) describe it as a method used to interpret unstructured messages such as texts, images, symbols, or audio data.

The study involved 75 Grade 11 learners from three schools in the Pongola Circuit in KwaZulu-Natal Province. The sample comprised learners who were studying Mathematics and who voluntarily agreed to participate in the study. The schools were selected based on their proximity to the researcher's place of residence in Mpumalanga Province, as they are located near the provincial boundary between Mpumalanga and KwaZulu-Natal.

In addition, these schools were considered appropriate for the study because they had consistently demonstrated strong performance in Grade 12 Mathematics over the preceding five years. This performance profile informed the researcher's expectation that participating learners would be able to provide relevant and meaningful data for the study.

5.2.1 Test used in the study

1. Mr Nene wants to plant seedlings in his rectangular garden with an area of 54 m^2. The length of the garden is longer than its width by 3m. What are the dimensions of his garden? (6)

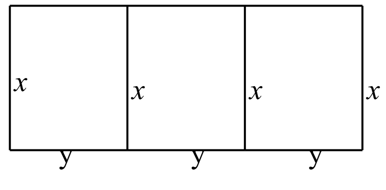
2. To protect his vegetables, a farmer buys 32m of fencing for his rectangular garden. Determine the dimensions of the vegetable garden if it should cover a maximum area (6)

3. A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 72m of wire fencing:

3.1. Determine the dimensions that would create a garden of maximum area. (5)

3.2. What is the maximum area? (1)

4. A farmer has 600m of fence to create a rectangular pasture which has to be divided into three camps (as shown in the sketch below) all with the same dimensions. What is the maximum area that can be enclosed with this fence? (7)



TOTAL: 25 MARKS

The test described above was used to generate data for this study. Although the initial plan was for learners to work in groups, this was revised to individual completion due to the COVID-19 pandemic. Conducting group work would have compromised social distancing requirements and increased health risks for participants. The task required learners to apply knowledge of formulas for calculating the perimeter and area of rectangular figures, as well as determining the coordinates of the maximum turning point of a quadratic function. In essence, learners were expected to draw on their prior knowledge from Algebra, Measurement, and Functions to solve the problems effectively.

5.2.2 Mathematical modelling competencies

The mathematical modelling process shown in Figure 5.1, together with the sub-competencies proposed by Maaß (2006), was used as the framework for data analysis in this study. To simplify the analysis process, four tables (Tables 5.2 to 5.5) were developed, with each question assigned its own table for systematic analysis of learners' responses and mathematical competencies.

The analysis focused on the following key competencies:

1. **Understanding and simplifying the problem:** This involved determining whether learners had correctly understood the question, based on their explanations and simplification of the problem situation. It also included identifying whether appropriate assumptions were made.
2. **Formulating a mathematical model:** This required examining how learners represented the situation using formulas, mathematical expressions, sketches, or tables. It also involved checking whether learners were able to identify and express relationships between variables in mathematical form.
3. **Applying appropriate mathematical procedures:** This stage focused on evaluating the methods learners used to solve the formulated mathematical model and whether these methods were appropriate and correctly applied.
4. **Interpreting results:** Here, learners' ability to interpret their mathematical solutions in relation to the original context of the problem was examined.

5. **Validating results:** The final stage involved assessing how learners justified their solutions and verified whether their results were reasonable and appropriate within the given context.

As indicated in the previous chapters, mathematical modelling is a cyclical process consisting of interconnected stages. Therefore, mathematical modelling competence refers to a learner's ability to successfully engage with and implement all these processes in an integrated manner.

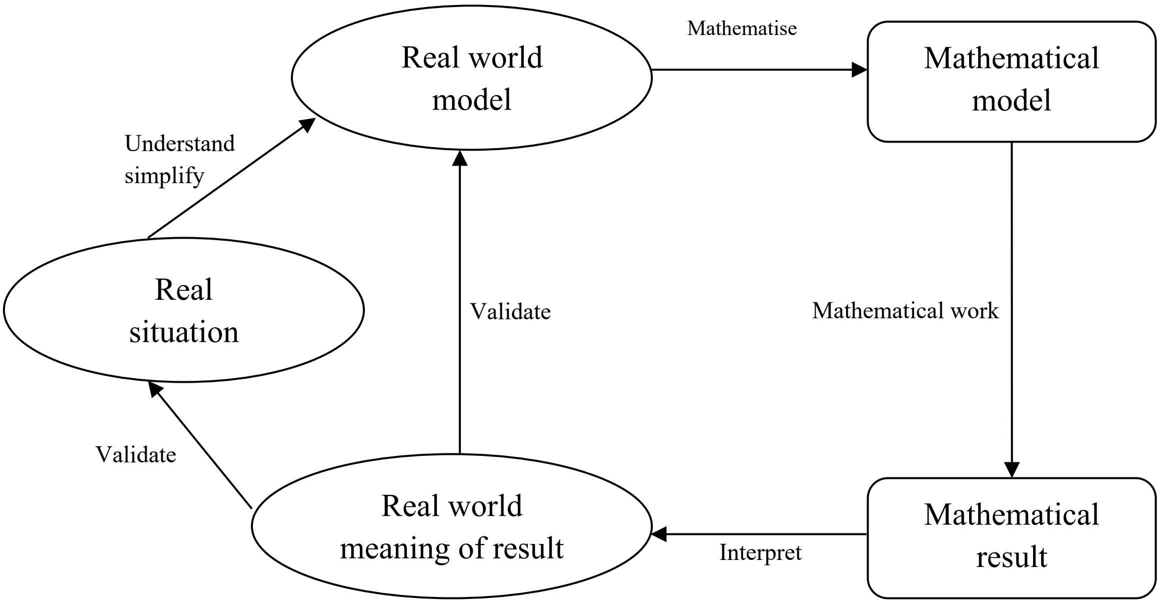


Figure 5.1: A five-step modelling cycle used in the study

Lingefjard (2006) notes that for learners to excel in mathematical modelling, they must be able to solve real-life problems effectively, validate mathematical models, reflect on and refine models, and clearly communicate ideas through mathematical representations. He further outlines the stages of the modelling process. From these perspectives, mathematical modelling competency can be understood as the ability to successfully apply the processes of mathematical modelling in solving problems.

According to Maaß (2006), a clear definition of the modelling process is closely linked to an accurate understanding of modelling competencies and skills. This implies that a strong grasp of modelling competencies enables a clearer explanation and application of the modelling process. In this study, learners' competencies were examined by analysing how they engaged with each stage of the modelling process when solving the test problems. As indicated in Section 3.7, learners' difficulties may emerge at different stages of the modelling cycle. For example, some learners may struggle with understanding the problem, others with mathematisation (translating a real-world situation into a mathematical model such as an equation or formula), while others may encounter challenges in solving the mathematical model or interpreting the results.

In addition to the five-step modelling cycle proposed by Kaiser and Stender (2013), this study also drew on the modelling competencies identified by Maaß (2006), which include specific sub-skills closely related to understanding and applying the modelling process. Accordingly, learners were expected to demonstrate the following competencies and associated indicators:

1. **Understanding the real-world problem and constructing a model based on reality.** This was identified through learners' ability to:
    - Develop appropriate assumptions to simplify the situation;
    - Identify and define relevant variables and influencing factors;
    - Establish relationships between the variables involved.
2. **Formulating a mathematical model from the real-world situation.** This was assessed by learners' ability to:
    - Mathematically represent quantities and relationships;
    - Simplify and structure key variables and relationships into a coherent model.
3. **Solving mathematical problems within the model.** This was evidenced when learners were able to:
    - Use heuristic strategies to solve problems efficiently;
    - Apply appropriate mathematical procedures and knowledge to obtain solutions.
4. **Interpreting mathematical results in a real-world context.** This was demonstrated when learners could:
    - Explain the meaning of mathematical results in context;
    - Generalise results from a specific situation;
    - Communicate findings using appropriate mathematical language.
5. **Validating the solution.** This was observed when learners were able to:
    - Critically check and evaluate their solutions;
    - Reconsider and refine the model where results were unreasonable or invalid, including repeating the modelling process where necessary.

In this study, a marking guideline (memorandum) and a structured marking rubric were used to ensure accurate and consistent assessment of learners' competencies. The rubric presented in Table 5.1 was developed based on the competencies identified by Maaß (2006) and the modelling cycle proposed by Kaiser and Stender (2013).

5.2.3 Marking rubric

Learners' written responses to the test questions were assessed using a marking guideline (memorandum) and a rubric, presented in Appendices B and C, respectively. An extract of the rubric is also shown in Table 5.1. All questions were evaluated using the same set of criteria to ensure consistency in marking.

The rubric consisted of three performance levels. Level 1 indicated unsatisfactory performance and was awarded zero marks. Level 2 represented partially achieved responses and was awarded one mark. Level 3 reflected fully achieved performance, where the learner met all expected criteria for a given aspect and was awarded three marks.

For example, in the validation criterion, a learner who did not attempt to validate the solution received zero marks. A learner who tried but failed to clearly justify why the solution was acceptable or unacceptable received one mark. A learner who fully validated the solution and provided a clear justification of its correctness or incorrectness received three marks.

Each question was assessed out of a maximum of 10 marks using this rubric, and the total mark for the entire test was 40 marks.

Table 5.1: The three-level rubric used to assess learners' responses

QUESTION 1		
Level	**Definition**	**Score/ Marks**
	Understanding the Problem	
Level 1	Involves the expressions indicating that the individual did not comprehend the problem and did not make an assumption, failed to establish a connection between the variables.	0
Level 2	Includes the expressions showing that s/he understood the problem to some extent, did not make an assumption, but established a connection between the variables	1
Level 3	Includes the expressions showing that s/he understood the problem completely, made a realistic assumption, formed a relationship between variables.	2
	Mathematise (formulating a model)	
Level 1	Fails to select appropriate mathematical symbols, fails to develop a mathematical model (equation or expression); makes no attempt to simplify the relevant quantities and their relationships.	0
Level 2	Selects appropriate mathematical symbols, develops a mathematical problem (equation or expression) that is partial correct and simplifies the quantities and their relations to some extent.	1
Level 3	Selects appropriate mathematical symbols, develops a mathematical problem (equation or expression) that is correct to represent the real model simplified the quantities and their relations	2
	Mathematical work (Solving the mathematical problem)	
Level 1	Makes no utilisation of mathematical knowledge to solve the problem, does not apply proper mathematical methods to find solutions, does not solve the constructed model to get solutions	0

Level 2	Creates errors or deficiencies in the answers of the mathematical models that are correctly formed.	1
Level 3	Obtains the correct mathematical solution by solving the correctly constructed mathematical models using proper methods or strategies	2
Interpreting the results		
Level 1	Misinterprets, or does not interpret the obtained mathematical solution in a real-life context. Gives no general statement regarding solution(s) obtained.	0
Level 2	Presents a partial explanation of the proper mathematical solution in a practical setting, makes attempts to make a general statements regarding solutions to certain extent.	1
Level 3	Provides a clear interpretation of the achieved accurate mathematical solution in a practical setting. Gives a clear general statements regarding results.	2
Validate the results		
Level 1	Not validating or making an invalid validation.	0
Level 2	Validates partially by not giving the reason as well as not correcting the determined mistakes. States that the obtained solution is invalid but no reason given.	1
Level 3	Validates completely, giving the correct reason why the solutions is acceptable/not acceptable and correcting the determined mistakes, verifies that the multiplication of the two obtained dimensions gives an area of 54m Considers alternative ways of solving the problem.	2
	Question 1 subtotal	/10

The rubric shown in Table 5.1 represents only a section of the full assessment tool and was used specifically to evaluate learners' competencies in Question 1. The same rubric structure was applied to each of the four questions in the test. Since the assessment consisted of four questions, evaluating a single learner's work required the completion of four separate rubric forms like the one illustrated in Table 5.1. With 75 learners participating in the study, a total of 300 completed rubric documents were produced for analysis. Despite the use of the rubric, no learner achieved the full 40 marks. This was mainly due to limited performance in the final two stages of the rubric, namely interpreting and validating results. This pattern is also reflected in Tables 5.2 to 5.5. Overall, the results indicate that while most learners were able to understand the problem, formulate a mathematical model, and solve the resulting equations, they struggled with interpreting and validating their solutions. A fully completed version of the rubric used to assess learners' mathematical modelling competencies is provided in Appendix C, which includes all four questions.

5.2.4 An analysis of the learners' scripts question by question

To determine learners' competencies, all scripts were marked on a question-by-question basis. For each learner's script, I examined whether all the modelling processes outlined in Figure 5.1 had been correctly followed. To gain a deeper understanding of the phenomenon under study—namely learners' competencies in mathematical modelling—I employed data triangulation in the analysis. This involved the use of more than one method of data analysis. In addition to the marking guideline (memorandum) used to assess learners' responses, I also applied a rubric to evaluate their modelling competencies. The rubric was completed for each learner, and a summary of the findings for each question was recorded on the back of the rubric sheet (see an example of a completed rubric in Appendix B).

QUESTION 1

Mr Nene wants to plant seedlings in his rectangular garden with an area of 54m^2. The length of the garden is longer than its width by 3m. What are the dimensions of his garden? (6)

For this question, learners were required to determine the dimensions of a rectangular garden with an area of 54 m^2, given that the length is 3 m longer than the width. To demonstrate understanding of the problem, learners were expected to follow the steps outlined below:

1. **Represent the situation diagrammatically and define variables:** Learners were expected to draw a rectangle to represent the garden, define appropriate variables for length and width, and state relevant assumptions to simplify the problem. For example, it should be assumed that the entire area is usable for cultivation (i.e., no rocks or obstacles reduce the cultivable space), ensuring that the full 54 m^2 is considered.
2. **Formulate a mathematical model:** Learners were required to construct an equation based on the relationships between the variables. Specifically, they had to indicate that the length is 3 m longer than the width and that the product of length and width equals 54 m^2.
3. **Solve the mathematical model:** Learners were expected to correctly solve the resulting equation to determine the possible dimensions of the garden.
4. **Interpret the solution in context:** Learners needed to express their answers in complete sentences, clearly relating the mathematical results to the real-life situation. For example, a correct interpretation would be: "The width of the garden is 6 metres, and the length is 9 metres."
5. **Validate the solution:** Learners were expected to verify their answers by checking whether the obtained dimensions satisfy both conditions of the problem (i.e., that one dimension is 3 m longer than the other and that their product equals 54 m^2). They were also expected to identify and correct any incorrect or extraneous solutions.

Prior knowledge from earlier grades in Algebra and Measurement was essential for successfully completing the test. In Algebra, learners were expected to apply skills in solving linear, literal, and quadratic equations. In Measurement, they were expected to understand and apply formulae for the perimeter and area of geometric shapes such as triangles, quadrilaterals, circles, and cylinders. Given that this task involved a rectangular figure, knowledge of the area formula for a rectangle was particularly important.

Figure 5.2 presents an example of a learner's response (Learner L27) to Question 1. The learner successfully produced a sketch of the garden and established a relationship between the variables $l$and b. However, the variables were not clearly defined, and no explicit assumptions were stated, which are essential components of mathematical modelling.

In the modelling process, the learner introduced a variable $x$but did not clarify its relationship to $l$or b. In addition, a mathematical error occurred when expanding the expression, as brackets were omitted. Despite this, the learner proceeded correctly in subsequent steps and arrived at the correct solutions. However, the learner did not identify or justify why the negative solution was not valid in the context of the problem, nor was there any attempt to verify the final answer. Because the learner did not successfully complete all five stages of the modelling process, the performance was classified as partially competent in mathematical modelling. In terms of mathematical proficiency, the learner demonstrated strengths in strategic competence, conceptual understanding, and procedural fluency, but showed weakness in adaptive reasoning. This is evident in the learner's ability to understand the problem, develop an appropriate model (despite notation errors), and execute procedures correctly, while failing to interpret, justify, and validate the solution within the real-world context.

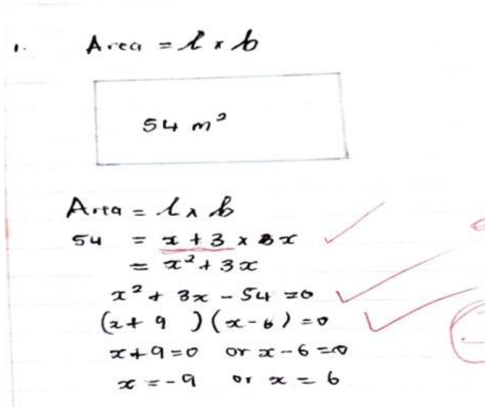


Figure 5.2: L27 response to Question 1

Figure 5.3 presents a learner's response in which no sketch was drawn to represent the garden. In addition, the learner did not provide a clear statement defining the variables $l$and b. A statement such as "Let $l$represent the length and $b$represent the breadth" would have made the solution more transparent and easier to follow, as it clearly specifies the meaning of each variable. Despite the absence of a diagram and explicit variable definitions, the learner was still able to formulate correct mathematical expressions and proceed with the solution accurately. However, as in the case of the previous learner in Figure 5.2, no assumptions were stated to simplify the problem, which is an essential component of the modelling process.

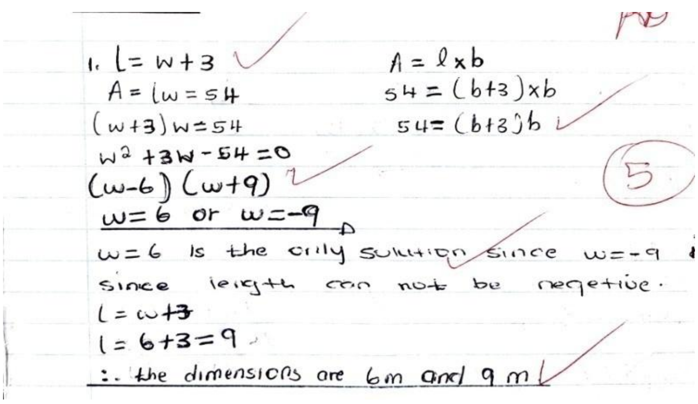


Figure 5.3: L36 response to Question 1

Learner L36 in Figure 5.3 was able to obtain both solutions and provided a clear explanation for why the negative solution was invalid. However, the learner did not achieve full marks, as they were unable to correctly determine the value of the length and did not complete the validation of the solution.

Another notable issue was the inconsistent use of variables $b$and $w$interchangeably. If the learner had chosen $b$to represent breadth, this notation should have been used consistently throughout the solution. Switching between variables may create confusion and reduce clarity in mathematical communication.

Despite these challenges, Learner L36 demonstrated strong strategic competence, adaptive reasoning, and conceptual understanding. This is evident in their ability to formulate a correct mathematical model, solve the equation accurately, and justify the validity of one of the solutions. However, the learner displayed only partial procedural fluency, as they did not complete all required computations, specifically failing to determine the length of the garden.

Figure 5.4 presents a learner’s response that reflects carelessness in the solution process. Although the learner correctly formulated a mathematical model representing the given information, an error occurred during the solution process. Specifically, a zero was omitted in Step 5, resulting in a breakdown of equivalence between Step 4 and Step 5. This omission led to an incorrect expression, where the learner obtained an invalid transformation of the original equation.

Despite this procedural error, the learner still arrived at the correct final answers. However, the explanation provided for why the solution $w = -9$ was invalid was incorrect and contradictory, as the learner effectively cancelled out their own justification. As with previous cases, there was no evidence of assumption-making or verification of the solution. In terms of mathematical proficiency strands, this learner demonstrated partial proficiency. The learner showed conceptual understanding through the correct formulation of the equation and appropriate substitution, indicating a sound grasp of the problem context. Strategic competence and procedural fluency were also evident in the correct setup and continuation of the solution process. However, the lack of validation and incorrect reasoning regarding the rejected solution indicated weak adaptive reasoning. Overall, the learner was classified as partially proficient across the mathematical proficiency strands.

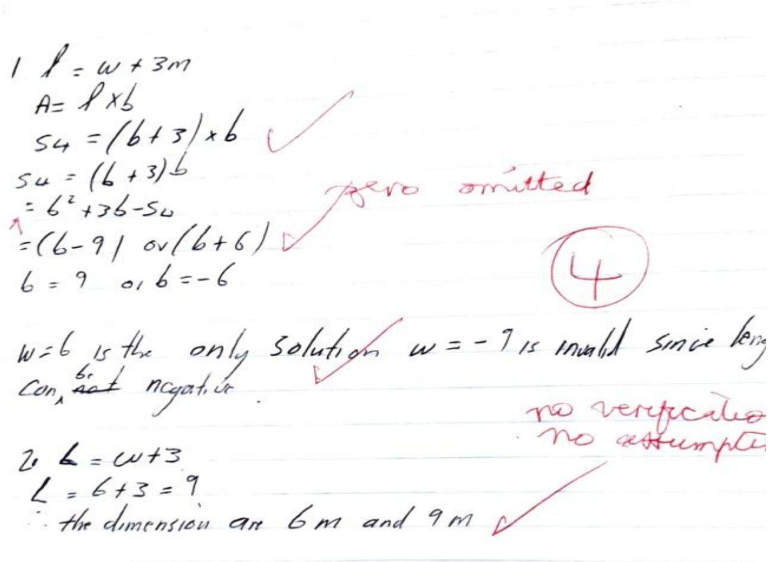


Figure 5.4: learner L46 response having a slip in solving

Figure 5.5 also presents a learner's response to Question 1. Learner L46 successfully formulated the correct expression for the length in terms of the breadth and ultimately developed the correct quadratic equation. However, in terms of the mathematical modelling process, the learner did not make explicit assumptions that would ensure the validity of the solution across relevant real-life contexts.

In addition, the learner did not provide a justification for why the solution $-9$was invalid within the context of the problem. There was also no evidence of verification or validation of the final solutions. Since not all stages of the modelling process were completed, the learner was classified as partially competent in mathematical modelling.

Regarding mathematical proficiency, the learner was also considered partially proficient, as adaptive reasoning was not demonstrated. The other three strands—conceptual understanding, procedural fluency, and strategic competence—were appropriately displayed. However, adaptive reasoning was lacking, particularly in the failure to justify why $-9$was not a valid solution in the given context.

In this study, a learner was regarded as mathematically proficient if all strands were demonstrated, except productive disposition, which could not be assessed through the single test instrument used to generate the data. Further analysis is presented in Table 6.3.

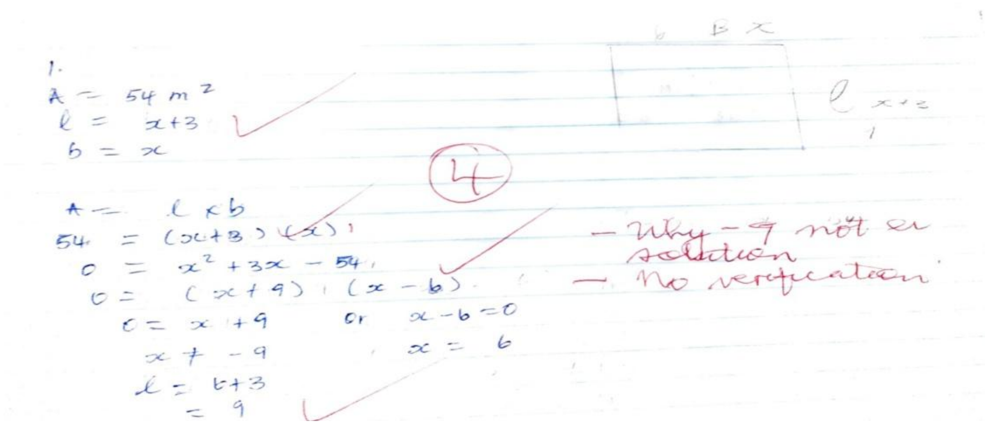


Figure 5.5*:* L25 response to Question 1

Figure 5.6 presents Learner L40's response to Question 1. Like Learner L25 (as shown in Figure 5.5), this learner was classified as partially competent in mathematical modelling and partially proficient in the mathematical proficiency strands. This classification was based on the same reasons identified for Learner L25. Further analysis of the mathematical proficiency strands is presented in Table 5.3.

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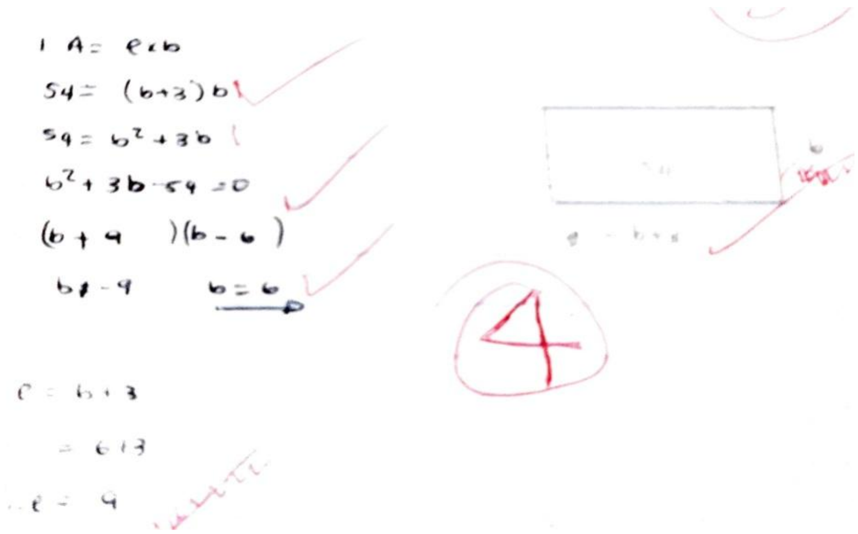


Figure 5.6: Learner L40’s response to Question 1

The extract in Figure 5.7 presents Learner L67’s response to Question 1. The learner successfully formulated the correct expression showing the relationship between length and breadth, developed the appropriate equation representing the area of the garden, and proceeded to solve it correctly. However, the learner did not state any assumptions and did not verify or validate the obtained solutions. Due to the absence of assumptions and validation, the learner was classified as partially competent in mathematical modelling.

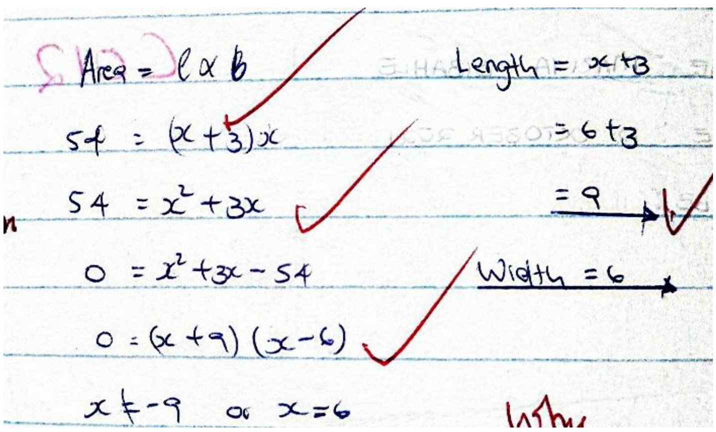


Figure 5.7: Learner L67’s response to Question 1

From the extract, it is evident that the learner demonstrated proficiency in conceptual understanding, as the problem was clearly comprehended. The learner also showed strategic competence through the correct formulation of the mathematical model, as well as procedural fluency by solving the model accurately. However, the learner did not verify or validate the solutions. As a result, this learner was classified as partially proficient across the mathematical proficiency strands. Despite the absence of clearly defined variables and stated assumptions, all learners were still able to construct accurate mathematical models for this question. The ability to develop correct models suggests a sound understanding of the problem context. However, none of the learners engaged in the validation of their solutions or the models they formulated.

QUESTION 2

To protect his vegetables, a farmer buys 32m of fencing for his rectangular garden. Determine the dimensions of the vegetable garden if it should cover a maximum area (6)

Figure 5.8 presents a learner's response to Question 2. The learner completed most of the required steps correctly; however, they did not make any assumptions and did not verify or validate their solutions, which are essential components of the modelling process. In this study, a learner was classified as competent if they demonstrated understanding of the problem, made appropriate assumptions, formulated a mathematical model, solved the problem correctly, and interpreted the results meaningfully. Since the learner did not complete all these stages, they were classified as partially competent in mathematical modelling. Furthermore, the absence of verification contributed to the classification of the learner as partially proficient in the mathematical proficiency strands, as adaptive reasoning—particularly the ability to validate and justify solutions—was not demonstrated.

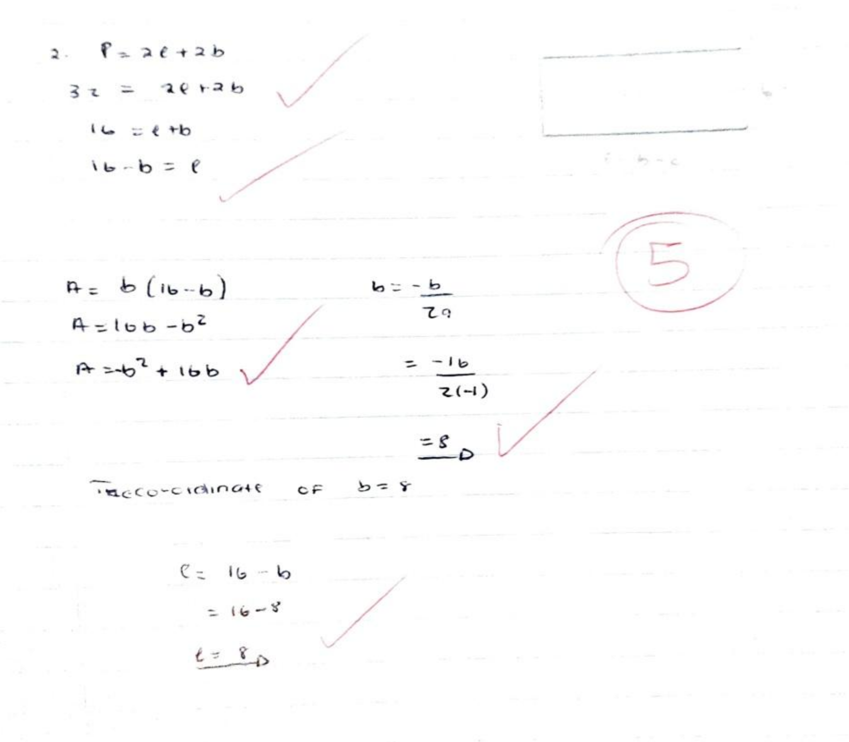


Figure 5.8: Learner L67's response to Question 2

Prior knowledge about the area and perimeter was needed to respond to this question in addition to concepts learnt in Functions, where they learnt about the maximum and minimum values of parabolas. In assessing the learners' competencies, I followed the five steps used for assessing Question 1. The learner in Figure 5.8 made a sketch of a rectangular garden and then determined the right or proper formula for a perimeter of a rectangle, substituted correctly to get $2l+2b=32$, and made the length the subject of the formula to get $l=16-b$. They wrote an expression for an area of a rectangle in terms of l and b which yielded the expression $A=-b^2+16b$ which is an expression of a parabola with a maximum turning point. Next, the learner used the formula $x=\frac{-b}{2a}$ where a is the coefficient of a term with a quadratic variable and b represents is the coefficient of a variable that is linear to find the x value of the maximum turning point. Since the learner used variable b to represent the breadth, it was confusing to express the x-coordinate as $b=\frac{-b}{2a}$ because the same variable represented different quantities. It would be better if the learner had used the letter w for the width.

Regarding mathematical proficiency, the learner in Figure 5.8 demonstrated proficiency in conceptual understanding, strategic competence, and procedural fluency, but not in adaptive reasoning, as no verification of the results was carried out. The other strands were appropriately demonstrated. Figure 5.9 presents a learner's solution to Question 2 in which the maximum area was determined by completing the square. Like other learners, this learner did not explicitly define what each variable represented, and no assumptions were stated. After determining the maximum x-coordinate, the learner did not proceed to calculate the corresponding y-value, which was required to establish both dimensions that produce the maximum area. Instead, the learner provided the maximum area as the final answer. Furthermore, there was no verification of the solution. In terms of mathematical proficiency strands, the learner demonstrated conceptual understanding, strategic competence, and procedural fluency, but lacked adaptive reasoning. An attempt to verify the solution would likely have helped the learner realise that the question required the dimensions (length and width) that yield the maximum area, rather than the area itself. The expected solution was a length of 8 metres and a width of 8 metres.

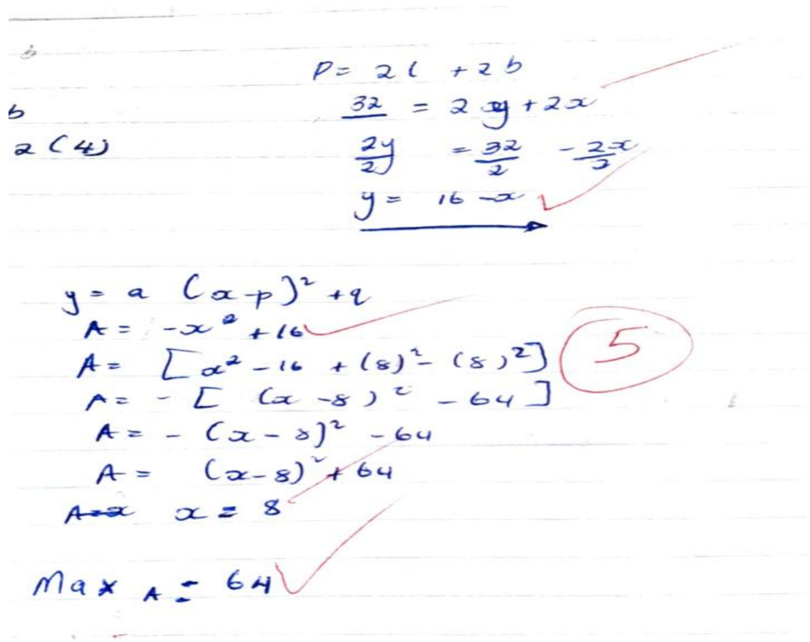


Figure 5.9: Learner L14's response to Question 2 by completing a square

Figure 5.10 presents a learner's response in which the trial-and-error method was used to determine the correct dimensions. The learner attempted various values that could be substituted into the perimeter formula of a rectangle to obtain a perimeter of 32 m. In this case, the selected values of 8 and 8 were eventually used, resulting in an area of 64 m². However, most of the modelling processes were not followed. There was no explicit formulation of a mathematical model or equation, no stated assumptions, no interpretation of the results, and no verification of the solution.

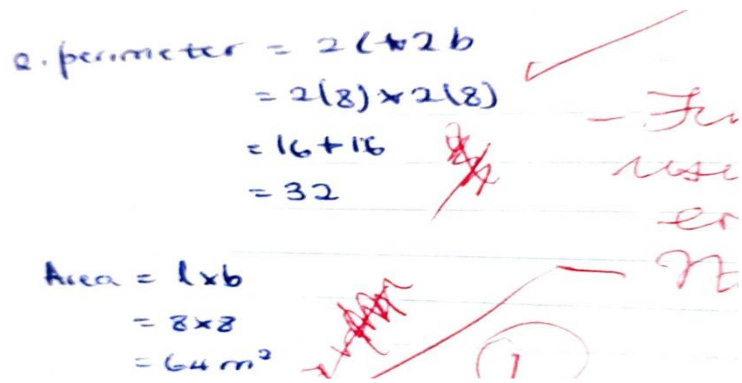


Figure 5.10: Learner L1 used the trial and error method to find dimensions

In terms of mathematical proficiency strands, the learner demonstrated conceptual understanding, as they appeared to understand what the problem required. It is possible that the learner explored different combinations that satisfy a perimeter of 32 m during rough work and only recorded the correct dimensions on the answer sheet. However, these steps were not sufficiently evidenced in the written work to conclude that the learner demonstrated proficiency in the other mathematical strands.

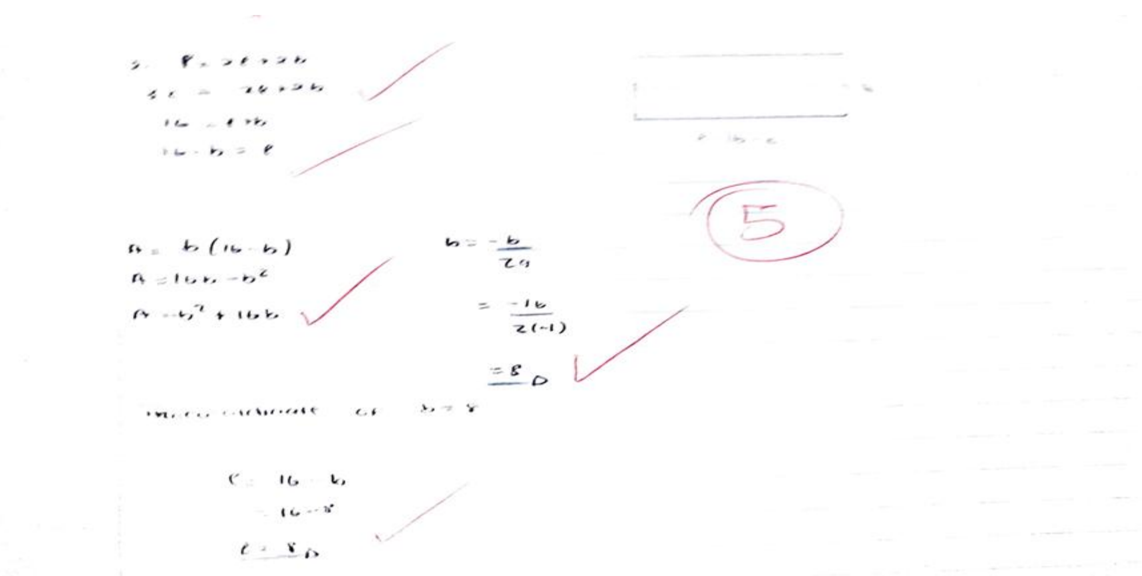


Figure 5.11: Learner L40’s response to Question 2

Figure 5.11 presents a learner who performed exceptionally well in key aspects of the modelling process, including understanding the problem, formulating the model (perimeter and area formulae), and solving the mathematical equations correctly. However, the learner's failure to make explicit assumptions and to verify the solution led to the classification of this learner as partially competent in mathematical modelling. As with other learners who did not validate their solutions, this learner was also considered partially proficient in the mathematical proficiency strands. Figure 5.12 shows a learner who clearly understood the problem, as evidenced by the correct formulation of the perimeter equation, the correct expression for the area, and the successful use of completing the square to determine the dimensions that maximise the area. Nevertheless, the learner did not make explicit assumptions or provide justification for the results obtained. Consequently, this learner was classified as partially competent in mathematical modelling and simultaneously partially proficient in the mathematical proficiency strands.

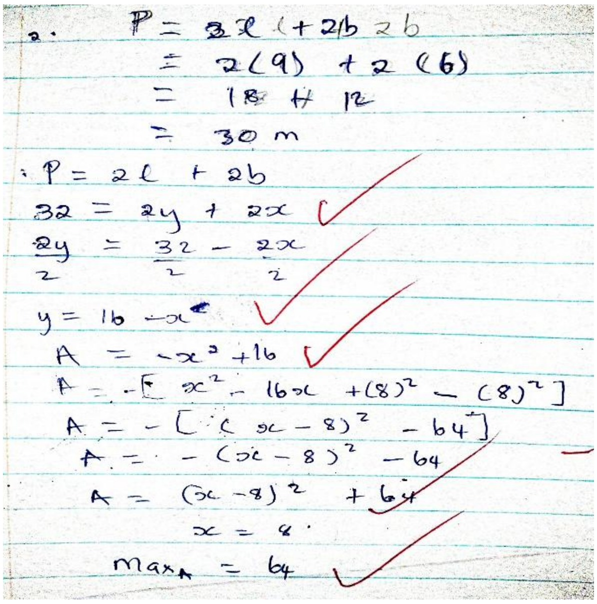


Figure 5.12: Learner L25's response to Question 2

Although the learner demonstrated proficiency in the other three strands—conceptual understanding, procedural fluency, and strategic competence—they did not perform well in adaptive reasoning, as they were unable to justify, verify, or validate the solution. In terms of mathematical modelling, the learner successfully formulated a correct mathematical model, which indicates an understanding of the problem. Appropriate mathematical procedures were then followed to obtain the correct solution; however, the results were neither interpreted within the context of the original problem nor validated. Furthermore, the learner did not state any assumptions during the modelling process.

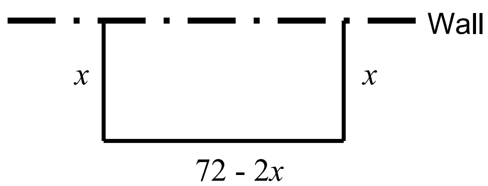
QUESTION 3

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 72m of wire fencing:

3.1. Determine the dimensions that would create a garden of maximum area. (5)

3.2. What is the maximum area? (1)

Like the previous two rectangle-based questions, learners were required to determine the area and perimeter of a rectangle to identify the dimensions that would produce the maximum possible garden area. They were expected to read the problem with understanding and recognise that, although the garden is rectangular, only three sides are to be fenced. The prior knowledge required to solve this question included completing the square of a quadratic expression and determining the maximum value (turning point) of a parabola. In assessing learners' competencies in mathematical modelling, the same five-step process used in Questions 1 and 2 was applied. This involved determining whether learners understood the problem, simplifying it appropriately, identifying relevant variables and establishing relationships between them, formulating and solving the mathematical model, interpreting the solution in a real-world context, and finally verifying the solution by justifying its correctness or identifying and correcting errors. To approach this question effectively, it was advisable to begin by drawing a sketch of the scenario and assigning variables to represent the relevant dimensions. For example, let x be the two sides that meet at 90° to the brick wall and the other dimension that is parallel to the brick wall would be $(72-2x)$ metres. In the diagram, the sides to be fenced were indicated by a solid line and the already existing wall by a solid line.



Then the area of the enclosed field $= l \times b$

$$= x(72 - \mathbf{2}x)$$
$$= -2x^2 + 72x$$

The area is a quadratic expression with the highest point of turning. Then the x coordinate of the turning point is given by $x = \frac{-72}{2(-2)} = 18m$

Substituting $x = 18$into the expression for the length of the other side gives: $y = 72 - 2(18) = 36$m.

Thus, the dimensions of the largest possible rectangular area are a length of 36 m and a width of 18 m.

To verify the correctness of the solution, learners needed to add the lengths of the three fenced sides and confirm that they total 72 m:

18 m + 36 m + 18 m = 72 m.

Therefore, the solution is correct, as it satisfies the total length of fencing available.

The second part of the question required learners to determine the maximum area. This could be done using more than one method. One approach was to multiply the length by the width: Area = 36 m × 18 m = 648 m².

Alternatively, they could have used their expression for the area, $A = -2x^2 + 72x$ and then substitute x by 18 since the x-coordinate at the maximum turning point is 18 and get:

$$\begin{aligned} A &= -2x^2 + 72x \\ &= -2(18)^2 + 72(18) \\ &= -648 + 1296 \\ &= 648 \end{aligned}$$

$\therefore$ The maximum area is 648m^2.

Figure 5.13, displays a learner's response that is correct except that no assumption was made and the solution was not verified, The learner produced a correct sketch of the situation in terms of labelling, but did not indicate on the sketch that the other side opposite the one of length $(72 - x)$ was not going to be fenced, as it has a rock wall. The learner was supposed to indicate with a broken line that that side was not part of the sides that were to be fenced. Lastly, there was no concluding sentence that said the dimensions were 18m and 36m.

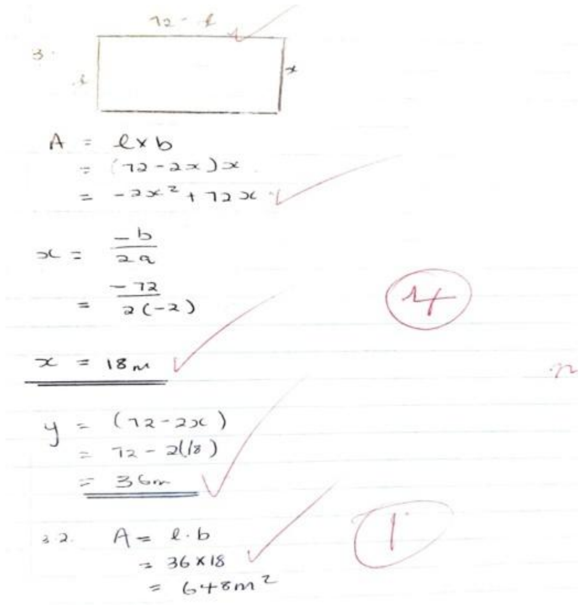


Figure 5.13: Correct response but that no assumption or verification made

The learner’s response in Figure 5.14 indicates a lack of understanding of the problem. In this case, the learner incorrectly included all four sides of the garden, despite the instruction that one side is bordered by a rock wall and therefore does not require fencing. As a result, the use of the standard perimeter formula for a rectangle led to an incorrect solution from the outset. For this question, the learner was expected to begin by drawing a sketch of the garden and clearly indicating the side covered by the rock wall. In addition, the learner should have made appropriate assumptions during the formulation of the mathematical model. However, no such assumptions were stated. It is evident that the learner did not fully comprehend the question, as they failed to account for the fact that only three sides of the garden require fencing. Consequently, the correct model should have included only the three fenced sides rather than all four sides of the rectangle. Due to this misunderstanding, the learner was unable to proceed further with the solution. In terms of the mathematical modelling process, this learner failed at the stage of problem comprehension. As a result, they were unable to formulate an appropriate mathematical model and could not progress to solving the problem.

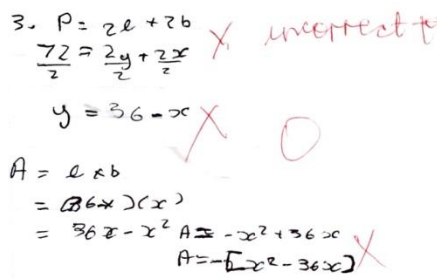


Figure 5.14: Wrong method that led to incorrect solution

The learner in Figure 5.15 demonstrated a clear understanding of the question, as evidenced by the working shown. However, like most participants in this study, the learner did not define the variables or indicate what each variable represented. In addition, the learner did not make any assumptions, which are essential when solving mathematical problems situated in real-life contexts. Furthermore, there was no evidence of validation or interpretation of the final solution. The failure to verify and interpret the results contributed to the classification of this learner as partially competent in mathematical modelling. In terms of mathematical proficiency, the learner did not demonstrate adaptive reasoning. Therefore, the learner was considered partially proficient across the mathematical strands, as only conceptual understanding, procedural fluency, and strategic competence were evident.

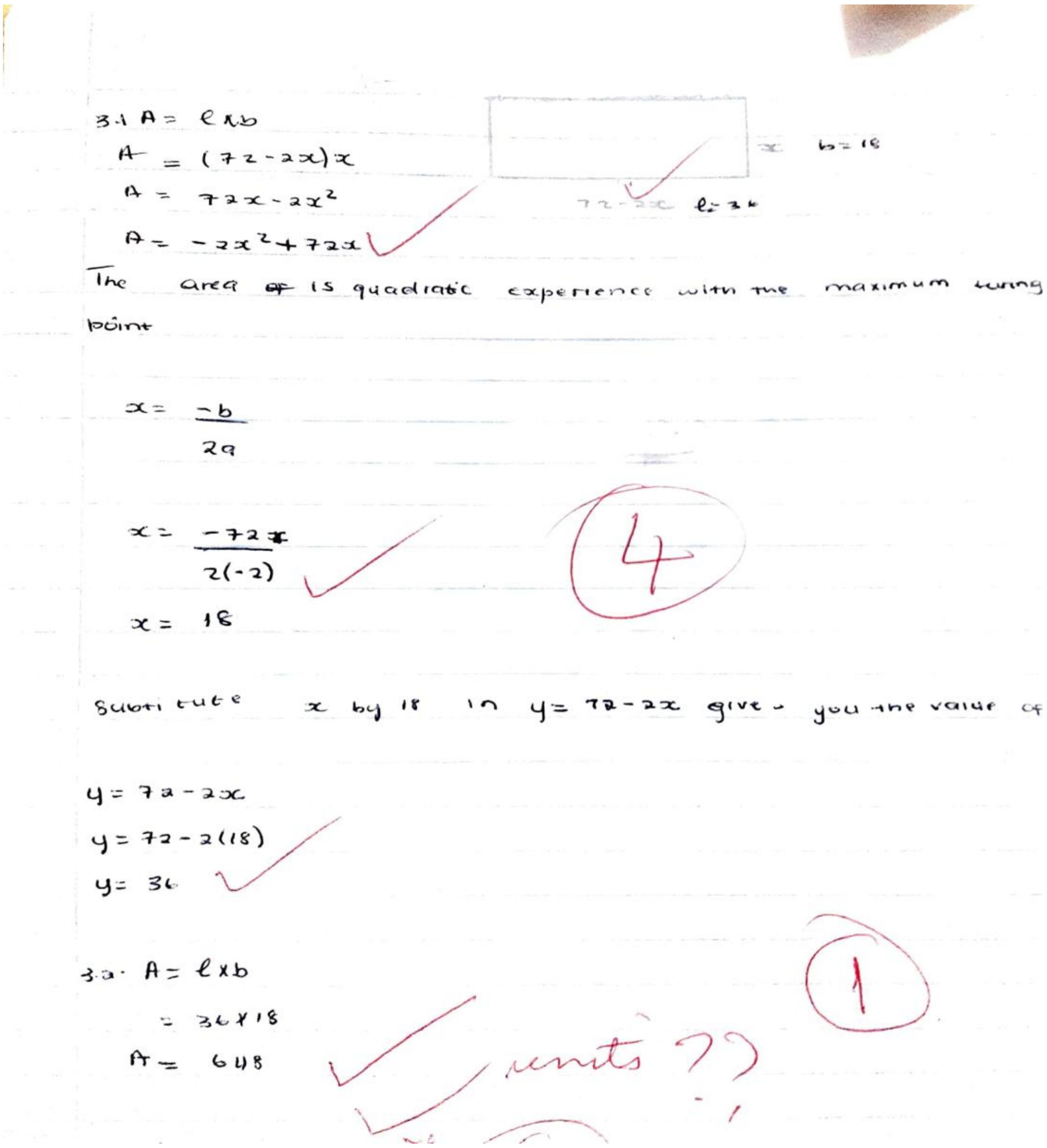


Figure 5.15: Learner L40’s response to Question 3

Figure 5.16 illustrates how another learner demonstrated a clear understanding of the problem. This is evident from the diagram, in which the side covered by the rock wall is clearly distinguished by shading in bold, making it visually distinct from the other sides that require fencing.

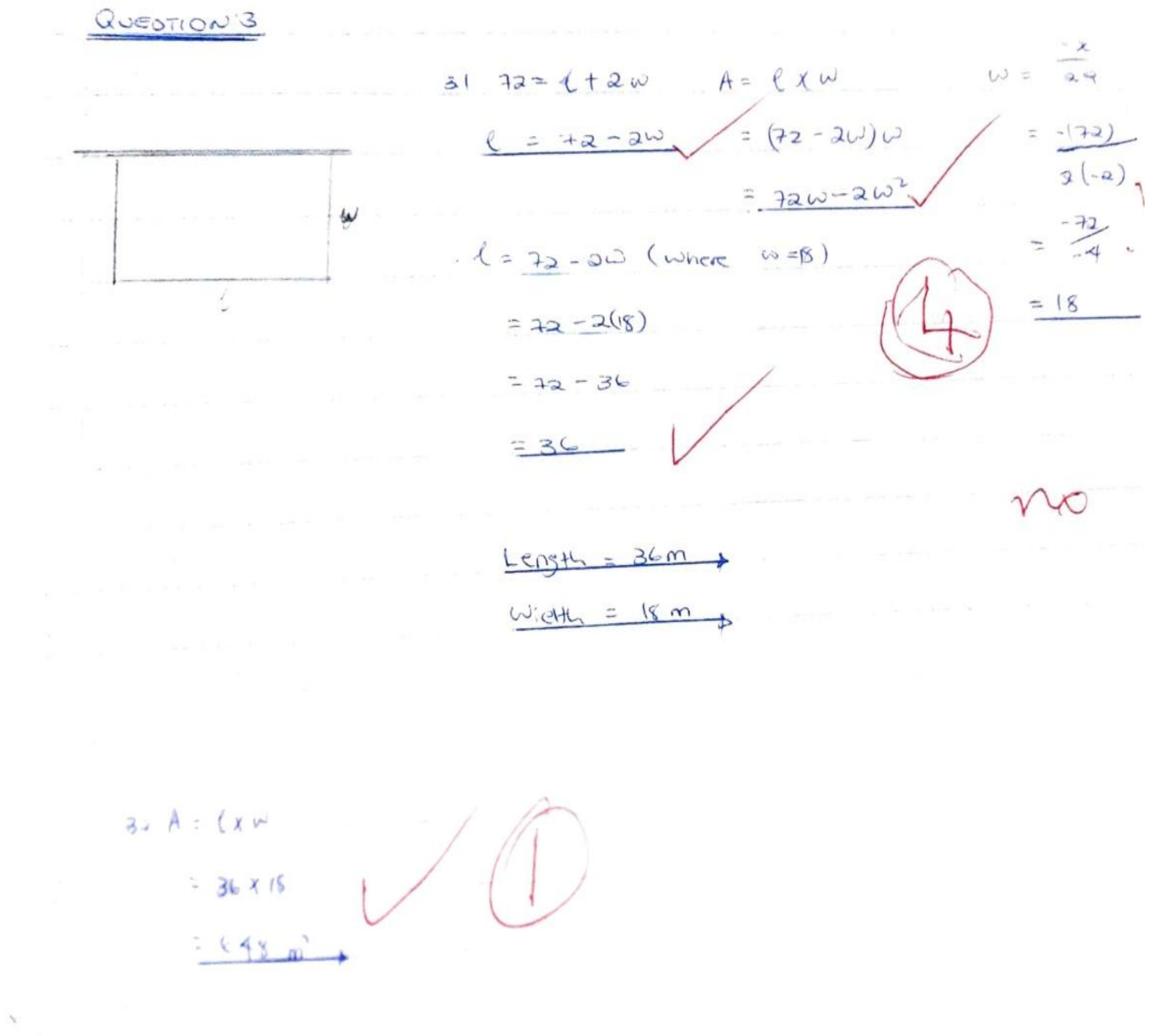


Figure 5.16: Learner L67's response to Question 3

Figure 5.17 shows a learner that could formulate a correct athematical equation but could not proceed correctly. After obtaining the quadratic expression, the learner was supposed to use the formula $x = \frac{-b}{2a}$ to get the x-coordinate that would give the maximum area. The obtained value of x is the dimension of the two sides perpendicular to the side fenced by a rock. From there, the learner should have substituted the x value into the equation $l = 72 - 2x$ to get the length, the side opposite to the side fenced by the rock.

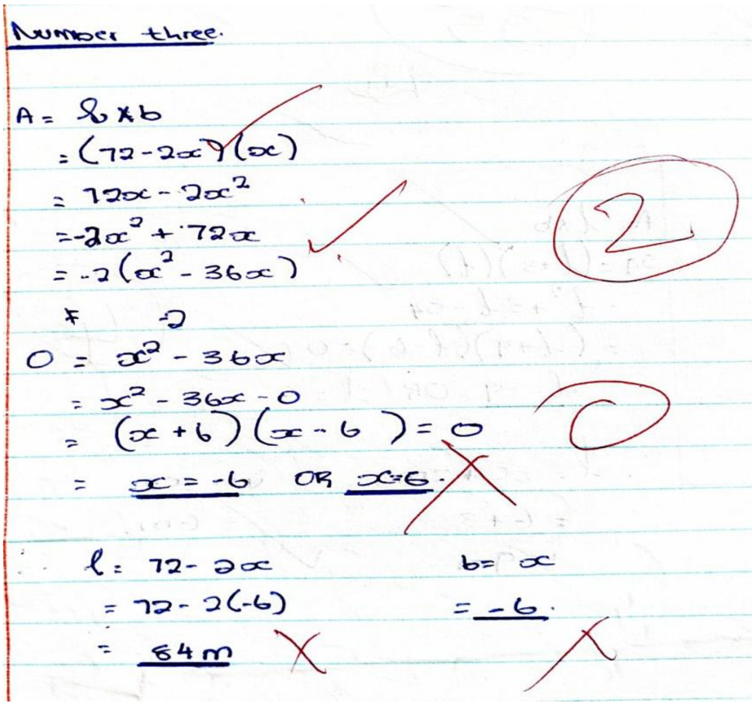
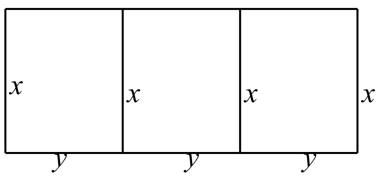


Figure 5.17: Learner L32's response to Question 3

The extract in Figure 5.17 shows that although learners were able to formulate a correct mathematical model, they encountered difficulties in completing the solution process. As a result, they resorted to incorrect methods or approaches, which ultimately led to incorrect solutions.

QUESTION 4

A farmer has 600m of fence to create a rectangular pasture which has to be divided into three camps all with the same dimensions as shown in the diagram. What is the maximum area that can be enclosed with this fence? (7)



To be able to work out Question 4, learners needed the same prior knowledge to work out Question 2 and 3. To work out this problem, learners could have seen that the length of the whole pasture was $3y$ while the width or breadth was x. Also, the pasture was divided into three camps, each one has width of x and length y units. Therefore, the distance to be fenced could be defined as $4x+6y$. Given that a farmer had a fence that was 600m long, the learners were supposed to make the equation $4x + 6y = 600$ as indicated as equation (1) below. From here, learners were supposed to make y the subject of the formula as shown in equation (2), which was the length of each camp and then use that in finding the area of each camp using the formula $A = l \times b$.

$$600 = 4x + 6y \quad ......(1)$$

$$6y = 600 - 4x$$

$$y = 100 - \frac{2}{3}x \quad .....(2)$$

Area of the pasture is given as:

$$\text{A} = xy$$

$$A = x(100 - \frac{2}{3}x)$$

$$= 100x - \frac{2}{3}x^2$$

$$= -\frac{2}{3}x^2 + 100x$$

This is a quadratic function with the maximum turning point with a $= -\frac{2}{3}$ and b = 100

The value of the x-coordinate of the turning point could then be calculated as:

$$x = \frac{-b}{2(a)}$$

$$= \frac{-100}{2(-\frac{2}{3})}$$

$$= \frac{-100}{-\frac{4}{3}}$$

$$= 100 \times \frac{3}{4}$$

$$x = 75$$

The value of y could then be calculated by substituting x = 75 in equation (2)

$$y = 100 - \frac{2}{3}x$$

$$y = 100 - \frac{2}{3}(75$$

$$y = 100 - 50$$

$$y = 50$$

From the diagram, area = xy is the area of one camp. To find the total area of the pasture, they were supposed to multiply the area of one camp by 3 since there are three camps.

$$\begin{aligned} \text{Area} &= 3xy \\ &= 3(75)(50) \\ &= 11250 \end{aligned}$$

$\therefore$ Area = 75 × 150 = 11 250

To check the validity of the answer, they could have added all the dimensions to see if they add up to 600m. Length of wire = x + x+ x + x + y + y

$$\begin{aligned} &= 4(75) + 2(150) \\ &= 300 + 300 \\ &= 600 \end{aligned}$$

The extract in Figure 5.18 presents a learner's response to Question 4. The learner completed the solution correctly up to the final step; however, they failed to recognise that the area calculated represented only one camp. To determine the total surface area, the learner was expected to multiply this value by 3, since the pasture had been divided into three equal camps. Another error made by the learner was the incorrect use of units, where cm^2 was written instead of m^2.

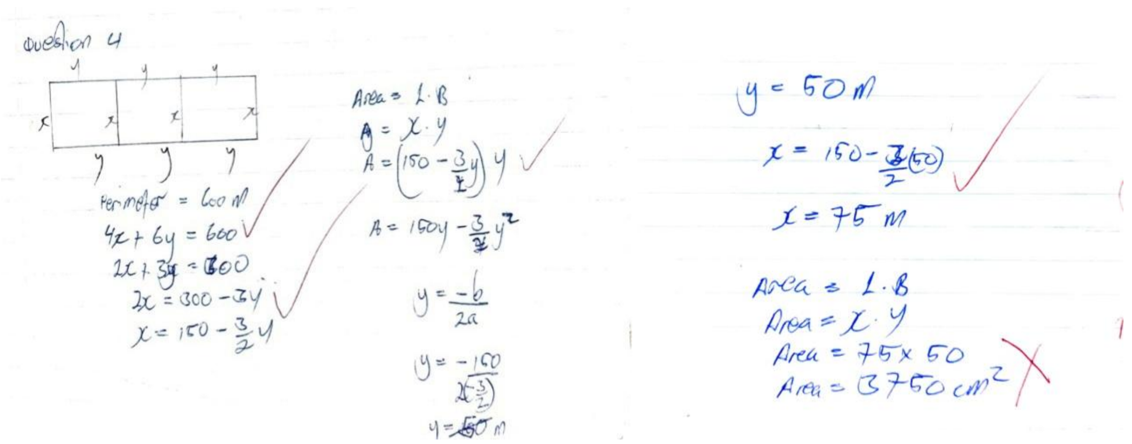


Figure 5.18: Learners that performed well except finding the maximum area

The illustration in Figure 5.19 indicated the response of a learner that did not comprehend the question very well. Firstly, the learner came out with the incorrect formula for the total length of the fence that could divide the pasture. From the given sketch, there were four sides with the length of x metres and six sides of length y metres, and this information should have helped the learner to see that the total length of 600m was equal to (4x + 6y) metres. After producing a wrong formula, the learner did not proceed accurately, as seen in the expression for the area in the fifth step. It is not clear where the learner got $600-4x$ since their expression for y was then different from the one calculated in step three, which was $y=200-\frac{4}{3}x$. From here, everything was incorrect and therefore could not earn a mark. Furthermore, the learner did not complete solving their incorrect solutions.

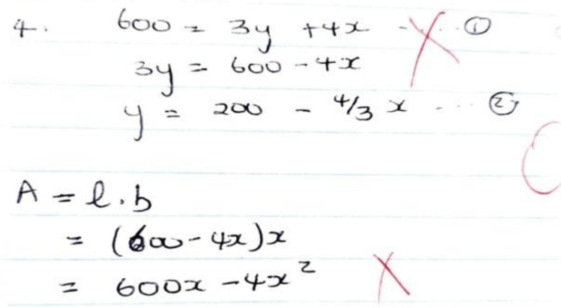


Figure 5.19: Response displaying that the learner did not grasp the problem

The extract in Figure 5.20 presents the work of a learner who incorrectly considered the entire length as $y$instead of $3y$. The learner was able to proceed as far as determining the breadth, x, but could not continue to correctly determine the value of y. Had the learner correctly interpreted the relationship between the variables, the value of $y$would have been 150 metres, which would have led to the correct area of 11 250 square metres.

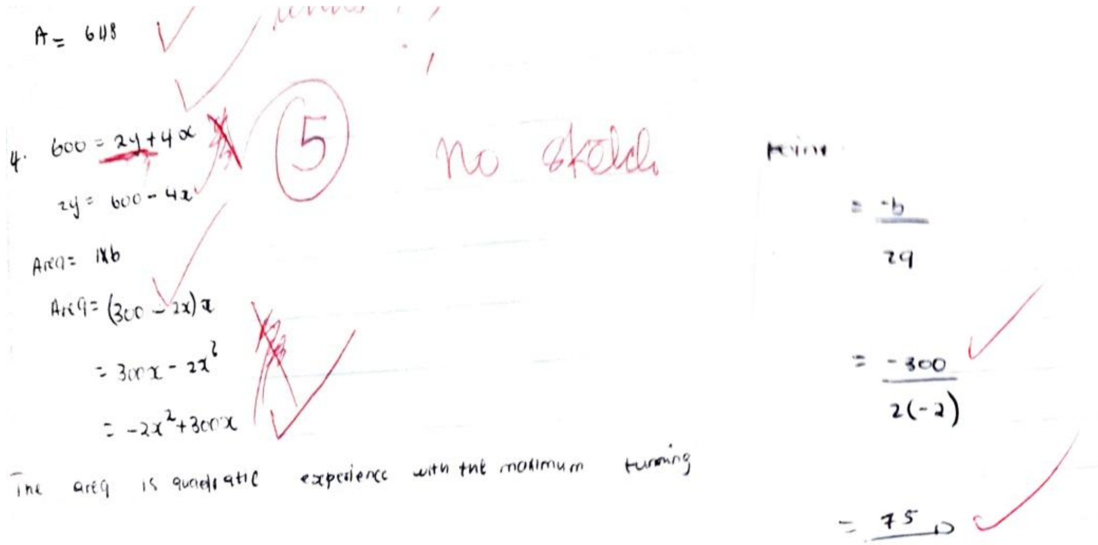


Figure 5.20: Learner L40's response to Question 4

Figure 5.21 shows a learner who performed in a similar manner to the learner in Figure 5.17. The solution was correct in all steps except the final stage of calculating the area. The learner's failure to make assumptions and to verify the solution resulted in their classification as partially competent in mathematical modelling. The error in calculating the area was mainly due to the learner not referring to the given diagram. As a result, the learner computed the area of a single camp rather than the total area of the three camps. To obtain the correct answer, the learner should have multiplied their result by three to represent the total area of the entire pasture.

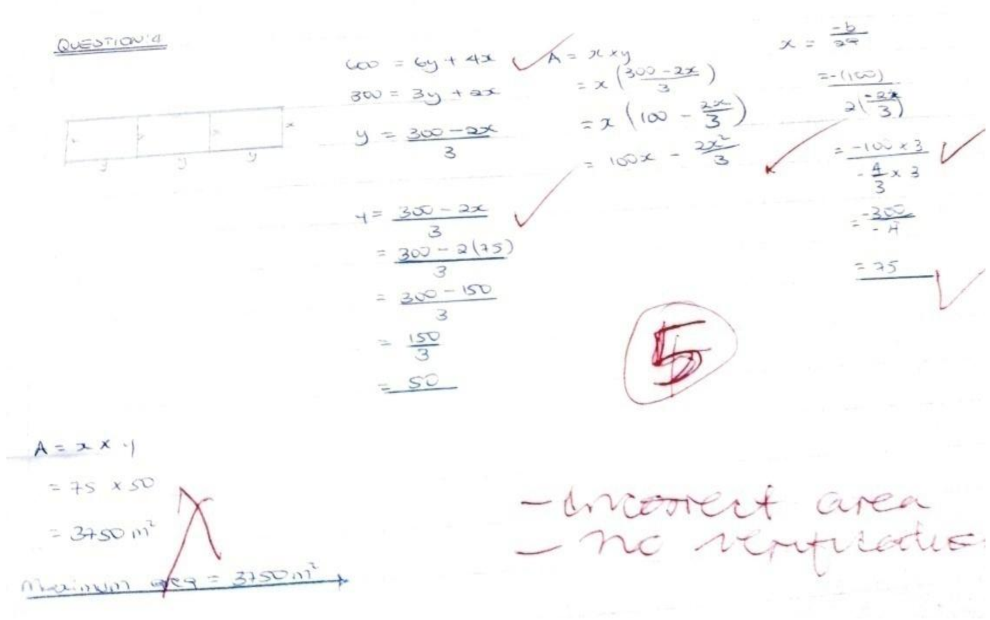


Figure 5.21: Learner L67's response to Question 4

Regarding mathematical proficiency, this learner was classified as partially proficient across the mathematical proficiency strands. This classification resulted primarily from the learner's failure to verify the solutions. Validation could have enabled the learner to recognise that the area calculated represented only a single camp rather than the entire pasture. There was no need to present all individual extracts from participants, as the emerging patterns were consistent across all analysed scripts, and no new categories emerged. An analysis of all scripts (excluding those set aside for referential adequacy—three from each school) consistently revealed similar findings. The data repeatedly showed that learners struggled with key aspects of the modelling process. Learners frequently omitted: (1) making assumptions; (2) clearly defining what each variable represented; (3) interpreting their solutions in a real-life context; (4) making general statements based on their results; and (5) validating and verifying their solutions to determine whether they made sense in the given context. The absence of new insights suggested that data saturation had been reached. Lowe, Norris, Farris, and Babbage (2018) define data saturation in qualitative research as the point at which additional data no longer yields new or relevant information in relation to the research questions. In other words, at saturation point, further analysis does not reveal new patterns or contribute to a deeper understanding of the phenomenon under investigation.

5.3 Data Analysis

It was initially difficult to identify clear themes, as it was necessary to examine each learner's script carefully to detect emerging patterns. Each script had to be reviewed multiple times. During the initial marking phase, I read through all scripts and noted instances where learners had not completed specific steps of the modelling process. These observations were recorded directly on the scripts. For example, where a learner did not assume, I wrote "no assumption made," and where solutions were not verified, I indicated "no verification of solutions."

At this stage, all omissions related to the mathematical modelling process were documented, using the five-step modelling framework as a guiding structure to maintain focus on the objectives of the study. Data analysis in this study was conducted using multiple lenses, including mathematical modelling competencies, mathematical proficiency, APOS theory, concept image and definition, and multiple representations.

The analysis of mathematical modelling competencies was carried out in three stages:

1. A detailed examination of learners' scripts, during which all actions related to the modelling process were recorded (see Table 5.2).
2. Data reduction, where each learner's performance was summarised by indicating whether each modelling step had been achieved or not (see Table 5.3).
3. Identification of themes and patterns, leading to the development of key categories.

Learners' competencies in mathematical modelling were classified using the following criteria:

- **Competent:** All five modelling steps were correctly demonstrated.
- **Partially competent:** At least three but fewer than five modelling steps were correctly demonstrated.
- **Not competent:** Fewer than three modelling steps were correctly demonstrated.

The analysis of learners' competencies was conducted on a question-by-question basis, with findings presented in Tables 5.2 to 5.5. Although this was not a comparative study, each school had to be analysed separately because the time allocated for the test in School A differed from that of Schools B and C. After this discrepancy was identified, additional time was negotiated for Schools B and C.

The analysis of Question 1 using the above criteria revealed that none of the learners were fully competent in mathematical modelling for this question. The analysis of learners' scripts was used to construct Table 5.2, where the column headings represent the five steps of the modelling process. Although all 75 scripts were analysed, only nine cases are presented in Table 5.2 for illustration purposes. Given that the analysis of nine learners already extended over several pages, including all participants for Question 1 alone would have resulted in more than 20 pages of data.

For each question, three learners per school were selected for detailed presentation, and different learners were used across Questions 1 to 4. In each case, it was determined whether learners had defined variables, made assumptions, formulated a mathematical model, solved the problem, verified their solutions, and interpreted their results in context.

Table 5.2: Portion of Analysing Question 1

Mr Nene wants to plant seedlings in his rectangular garden with an area of 54 m^2. The length of the garden is longer than its width by 3m. What are the dimensions of his garden?						
CRITERIA						
Learner	**Understanding the Problem**	**Mathematise (formulating a model)**	**Mathematical work (Solving the mathematical problem)**	**Interpreting the results**	**Validate the results**	**Competency:** ➢ Competent ➢ Partly Competent ➢ Not Competent
L1	• Did not define the variables • Could not present the situation using a sketch or diagram • Made no assumption	• Formed a correct relationship between variables • Formulated a correct equation representing the situation	• Worked out the set up equation correctly • Used mathematical knowledge to solve the problem	• Did not explain why -9 is an invalid solution • No general statements of solutions	• No validation of results • Did not check whether the product of the two obtained solutions gives an	**Partial competent**

					area of $54m^2$	
L2	• Did not define the variables • Presented the situation using a diagram • Made no assumption	• Formed a correct relationship between variables • Formulated a correct equation representing the situation	• Solved the formulated equation correctly • Used mathematical knowledge to solve the problem	• Did not explain why -9 is an invalid solution • No general statements of solutions	• No validation of results • Did not check whether the product of the two obtained solutions gives an area of $54m^2$	**Partial competent**
L3	• Did not define the variables • Presented the situation using a sketch • Made no assumption	• Formed a correct relationship between variables • Formulated a correct equation representing the situation	• Could not solve correctly the set up equation correctly • Did follow correctly the steps used to solve quadratic	• Did not explain why -9 is an invalid solution • No general statements of solutions	• No attempt was made to verify the solutions • Did not check whether the product of the two obtained solutions gives an area of $54m^2$	• **Partial competent**
L41	• Did not define the variables • Presented the situation using a diagram • Made no assumption	• Formed a correct relationship between variables • Formulated a correct equation representing the situation	• Solved the formulated equation correctly • Used mathematical knowledge to solve the problem	• Did not explain why -9 is an invalid solution • No general statements of solutions	• No validation of results • Did not check whether the product of the two obtained solutions gives an area of $54m^2$	• **Partial competent**
L42	• Did not define the variables • Could not present the situation using a sketch or diagram • Made no assumption	• Formed a correct relationship between variables • Formulated a correct equation representing the situation	• Worked out the set up equation correctly • Used mathematical knowledge to solve the problem	• Did not explain why -9 is an invalid solution • No general statements of solutions	• No validation of results • Did not check whether the product of the two obtained solutions gives an	• **Partial competent**

					area of $54m^2$	
L43	• Did not define the variables • Could not present the situation using a sketch or diagram • Made no assumption	• Formed a correct relationship between variables • Formulated a correct equation representing the situation	• Worked out the set up equation correctly • Used mathematica l knowledge to solve the problem	• Did not explain why -9 is an invalid solution • No general statement s of solutions	• No validation of results • Did not check whether the product of the two obtained solutions gives an area of $54m^2$	• **Partial competent**
L73	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Formulated a correct equation representing the situation • Could not present the situation using a sketch or diagram	• Solved the formulated equation correctly • Used mathematica l knowledge to solve the problem	• Did not explain why -9 is an invalid solution • No general statement s of solutions	• No validation of results • Did not check whether the product of the two obtained solutions gives an area of $54m^2$	• **Partial competent**
L74	• Did not define the variables • Could not present the situation using a sketch or diagram • Made no assumption	• Formed a correct relationship between variables • Formulated a correct equation representing the situation	• Solved the formulated equation correctly • Used mathematica l knowledge to solve the problem	• Did not explain why -9 is an invalid solution • No general statement s of solutions	• No validation of results • Did not check whether the product of the two obtained solutions gives an area of $54m^2$	• **Partial competent**
L75	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Formulated a correct equation representing the situation • Could not present the situation using a sketch or diagram	• Solved the formulated equation correctly • Used mathematica l knowledge to solve the problem	• Did not explain why -9 is an invalid solution • No general statement s of solutions	• No validation of results • Did not check whether the product of the two obtained solutions gives an area of $54m^2$	• **Partial competent**

Despite the qualitative nature of the study, it was necessary to use simple numerical summaries to illustrate how learners from the three schools performed in Question 1. Although none of the learners were able to define variables or make assumptions, their ability to construct a correct mathematical model was nevertheless considered as partial evidence of understanding the problem. Table 5.3 summarises learners' performance in this regard and shows that none of the learners successfully completed all five stages of the modelling process. Based on these findings, I concluded that the participants were only partially competent in solving the problem using mathematical modelling processes.

Table 5.3: Mathematical modelling competencies for Question 1

QUESTION 1 ANALYSIS						
CRITERIA						
	Understanding the Problem	**Mathematise (formulating a model)**	**Mathematical work (Solving the mathematical problem**	**Interpreting the results**	**Validate the results.**	**Competency:** Competent (C) ➢ Partly Competent (PC) ➢ Not Competent (NC)
Number of Learners from School A = **30**	28	28	26	0	1	**0 (C)** **26 (PC)** **4(NC)**
Number of Learners from School B = **25**	25	25	25	5	0	**0 (C)** **25 (PC)** **0 (NC)**
Number of Learners from School C = **20**	20	20	20	0	0	**0 (C)** **20 (PC)** **0 (0NC)**

A detailed analysis of the results revealed the following:

• **School A:** Only one learner managed to verify the solution but did not interpret it. As a result, no learner successfully completed all five steps of the modelling process. Twenty-six learners demonstrated performance in at least three but fewer than five steps, indicating that they were partly competent. Further analysis showed that four learners were not competent, as they correctly completed fewer than three steps.

• **School B:** Twenty learners successfully completed exactly three steps of the modelling process. Five learners completed four steps correctly. This implies that all 25 learners in this school were partly competent, as none managed to complete all five steps successfully.

• **School C:** All learners managed to complete only the first three steps, meaning they progressed only up to the problem-solving stage but did not proceed to verification or validation of their solutions. Consequently, all learners in this school were also classified as partly competent.

The same procedure used for the analysis of Question 1 was applied to Question 2. This involved recording each learner's actions at every stage of the modelling process. Table 5.4 presents a portion of the findings from the analysis of Question 2. Although the full dataset covered all 75 learners, it was not feasible to include all scripts due to space constraints. Therefore, only nine scripts were presented per question, with three selected from each school. From Table 5.4, it is evident that no learner successfully completed all five stages of the modelling process.

Table 5.4: Portion of Analysing Question 2

QUESTION 2 ANALYSIS						
In order to protect his vegetables, a farmer buys 32m of fencing for his rectangular garden. Determine the dimensions of the vegetable garden if it should cover a maximum area						
CRITERIA						
Learner	**Understanding the Problem**	**Making assumptions and mathematise (formulating a model)**	**Mathematical work (Solving the mathematical problem)**	**Interpreting the results**	**Validate the results**	**Competency:** ➢ Competent ➢ Partly Competent ➢ Not Competent

L6	• Did not define the variables • Could not form a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Could not present the situation using a sketch or diagram • Formulated an incorrect mathematical equation for the perimeter	• Could not proceed further after failing to make a correct formula for the perimeter •	• No interpretation meaning of the results since the learner failed to formulate the equations of the perimeter and area	• No validation of results as there were no solutions found	• **Partial competent**
L7	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Formulated a correct mathematical equation for the perimeter • Could not present the situation using a sketch or diagram	• Solved the formulated equation correctly • Used mathematical knowledge to solve the problem	• No general statements of solutions	• No validation of results ▪ Did not consider alternative ways of solving the problem	• **Partial competent**
L8	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Formulated a correct mathematical equation for the perimeter • Could not present the situation using a sketch or diagram	• Solved the formulated equation correctly • Used mathematical knowledge to solve the problem	• No general statements of solutions	• No validation of results ▪ Did not consider alternative ways of solving the problem	• **Partial competent**
L37	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Formulated a correct mathematical equation for the perimeter	• Could not solve the formulated equation of the area to get dimensions • Did not calculate the area	• No general statements of solutions	• No validation of results ▪ Did not consider alternative ways of solving the problem	• **Partial competent**

		• Could not present the situation using a sketch or diagram	• Used mathematical knowledge to solve the problem			
L38	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Formulated a correct mathematical equation for the perimeter • Could not present the situation using a sketch or diagram	• Solved the formulated equation correctly to get only the dimensions • Did not calculate the area • Used mathematical knowledge to solve the problem	• No general statements of solutions	• No validation of results • No validation of results ▪ Did not consider alternative ways of solving the problem	• **Partial competent**
L39	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Formulated a correct mathematical equation for the perimeter • Could not present the situation using a sketch or diagram	• Solved the formulated equation correctly to get only the dimensions • Did not calculate the area • Used mathematical knowledge to solve the problem	• No general statements of solutions	• No verification of results • Did not consider alternative ways of solving the problem	• **Partial competent**
L56	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Formulated a correct mathematical equation for the perimeter • Could not present the situation using a sketch or diagram	• Solved the formulated equation correctly to get only the dimensions • Did not calculate the area • Used mathematical knowledge to solve the problem	• No general statements of solutions	• No verification of results • Did not consider alternative ways of solving the problem	• **Partial competent**
LL57	• Did not define the variables • Formed a correct	• Selected appropriate symbols or variables	• Did not work out the problem beyond writing the	• No general statements of solutions	• No verification of results	• **Partial competent**

	relationship between variables • Made no assumption	• Formulated a correct mathematical equation for the perimeter • Could not present the situation using a sketch or diagram	quadratic expression for the area • Did not completely work out the solutions (dimensions)		• Did not consider alternative ways of solving the problem	
L58	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Could not present the situation using a sketch or diagram • Formulated a correct mathematical equation for the area	• Used the formula for finding the maximum turning point of the quadratic formula formed • Correctly calculated the dimensions of the garden	• No interpretation or explanation of the meaning of the results • Did not make a general statement of solutions that were obtained	• No validation of results • Did not test the solutions to check if they would give the perimeter of 32m	• **Partial competent**

Again, as in Question 1, a simplified analysis of Question 2 was conducted and summarised in Table 5.5. The findings indicated that none of the learners were fully competent in mathematical modelling, as none were able to correctly complete all five steps of the modelling process.

Table 5.5: Mathematical modelling competencies for Question 2

QUESTION 2 ANALYSIS						
CRITERIA						
	Understanding the Problem	**Mathematise (formulating a model)**	**Mathematical work (Solving the mathematical problem**	**Interpreting the results**	**Validate the results.**	**Competency:** ➢ Competent (C) ➢ Partly Competent (PC) ➢ Not Competent (NC)
Number of Learners from School	22	19	20	0	0	**0 (C)** **19 (PC)** **11 (NC)**

A = **30**						
Number of Learners from School B = **25**	25	25	18	0	2	**0 (C)** **20 (PC)** **5 (NC)**
Number of Learners from School C = **20**	20	20	14	1	0	**0 (C)** **15 (PC)** **5 (NC)**

School A: No learner was competent in mathematical modelling, as none of them successfully completed all five steps of the modelling process. Nineteen learners were partially competent, as they correctly completed three steps, while only one learner managed to complete four steps successfully. This further indicated that eleven learners were not competent, as they completed fewer than three steps correctly.

• **School B:** Only two learners were able to complete four steps of the modelling process, while eighteen learners correctly completed exactly three steps. This implies that five learners were not competent, as they completed fewer than three steps correctly. The two learners who completed four steps correctly failed only in interpreting their results.

• **School C:** No learner was competent in mathematical modelling, meaning that none successfully completed all five steps of the modelling process. One learner progressed as far as interpreting the result but failed to validate the solution. Fifteen learners completed more than three but fewer than five steps correctly and were therefore classified as partially competent. Five learners were not competent, as they completed fewer than three steps correctly.

Table 5.6 presents the results of the analysis of learners' performance in Question 3. As in Question 1, no learner completed all five steps of the modelling process correctly. As previously noted, learners in School A did not complete the test within the allocated time. Although 14 learners attempted this question, they were unable to complete any step of the modelling process correctly. It is evident that time constraints affected their ability to engage with the task. However, considering that these learners also struggled in earlier questions, the issue of time alone cannot fully explain their performance. Consequently, only data from Schools B and C were used for detailed analysis and reporting.

The patterns observed in Questions 1 and 2 were also evident in Question 3 and the remaining questions. Learners consistently failed to define the variables used in formulating their mathematical models. They also struggled to make appropriate assumptions that could support correct solutions. For example, a valid assumption could have been that the land is flat or that there are no physical obstacles, such as trees with large trunks, along the sides to be fenced. In addition, some learners demonstrated a lack of understanding of the problem, as reflected in their sketches of the garden. They were unable to correctly indicate the side already covered by a rock wall.

Table 5.6: Portion of Analysing Question 3

<table>
<tr><th colspan="7">QUESTION 3 ANALYSIS</th></tr>
<tr><td colspan="7">A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 72m of wire fencing:
• Determine the dimensions that would create a garden of maximum area.
• What is the maximum area?</td></tr>
<tr><td colspan="7">CRITERIA</td></tr>
<tr><td>Learner</td><td>Understanding the Problem</td><td>Making assumptions and mathematise (formulating a model)</td><td>Mathematical work (Solving the mathematical problem)</td><td>Interpreting the results</td><td>Validate the results</td><td>Competency:
➢ Competent
➢ Partly Competent
➢ Not Competent</td></tr>
<tr><td>L16</td><td>• Did not define the variables
• Could not form a correct relationship between variables
• Made no assumption
• The learner did not understand the problem</td><td>• Could not develop a correct equation representing the situation where one side of the garden was already protected by a rock
• Could not present the situation using a sketch or diagram</td><td>• Could not apply mathematical knowledge and strategies to solve the problem
• Could not proceed further to determine the solutions
• No correct solutions were obtained</td><td>• No general statement of solutions was made since the learner could not proceed to get the solutions</td><td>• No verification of results was made since no solutions were obtained</td><td>• Not competent</td></tr>
</table>

L17	Learner did not attempt this question					
L18	• Did not define the variables • Could not form a correct relationship between variables • Made no assumption • The learner did not understand the problem	• Could not develop a correct equation representing the situation where one side of the garden was already protected by a rock • Could not present the situation using a sketch or diagram	• Could not apply mathematical knowledge and strategies to solve the problem • Could not proceed further to determine the solutions • No correct solutions were obtained	• No general statement of solutions was made since the learner could not proceed to get the solutions	• No verification of results was made since no solutions were obtained	• **Not competent**
L38	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Formulated a correct mathematical equation for the area • Could not present the situation using a sketch or diagram	• Could not determine the x coordinate of the maximum turning point as one of the dimension Solved the • Did not calculate the area	• No general statements of solutions has been given	• No verification of results • Did not consider alternative ways of solving the problem	• **Partial competent**
L39	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Formulated a correct mathematical equation for the perimeter and area • Could not present the situation using a sketch or diagram	• Solved the formulated equation correctly to get both the values of x and y • Did not calculate the area • Used mathematical knowledge to certain extent solve the problem	• No general statements of solutions given	• No verification of results • Did not consider alternative ways of solving the problem	• **Partial competent**
L40	• Did not define the variables • Formed a correct relationship	• Selected appropriate symbols or variables • Formulated a correct	• Solved the formulated equation correctly to get the value of x but	• No general statements of solutions given	• No verification of results	• **Partial competent**

	between variables • Made no assumption	mathematical equation for the perimeter and area • Presented the situation using a sketch or diagram	could not proceed to get the value of *y* • Calculated the area • Used mathematical knowledge to certain extent solve the problem		• Did not consider alternative ways of solving the problem	
L59	• Did not define the variables • Could not form a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Could not formulate a correct mathematical equation for the perimeter and area • Could not present the situation using a sketch or diagram	• Could not solve So the formulated equation • Could not calculate the area • Could not use mathematical knowledge to solve the problem	• No general statements of solutions	• No verification of results • Did not consider alternative ways of solving the problem	• **Not competent**
L60	• Did not define the variables • Formed a correct relationship between variables • Made no assumption	• Selected appropriate symbols or variables • Formulated correct mathematical equations for the perimeter and the area • Could not present the situation using a sketch or diagram	• Solved the formulated equation correctly to get only the *x* value. Could not work out the value of *y* • Did not calculate the area • Used mathematical knowledge to solve the problem	• No general statements of solutions	• No verification of results • Did not consider alternative ways of solving the problem	• **Partial competent**
L61	Did not attempt this question					

Another notable observation was that learners generally did not complete the final two stages of the modelling process, namely interpretation of results and verification of solutions. In most cases, learners were able to formulate a correct mathematical model and proceed to solve it but failed to interpret or validate their answers.

As with the previous questions, a summary of the analysis for Question 3 was necessary. The findings are presented in Table 5.7. In examining learners' scripts to determine their competencies in mathematical modelling, the following observations were made:

- School B: No learner successfully completed all five steps of the modelling process. Only 11 learners completed at least three but fewer than five steps correctly. According to the competency criteria used in this study, these learners were classified as partially competent. Furthermore, 14 learners were not competent in mathematical modelling, as they completed fewer than three steps correctly.

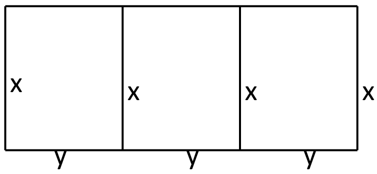
Table 5.7: Mathematical modelling competencies for Question 3

QUESTION 3 ANALYSIS						
CRITERIA						
	Understanding the Problem	**Mathematise (formulating a model)**	**Mathematical work (Solving the mathematical problem**	**Interpreting the results**	**Validate the results.**	**Competency:** ➢ Competent (C) ➢ Partly Competent (PC) ➢ Not Competent (NC)
Number of Learners from School A = **30**	0	0	0	0	0	**0 (C)** **0 (PC)** **30 (NC)**
Number of Learners from School B = **25**	24	24	7	4	0	**0 (C)** **11 (PC)** **14 (NC)**
Number of Learners from School C = **20**	15	15	11	0	0	**0 (C)** **11 (PC)** **9 (NC)**

- **School C:** No learners successfully completed all five steps of the modelling process accurately. This indicates that, according to the competency criteria used in this study, no learner was classified as competent in mathematical modelling. Furthermore, 11 learners were classified as partially competent, while 9 learners were not competent, as they completed fewer than three steps correctly.

Findings from the analysis of learners' performance in Question 4 are presented in Table 5.8. No learners successfully completed any step of the modelling process correctly. As in Question 3, very few learners attempted this question, and those who did generally performed poorly. Overall, this question was poorly attempted across all schools. The weak performance may be attributed to learners' inability to correctly identify the sides that needed to be fenced. In addition, as with previous questions, learners struggled with defining variables, making assumptions, and interpreting results, particularly in formulating general statements based on their solutions. A major difficulty was the incorrect formulation of the mathematical expression for the perimeter. This led to further errors in determining the area. Most learners considered only the dimensions enclosing the entire pasture and failed to account for the internal divisions that created smaller camps within the pasture.

Table 5.8: Portion of Analysing Question 4

<table>
<tr><td colspan="7">QUESTION 4 ANALYSIS</td></tr>
<tr><td colspan="7">A farmer has 600m of fence to create a rectangular pasture which has to be divided into three camps all with the same dimensions. What is the maximum area that can be enclosed with this fence?

</td></tr>
<tr><td colspan="7">CRITERIA</td></tr>
<tr><td>Learner</td><td>Understanding the Problem</td><td>Making assumptions and mathematise (formulating a model)</td><td>Mathematical work (Solving the mathematical problem)</td><td>Interpreting the results</td><td>Validate the results</td><td>Competency:
➢ Competent
➢ Partly Competent
➢ Not Competent</td></tr>
<tr><td>L6</td><td>• Did not define the variables
• The learner did not</td><td>• Could not form a correct relationship between variables</td><td>• Could not apply mathematical knowledge and strategies to</td><td>• No general statement of solutions was</td><td>• No verification of results was made since no solutions</td><td>• Not competent</td></tr>
</table>

	understand the problem • Made no assumption	• Could not develop a correct equation representing the situation	solve the problem since • No correct solutions were obtained	made since the learner did not solve the incorrect formula he/she has made	were obtained	
L7	• Learner did not attempt this question					
L9	• Did not define the variables • The learner did not understand the problem • Made no assumption	• Could not form a correct relationship between variables • Could not develop a correct equation representing the situation	• Could not apply mathematical knowledge and strategies to solve the problem since • No correct solutions were obtained	• No general statement of solutions was made since the learner did not solve the incorrect formula he/she has made	• No verification of results was made since no solutions were obtained	• **Not competent**
L36	• Did not define the variables • The learner did not understand the problem • Made no assumption	• Formed an incorrect relationship between variables • Formulate an incorrect mathematical equation for the area based on the incorrect formulated perimeter (worked consistent)	• Solved the formulated equation correctly to get the value of x but could not proceed to get the value of y which were based on the • Did not calculate the area of the pasture • Used mathematical knowledge to solve the problem	• General statement of solutions has not been given	• Solutions were verified by adding the lengths and width and they added to 600m	• **Partial competent**
L37	• Did not define the variables • The learner did not understand the problem • Made no assumption	• Could not form a correct relationship between variables • Could not formulate a correct mathematic	• Could not solve the incorrect formulated equation consistently and correct y • Did not calculate the area	• No general statements of solutions	• No verification of results • Did not consider alternative ways of solving the problem	• **Not competent**

		al equation for the perimeter and area	• Could not use mathematical knowledge clearly.			
L38	• Did not define the variables • Made no assumption	• Formed a correct relationship between variables • Formulate a correct mathematical equation for the area	• Solved the formulated equation correctly to get the value of *x* and *y* • Did not calculate the area of the pasture • Used mathematical knowledge to solve the problem	• No general statements of solutions has been given	• No verification of results • Did not consider alternative ways of solving the problem	• **Partial competent**
L60	• Did not define the variables • Made no assumption	• Formed an incorrect relationship between the variables • Formulated an incorrect mathematical equation for the perimeter and area	• Worked consistently only to find the equation of length y • Did not proceed further • Used mathematical knowledge to certain extent to solve the problem	• No general statements of solutions has been given	• No verification of results • Did not consider alternative ways of solving the problem	• **Partial competent**
L61	Learner did not attempt this question					
L62	• Did not define the variables • Made no assumption	• Formed a correct relationship between variables • Formulated a correct mathematical equation for the perimeter and area • Could not present the situation using a sketch or diagram	• Solved the formulated equation correctly to get the value of *x* but could not proceed to get the value of *y* • Did not calculate the area of the pasture • Used mathematical knowledge to certain	• No general statements of solutions has been given	• No verification of results • Did not consider alternative ways of solving the problem	• **Partial competent**

			extent to solve the problem			

It was surprising that learners failed to consider the sides dividing the pasture into camps, even though a diagram or sketch of the pasture was provided. Further analysis of learners' scripts led to the development of Table 5.14. The analysis of Question 4 revealed that, although only five learners from School A attempted this question, none were able to correctly complete even one step of the modelling process.

Further observations from the analysis of learners' scripts from Schools B and C showed that:

- School B: Only one learner correctly completed four steps of the modelling process, while another learner correctly completed three steps. This means that only two learners achieved more than three steps successfully and were therefore classified as partially competent. Consequently, 23 learners were not competent, as they were unable to complete more than two steps correctly.

- School C: Only five learners correctly completed three steps of the modelling process and were therefore classified as partially competent based on the competency criteria used in this study. No learner successfully completed all five steps, meaning that no learner in this school was competent in this question. Furthermore, 15 learners were not competent, as they were unable to complete at least three steps correctly.

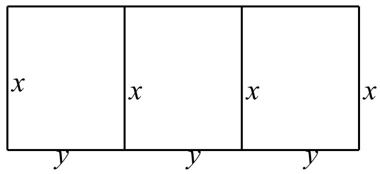
Table 5.9: Mathematical modelling competencies for Question 4

QUESTION 4 ANALYSIS						
CRITERIA						
	Understanding the Problem	**Mathematise (formulating a model)**	**Mathematical work (Solving the mathematical problem**	**Interpreting the results**	**Validate the results.**	**Competency:** ➢ Competent (C) ➢ Partly Competent (PC) ➢ Not Competent (NC)
Number of Learners from School A = **30**	0	0	0	0	0	**0 (C)** **0 (PC)** **0 (NC)**
Number of Learners from School B = **25**	7	7	2	0	1	**0 (C)** **2 (PC)** **23 (NC)**
Number of Learners from School C = **20**	15	14	5	0	0	**0 (C)** **5 (PC)** **15 (NC)**

It can be observed from Tables 5.3, 5.5, 5.7, and 5.9 that none of the learners were competent in mathematical modelling, as no learner successfully completed all the steps of the modelling process correctly. Most learners did not define variables, made no assumptions, did not interpret their results, and did not verify their solutions. However, many learners were able to formulate correct mathematical equations or expressions representing the given scenarios and proceeded to solve them using appropriate mathematical procedures. The poor performance in the test was not solely due to learners' failure to follow the steps of the modelling process but also resulted from a combination of errors across different stages of the process. Table 5.10 provides a summary of some of the common errors made by learners, which ultimately led to the loss of marks and contributed to their lack of competence in mathematical modelling.

Table 5.10: Summary of errors that led to poor performance in this study

Type of Errors	**Explanations**
Mathematical errors caused by working carelessly	These errors include mistakes that are called slips. It was evident that these slips resulted from excitement or learners rushing to finish the Test. For example in Question 1 a mathematical error occurred when a learner tried to multiply $x+3$ by x but omitted the brackets in $x+3$ (See Figure 5.2). Another learner made a mistake while solving the formulated equation. He/she omitted a zero at a certain step of the working and this made the working to be mathematically incorrect. By omitting the zero the learner ended up with the expression $b^2+3b-54$ instead of the equation $b^2+3b-54=0$ (See Figure 5.4)
Not defining variables	None of the participants defined the variables they have used in their working. For example, in Question 1, most learners used *l* and *b* for length and breadth or width, though there were some who used w for width. A proper way for Question 1 was to say: Let *l* be the length and *b* the breadth. In expressing the known relationships in equation form, variables are used. it is then required to define the variables utilised in the equations and state what they represent in the real world.
Failure to make assumptions	In this study most learners managed to formulate a mathematical model that represented the situation but none of them made any assumption, which is mandatory in mathematical modelling. Not a single learner who took part in this study tried to make assumptions in any of the questions. In Chapter Two, we discussed the importance and role of assumptions.
Incomplete solutions	Some learners did not complete working out or solving the problem fully. In Question 2 up to 4, learners were expected to give two values as the solutions but some once obtaining the first solution they

	thought they have finished. For example, in Question 3 learners had to calculate dimensions that would give the maximum area, some learners only calculated the size of one side instead of two.
Wrong interpretation of the diagram in Question 4	Some learners demonstrated the lack of understanding of the properties of a rectangle. They did not consider the other side that was not labelled was also 3y because opposite sides of a rectangle are equal. They only equated the given sides to 600m. The diagram is given below.  The perimeter of this figure was supposed to be $4x+6y$ instead of $4x+3y$. This was an indication that these learners did not understand the properties of a rectangle.
Formulating of incorrect mathematical model because of failing to understand the situation or problem	In Question 3, it was stated that only three sides of the rectangular garden were to be fenced since the fourth side was covered by a rock. Some learners showed that they had not understood or comprehended the problem very well as they included the four sides of the garden instead of the three sides. This resulted in a wrong mathematical model (see Figure 5.14).
Failure to solve the formulated mathematical model	Some learners formulated correct mathematical model or formula that represented the given situation. However, they could not proceed correctly up to the end to get the correct solution (see Figure 5.17).
Failure to verify or validate their solutions	Most of the learners did not verify their obtained solutions to check if they do satisfy the given situation. Validation in mathematical modelling occurs after a problem solver has obtained the solutions and then tries to check the reasonableness and logic of his or her solutions. For example, if they had validated their solution in Question 1, they could have mentioned that -9m was an invalid

	solution because no dimension dimensions of any real figure or object can be negative. Verification or validation of results is crucial since it enables learners to ensure that the answers they have obtained are accurate.
No interpretations of results	Majority of learners did not interpret their results. They managed to formulate mathematical models and solved them but left their answers still attached to variables. For example, in Question 1, some learners left their solution as $b = 6$ and $l = 9$. In interpreting the results one would expect the learner to give the answers as: length = 9 metres and the breadth = 6 metres. What made the situation worse was that learners did not define the variables at the beginning. Interpreting the results is crucial because it enables learners to make the connection between the problem scenario and the solutions they have acquired. Using this technique can assist learners avoid making mistakes in reporting their solutions or forgetting to include units in their answers.
Failure to link or relate their solution to the real life context	Majority of learners in this study failed to link or relate their solution to the real life. This is seen by the failure of majority of learners to consider –9m as an invalid solution because no dimensions of any real figure or object can be negative. In real life the length of an object is positive.
Failing to follow all the modelling process or steps	Most learners faced problems in following all the steps of the modelling process. Analysis of results showed that majority of learners were able to do the first three phases (comprehending the problem, developing a mathematical representation, and resolving a mathematical problem) but failed to do the last two steps, the interpretation and validating of the results.

5.4 Analysis of learners' mathematical proficiencies

Although the primary aim of this investigation was to analyse learners' mathematical modelling competencies, it was also necessary to examine their mathematical proficiencies due to the close relationship between mathematical modelling processes and mathematical proficiency. Table 5.11 highlights some of the key overlapping elements between the two constructs. From Table 5.11, it is evident that deficiencies in mathematical proficiency negatively impact learners' modelling competencies. For example, a learner who lacks strategic competence is likely to struggle to formulate an appropriate mathematical expression or model that accurately represents the given situation.

Table 5.11: Similarities between mathematical modelling processes and mathematical strands

Mathematical modelling processes	Mathematical strands (Mathematical proficiency)	Common features
Understanding the problem	Conceptual understanding	Both demand that a learner must understand the problem and all mathematical concepts involved
Mathematisation (formulation of a mathematical model	Strategic competence	Both involve formulation of mathematical equations, or expressions
Mathematical work (Solving the mathematical problem)	Procedural fluency	Both refer to the correct procedures to be followed in solving the problem
Results interpretation, Validation of results	Adaptive reasoning	All of them pertain to matters related to reasoning reflection, explanation, validation and justification

Source: Adapted from Kilpatrick et al. (2001) and Kaiser & Stender (2013)

To determine learners' mathematical proficiency, a structured grid was used. A portion of this grid is presented in Table 5.12. In the analysis of learners' scripts, attention was given to the mathematical proficiencies demonstrated when solving real-life contextual problems. In each script, focus was placed on the four strands of mathematical proficiency, namely conceptual understanding, strategic competence, procedural fluency, and adaptive reasoning. The fifth strand, productive disposition, was not assessed in this study due to the nature of the data collection instrument (a written test).

As explained in Chapter Three, productive disposition refers to learners' tendency to recognise the value and usefulness of Mathematics. Developing this strand requires sustained exposure to experiences that enable learners to appreciate Mathematics as meaningful and worthwhile. A single written test, such as the one used in this study, is therefore not an appropriate instrument for assessing learners' attitudes or dispositions towards Mathematics. For this reason, the analysis focused only on the other four strands of mathematical proficiency.

Table 5.12 presents an analysis of the scripts of three learners for Question 1. These learners were selected to illustrate their mathematical proficiencies in this question. A learner with the highest score was selected from each school to ensure that sufficient and meaningful information could be obtained. However, no learner from School A was able to complete the test, as they could not progress beyond Question 2 due to time constraints. The initial allocated time of 30 minutes was insufficient; consequently, additional time was negotiated for the remaining two schools to ensure that learners could complete the test.

Table 5.12: Analysis of learners' scripts for Question 1

Learner	Mathematical strand	Proficiency	Comprehensive Description	Overall Proficiency
L25	Conceptual Understanding	Proficient	Learner understood the problem very well as seen from the established relations and sketch drawn	Partly Proficient
	Strategic Competence	Proficient	Formulated the correct quadratic equation representing the situation	
	Procedural Fluency	Proficient	Learner followed the procedures of calculating the dimensions correctly	

	Adaptive Reasoning	Not Proficient	The learner did not justify why -9 was not a valid solution. Also the learner did not verify the solutions	
L40	Conceptual Understanding	Proficient	Learner understood the problem very well as seen from the established relations and sketch drawn	Partly Proficient
	Strategic Competence	Proficient	Formulated the correct quadratic equation representing the situation	
	Procedural Fluency	Proficient	Learner followed the procedures of calculating the dimensions correctly	
	Adaptive Reasoning	Not Proficient	The learner did not justify why -9 was not a valid solution. Also the learner did not verify the solutions	
L67	Conceptual Understanding	Proficient	Learner comprehended the problem very well as revealed by the development of correct labelled diagram and correct formula representing the area	Partly Proficient
	Strategic Competence	Proficient	Correct equation was formulated that illustrated that the length was 3m longer than the width	
	Procedural Fluency	Proficient	Correct procedures were followed on how to solve the formulated quadratic equation	
	Adaptive Reasoning	Not Proficient	No explanation was given why -9 was the invalid solution and no verification was made to show that the calculated values gives the product 54 and one side was 3m longer than the other	

The three mathematical strands of conceptual understanding, strategic competence, and procedural fluency were the only areas in which learners demonstrated proficiency, as shown in Table 5.12. None of the learners were able to verify whether their calculated solutions were appropriate for the given context. A simple verification strategy, for example, would have been to multiply the two obtained values to check whether their product equalled 54 m^2.

Thus, in Question 1, learners demonstrated proficiency in three out of the four mathematical proficiency strands used in this study and were therefore classified as partially proficient. Across all scripts, no learner engaged in verification of their solutions. However, a few learners were able to explain why −9 metres was not a valid solution. These learners were nevertheless classified as not proficient in adaptive reasoning.

The analysis of learners' scripts for Question 2 is presented in Table 5.13. Like Question 1, learners demonstrated proficiency in procedural fluency, strategic competence, and conceptual understanding, but they did not demonstrate proficiency in adaptive reasoning. None of the learners addressed the adaptive reasoning strand.

Table 5.13: Analysis of learners' scripts for Question 2

Learner	Mathematical strand	Proficiency	Comprehensive Description	Overall Proficiency
L25	Conceptual Understanding	Proficient	Learner comprehended the problem very well as seen from relationships formed	Partly Proficient
	Strategic Competence	Proficient	Developed the correct formula for perimeter of the garden	
	Procedural Fluency	Proficient	Correct procedures were followed in determining the dimensions of the garden	
	Adaptive Reasoning	Not Proficient	Learner did not verify the solutions to see if calculated dimensions gives a perimeter of 32m	
L40	Conceptual Understanding	Proficient	Learner comprehended the problem very well as seen from relationships formed	Partly Proficient
	Strategic Competence	Proficient	Correct procedures were followed in determining the dimensions of the garden	
	Procedural Fluency	Proficient	Developed the correct formula for perimeter of the garden	
	Adaptive Reasoning	Not Proficient	Learner did not verify the solutions to see if calculated dimensions gives a perimeter of 32m	
L67	Conceptual Understanding	Proficient	Correct sketch and correct expression for perimeter were the indicators that the learner understood the problem	Partly Proficient
	Strategic Competence	Proficient	Correct expression for the perimeter and area were formulated	
	Procedural Fluency	Proficient	Learner carried out procedures appropriately and correctly. No mistakes were made	
	Adaptive Reasoning	Not Proficient	No verification or validations of the solutions were done. Solutions were not checked to see if they satisfied the expressions. At least they could have verified their	

			solutions to check if their solutions would give the perimeter of the garden to be 32 metres	

Since mathematical proficiency was assessed using four strands that could be identified through the written test, a fully proficient learner was defined as one who demonstrated competence across all four strands.

Table 5.14: Analysis of learners' scripts for Question 3

Learner	Mathematical strand	Proficiency	Comprehensive Description	Overall Proficiency
L25	Conceptual Understanding	Could not be determined as no attempt was made	No learner from School A attempted this question as they could not finish because of time	Could not be decided as the learner could not attempt the question due to lack of time
	Strategic Competence			
	Procedural Fluency			
	Adaptive Reasoning			
L40	Conceptual Understanding	Proficient	Learner comprehended the problem very well as seen from correct expression of an area	Partly Proficient
	Strategic Competence	Proficient	Developed the correct expression for the maximum area	
	Procedural Fluency	Proficient	Correct procedures were followed in determining the dimensions of the garden	
	Adaptive Reasoning	Not Proficient	Learner did not verify or confirm whether his/her solutions up to 72m	
L67	Conceptual Understanding	Proficient	Learner understood the problem perfectly as seen from correct expression of an area	Partially Proficient
	Strategic Competence	Proficient	Formulated the correct formula for perimeter and expression for the maximum area	
	Procedural Fluency	Proficient	Correct procedures were followed in determining the dimensions of the garden	

	Adaptive Reasoning	Not Proficient	Learner did not verify the solutions to see if the three calculated dimensions add up to 72m	

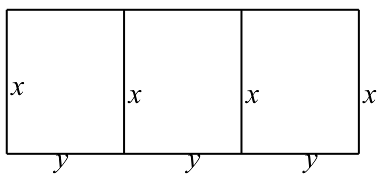
The situation presented in Table 5.9 is like that observed in the two preceding tables. Verification of solutions remained a significant challenge for learners. In this problem, learners were given 72 metres of fencing to be used on three sides of a rectangular garden to maximise the area. As with the previous questions, learners demonstrated proficiency in the first three mathematical strands. However, none of the participants verified whether their calculated dimensions correctly added up to 72 metres. The absence of verification and justification of solutions indicates that learners were not proficient in adaptive reasoning.

Table 5.15: Analysis of learners' scripts for Question 4

Learner	Mathematical strand	Proficiency	Comprehensive Description	Overall Proficiency
L25	Conceptual Understanding	Could not be determined as no attempt was made	No learner from School A attempted this question as they could not finish because of time	Could not be decided as the learner could not attempt the question due to lack of time
	Strategic competence			
	Procedural Fluency			
	Adaptive Reasoning			
L40	Conceptual Understanding	Proficient	Learner comprehended the problem very well as seen from correct expression of the length of the fence that can be used to divide the pasture as well as the expression for the area	Partially Proficient
	Strategic Competence	Proficient	Developed the correct expressions for the length of the fence and the area that can be enclosed	
	Procedural Fluency	Proficient	Correct procedures were followed in determining the area that can be enclosed with the given fence	
	Adaptive Reasoning	Not proficient	Learner did not verify the solutions to see if calculated dimensions add up to 600m	

L67	Conceptual Understanding	Proficient	Correct expression for the length of the fence was formulated as an indication that the problem had been understood	Partially Proficient
	Strategic Competence	Proficient	Correct expression for the length of the fence and of the area were correctly formulated	
	Procedural Fluency	Partially Proficient	Learner solved the problem correctly to a certain extent but could not determine the maximum area as the last step	
	Adaptive Reasoning	Not Proficient	No verification was made to show that 6y+4x would give 600m if x was 75m and y was 50m	

It can be concluded with confidence that the learners who participated in this study were generally partially competent. The main strand in which they struggled was adaptive reasoning. Although they were able to obtain solutions, they did not attempt to verify whether these solutions were reasonable or satisfied the given context. Overall, Question 4 was poorly performed by most learners. This may be attributed to its higher cognitive demand compared to the first three questions. Evidence from learners' responses suggests that some did not fully comprehend the problem and, as a result, were unable to formulate correct mathematical expressions. Figure 5.10 illustrates an example of a learner who was unable to complete this question correctly. Learners were presented with a diagram and were required to determine the maximum area that could be enclosed by a fence measuring 600 metres in total length. The learner in Figure 5.10 could see that the total length was 3y but failed to add the other side that was also 3y to get the expression of whole fence as (6y + 4x) metres.



There were no candidates who successfully demonstrated all four mathematical proficiency strands. This means that none of the participants were fully proficient according to the criteria used in this study.

5.5 Analysis of data using three philosophical frameworks

Data for this study were also analysed using the philosophical and theoretical frameworks outlined in Chapter Three, namely APOS theory (actions, processes, objects, and schema), concept image and concept definition, and multiple representations. The use of these three frameworks was intended to enhance the interpretation of the data and, consequently, provide a deeper understanding of the learners' thinking processes from three complementary perspectives.

5.5.1 Analysing data using APOS Theory

The APOS framework describes the development of understanding in Mathematics through the hierarchical construction of mental structures, namely actions, processes, objects, and schema. In this study, the theory was used to analyse the data to infer the possible cognitive processes that took place in the minds of Grade 11 learners as they solved mathematical problems involving area, perimeter, and optimisation. Using APOS theory, learners' scripts were analysed with reference to the criteria presented in Tables 5.16 and 5.17. Table 5.16 was used for the analysis of Question 1, while Table 5.17 was used for Questions 2 to 4.

Table 5.16: Indicators of activities at each level of APOS Theory for Question 1

Structure	Activities learners can do at each structure of APOS Theory
Action	Setting up variables
Process	Formulation of mathematical expression representing the length
Object	Formulating of an expression for the area
Action / Process	Equate an expression of the area to 54
Object	Determining the dimensions of the garden by solving the formed quadratic function
Schema	Connecting actions, processes, and objects to solve a problem

Adapted from Cahyani et al. (2019)

Different tables were used because the concept assessed in Question 1 differed from that in the remaining three questions. In Question 1, learners were required to determine the dimensions of a garden such that, when the length is three metres more than the breadth, the area equals 54 square metres. In contrast, Questions 2 to 4 focused on determining the maximum area that could be enclosed by a given length of fencing. In essence, the last three questions involved optimisation problems.

Analysis of Question 1

Question 1 of the Test required learners to determine the dimensions of the garden given that the area of a rectangular garden was $54m^2$ and its length was 3 metres longer than the breadth. In analysing learners' work in terms of the APOS Theory, I first defined each stage of the theory and relate it to what the learners did at that stage.

Action: According to Maharaj (2013), during the action stage, a person follows strict, sequential instructions to carry out a mathematical process. The learner's understanding of the basic concept is limited, and they find it difficult to move away from the prescribed method.

For example, one question in this study demanded learners to find maximum area of a rectangular garden that can be enclosed by a fence that is 32 metres long. Results showed that learners depended much on memorisation. This was shown by learners recalling the formulae for a perimeter and area of a rectangle $P = 2l + 2b$ and $A = l \times b$ respectively. They did not try to define what each variable represents, which I considered as a straight recalling of information

Process: According to Maharaj (2013), an action may get interiorised into a mental process as a person performs it again and thinks about it. The same function as an action is carried out entirely within the mind of the individual through a process, which is a mental structure. The person can specifically see doing the transition without explicitly carrying out each step. In this study, the process level was realised when learners were able to express the length of the garden as $l = b + 3$. Some learners did not write the step that indicates $l = b + 3$, but it was implied in the next step, the object level, when they came out with the formula for the area of the garden as $A = (b+3)b$. For those learners who did not write down that $l = b + 3$, it means they worked this step entirely within their minds.

Object: This is a stage of understanding a mathematical concept in which the learner can compose transformations to form cognitive objects through the application of activities at the action and process stages. In this study, the object level was reached when learners managed to write the area as $A=(b+3)b$ and equate it to 54 to get a quadratic equation $b^2+3b=54$. They then proceeded and wrote quadratic in standard form $b^2+3b-54=0$ and applied procedures of solving quadratic equation either by factorisation or quadratic formula to find magnitude of the breadth. The value of the length was obtained by adding 3 metres to the breadth since it was given that the length is longer than the breadth by three metres.

Schema: A schema is a set of processes and objects that occur when a student is able to organise and comprehend new and unfamiliar mathematical concepts. For example, at a schema level, the learners were able to conclude that the negative value was invalid, as length cannot be a negative value. At this level, learners can connect the properties of length and can draw a conclusion and understand a concept based on the connections.

Analysis of Question 2

Question 2 was analysed using Table 5.12. This question required learners to determine the dimensions of the rectangular garden that can be enclosed by a fence of 32m to cover the maximum area. In analysing learners' scripts, I looked at what was done by the learners at each stage of the APOS Theory.

Action: According to Table 5.12, the first level of the APOS theory was first accomplished when learners established the variables representing the dimensions of the garden. Most learners used l to represent the length and b the breadth. They then used the formula for the perimeter of a rectangle $P=2l+2b$

Process: This level was attained when learners produced the equation representing the perimeter. For example, if they had chosen l to represent the length and b the breadth, a learner could formulate the mathematical expression representing the perimeter as $2l+2b=32$

Object: The next stage, the object level, was achieved when the used a mathematical formula representing the perimeter of the garden, $2l+2b=32$ $2l+2b=32$ and proceeded to make the length, l, the subject of the formula to get $l=16-b$. From here, they formulated an expression for the area, $A=l\times b$ and eventually wrote the area in terms of b only to get

$A = (16 - b)b$
$= 16b - b^2$

Table 5.17: Indicators of activities at each level of APOS Theory for Question 2

Structure	Activities learners can do at each structure of APOS Theory
Action	Setting up variables
Process	Formulation of mathematical expression representing perimeter
Object	Formulating of an quadratic function with a maximum turning point
Action	Using t formula $x = \frac{-b}{2a}$ to find the x-coordinate that gives the maximum turning point
Object	Determining the dimensions that leads to maximum area
Action / Process	Finding the maximum area
Schema	Connecting actions, processes, and objects to solve a problem

Adapted from Cahyani et al (2019)

Schema: According to Asiala et al. (2004), schema is the entire body of knowledge that is either consciously or unconsciously related to a specific mathematical topic. In this study, learners had to connect knowledge from measurement (mensuration) and from functions to resolve a given word problem. For example, in this question, learners realised that the expression for the area $A = 16b - b^2$ is a quadratic function with a maximum turning point and to determine the x – coordinate of the turning point they had to use the formula $x = \frac{-b}{2a}$.They were ware that this vas the value of the width. After obtaining the width, they proceeded to find the length of the garden.

Analysis of Question 3 and 4

In analysing the two questions using APOS theory, I followed the same procedure used for Question 2. This was because these questions were similar in terms of the mathematical content assessed. All three questions involved optimisation and required learners to determine the maximum area that could be enclosed using a fixed length of fencing around a garden.

5.6 Analysing data using concept image and concept definitions

The theory of concept image and concept definition (Tall & Vinner, 1981) was useful in analysing learners' understanding of mathematical problems involving maximum area. Although the tasks in this study were designed to assess mathematical modelling competencies, it was possible to indirectly infer learners' concept definitions of a rectangle, as well as its perimeter and area.

In some instances, learners appeared to rely on concept definitions when responding to the questions. For example, all four questions involved a rectangular garden or camp, prompting learners to draw four-sided figures to represent the given situations. However, their sketches did not explicitly show key properties of rectangles, such as opposite sides being equal and parallel, or all interior angles being 90 degrees. Nevertheless, these representations assisted learners in visualising the problems and proceeding with their solutions. This aligns with Vinner's (1991) view that concept definitions should be concise and should not include information that can be logically derived from other properties. For instance, he argues that it is preferable to define a rectangle as a quadrilateral with three right angles rather than four right angles, since the fourth angle can be deduced in Euclidean geometry.

Learners' concept images of a rectangle and its properties were inferred from their written responses. The correctness of their solutions in relation to mathematical comprehension was examined, as well as the types of representations they used to solve the problems. In analysing the scripts in terms of concept image and concept definition, I applied the guidelines developed by Ojo and Olanipekun (2023). Although these guidelines were originally used in the context of Calculus, they were found to be equally applicable in this study. The guidelines include: (1) structure and synthesis; (2) understanding of keywords, labelling, and identification of relationships between mathematical concepts; and (3) use of visual representations as an explicit means of expressing mathematical ideas.

5.6.1 Structure and synthesis

For each question in the test, I examined whether learners identified and correctly interpreted all relevant information constituting the underlying concept. For example, in Question 1, I checked whether learners correctly represented the length as three metres longer than the breadth. In Question 3, I verified whether their calculations or sketches reflected that only three sides were to be fenced, given that the fourth side was obstructed by a rock. In Question 4, learners were provided with a diagram of a pasture divided into camps, and I assessed whether they included all the relevant sides to be fenced. Some learners, however, only considered the outer perimeter of the figure and excluded the internal dividing fences. Organising information in a logical and coherent sequence was regarded as part of the structure of the concept image, involving the identification of key components and the relationships among them.

I also analysed learners' work in terms of synthesis, which refers to the integration of different knowledge elements to form a coherent understanding. This involves making connections between previously learned concepts and applying them to solve problems or interpret situations. In this study, synthesis was evident in Question 1 when learners applied knowledge of solving quadratic equations to determine the dimensions of the garden. In Questions 2 to 4, synthesis was observed when learners drew on knowledge of functions and applied it to optimisation problems involving the determination of maximum area.

They used the formula $x = \frac{-b}{2a}$ to determine the x – coordinate of the maximum turning point, which turned to be one of the dimensions of the garden or pasture.

5.6.2 Comprehension of keywords, marking or labelling, and interrelations

Ojo and Olanipekun (2023) argue that, in constructing a concept image, it is important to identify and highlight key terms related to the problem, as this enables learners to better understand the task and focus on its essential elements. They illustrate this using the term "approach" in the context of limits. In the present study, all the problems involved rectangles, their perimeter, and area; thus, these three terms served as key concepts. The term "rectangle" required learners to understand its properties, including that opposite sides are equal and parallel, as well as the formulas for perimeter and area.

In analysing learners' work, I examined the correctness of the mathematical expressions used to represent perimeter and area. Although some learners drew sketches of the rectangle to represent the garden, no marks were awarded for the sketches themselves. Nevertheless, most learners who produced sketches were able to formulate correct expressions for both perimeter and area. Another important component in constructing a concept image, according to Ojo and Olanipekun (2023), is labelling. Effective labelling of visual representations supports clearer communication and enhances understanding of the problem, as it makes relationships between variables and mathematical expressions more explicit.

In this study, learners used variables to represent the length and breadth of the garden. Some of those who drew sketches labelled the sides of their diagrams. For example, in Question 1, they indicated the breadth as $b$and the length as $b + 3$, since the length was given as three metres more than the breadth. These learners generally did not make errors in deriving expressions for the area. However, it was observed that although variables were used, learners often failed to explicitly state what these variables represented. A correct use of variables requires clear definition; for instance, one would expect a statement such as, "Let $b$represent the breadth and $l$the length of the garden."

5.6.3 Visual Representation

Presmeg (2006), as cited in Ojo and Olanipekun (2023), emphasises that visual representation is essential for understanding complex mathematical concepts and problem-solving strategies. Visual tools such as graphs and diagrams provide a clear and meaningful way of communicating mathematical ideas. Similarly, Ojo and Olanipekun (2023) argue that learners can develop and express their understanding of mathematical concepts through such representations. When learners can integrate visual representations with their concept images, their understanding becomes deeper and more meaningful, particularly in areas such as calculus. This integration also supports problem-solving and mathematical reasoning, thereby enhancing overall mathematical proficiency.

In this study, the most common visual representation produced by learners was a rectangle, as all four questions involved either a rectangular garden or pasture. However, none of the learners constructed graphs of quadratic functions to represent the situation and identify the maximum turning point to determine the x-coordinate of the maximum value.

Instead, they calculated the maximum value of x algebraically using the formula $x = \frac{-b}{2a}$. I observed that learners who were able to construct an appropriate visual representation— in this case, a rectangle— and label it correctly was also able to formulate a correct mathematical model representing the given scenario. These learners were then able to proceed and successfully solve the resulting mathematical expression.

5.6.4 Analysing data in terms of multiple representations

One of the specific skills outlined in the CAPS document is the development of learners' ability to communicate mathematically using words, symbols, graphs, diagrams, and tables (DBE, 2011). This indicates that the South African Mathematics curriculum actively promotes the use of multiple representations in teaching and learning Mathematics. Consequently, it was important in this study to determine whether learners could accurately represent the given scenarios in the test using different representational forms when analysing their scripts.

According to Duval (1999), visual representations are widely used in Mathematics classrooms and are fundamental to mathematical understanding. Duval (2014) further explains that visual representations, such as geometric figures and diagrams, support visualisation, which is a cognitive process involving the construction of mental images that help shape mathematical thinking. In line with Duval (1999), Dreher, Kuntze, and Lerman (2016) emphasise that the ability to switch between different forms of representation is essential for developing an appropriate concept image.

Mathematical representations may be external and observable, such as diagrams, graphs, number lines, solid models, written words, mathematical expressions, equations, and formulae, as well as digital representations generated on calculators or computers. These external representations exist independently of the individual and can be viewed, interpreted, analysed, and discussed by others.

In analysing learners' use of multiple representations, one would expect to find drawings of rectangles across all questions, since all four tasks involved either a rectangular garden or pasture. However, very few learners produced such sketches. Another expected form of representation was the formulation of equations representing the perimeter and area of the gardens or pastures. In this regard, most learners were able to develop correct mathematical models (formulae). Diagrammatic representations were relatively scarce and were mainly limited to simple sketches of rectangles.

The correct use of representations was taken as an indicator that learners had a clear understanding of the given scenarios. According to Bal (2015), the ability to express mathematical ideas using multiple representations and to translate between them reflects mathematical thinking and communication. Such translational ability significantly enhances learners' conceptual understanding of mathematical ideas. Therefore, in this study, the ability to represent the same mathematical information using different correct modes was interpreted as evidence of learners' understanding of the problems.

The most used representations in this study were sketches of rectangles and algebraic models such as expressions and formulae. Unfortunately, none of the learners attempted to represent the problems using graphs of quadratic functions to determine maximum values in Questions 2 to 4. In Question 1, however, some learners, such as the one shown in Figure 5.22, were able to draw a sketch, express the length in terms of the breadth, develop a quadratic expression for the area, and ultimately determine the dimensions of the garden.

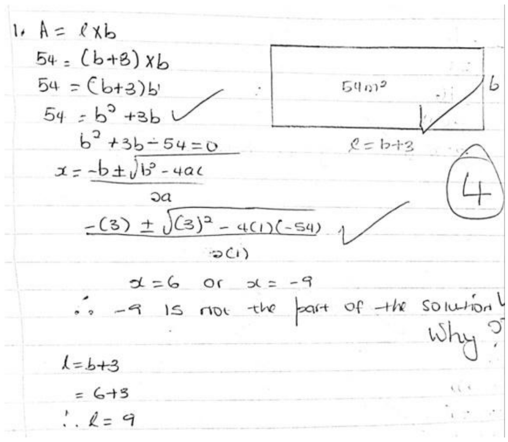


Figure 5.22: Response that included the diagram or sketch in Question 1

The analysis of data in terms of multiple representations was based on learners' ability to translate from one form of representation to another. Most learners were able to convert the given word problems into mathematical representations. Some learners progressed through multiple stages of translation, first converting the word problem into a diagram and then into a mathematical expression. After reading the word problems, many learners produced sketches of rectangles to represent the given situations, since all the tasks in the test involved rectangular gardens or pastures. Following the construction of these sketches, learners then formulated mathematical expressions representing the situations. For example, in Question 1, learners were presented with a scenario in which Mr Nene wanted to plant seedlings in a rectangular garden with an area of 54 m², where the length was 3 metres longer than the width. Learners were required to determine the dimensions of the garden.

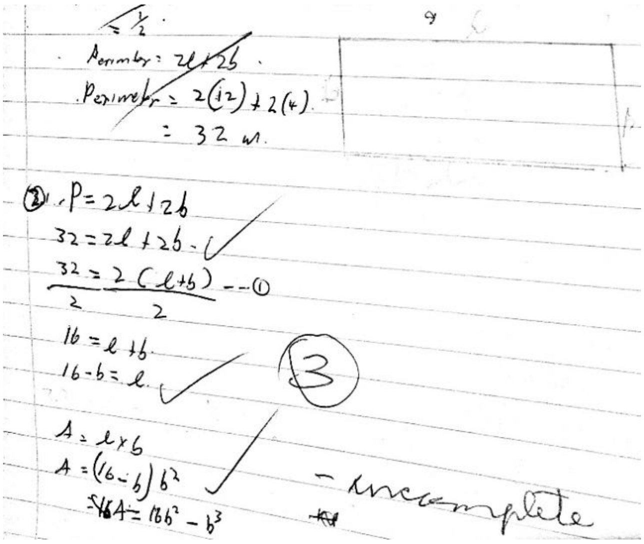


Figure 5.23: Learner L55 used more than one representation for Question 2

Figure 5.23 shows a learner who used more than one representation in Question 2 to illustrate the given scenario. The learner produced a sketch of a rectangle and formulated a mathematical equation representing the perimeter. However, the learner did not proceed further to determine the dimensions that would yield the greatest area. In Question 3, successful performance required learners to produce an accurate sketch, clearly indicating that one side of the garden was covered by a brick wall and that only three sides therefore required fencing. Figure 5.24 shows a learner who correctly produced such a sketch. This is another example of the use of multiple representations to illustrate the given scenario. The learner subsequently obtained the correct solution. However, in terms of mathematical modelling, the learner was still classified as partially competent, as they did not make explicit assumptions, verify or justify their solution, or interpret the results in relation to the given context.

Most learners limited their representations to only two, the diagrams and the formula. None of them used a table or graph to represent the given situations. One would expect to see learners drawing a parabola as they were able to employ the formula $x = \frac{-b}{2a}$ which is used to find the x-coordinate of the highest turning point.

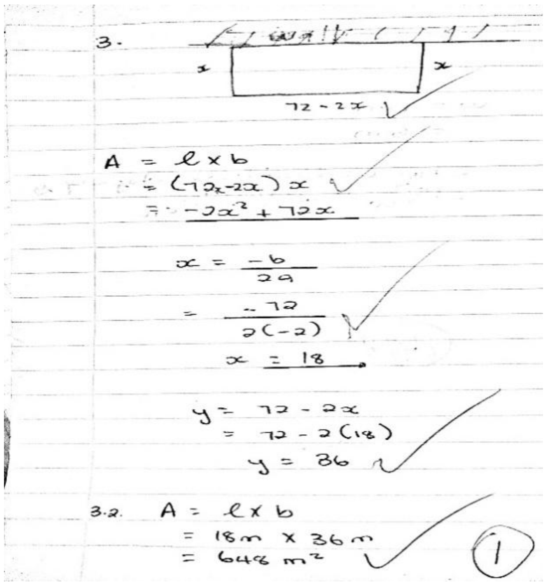


Figure 5.24: Learner L57 used more than one representation for Question 3

To answer Question 1, most learners first drew a rectangle to represent the given information diagrammatically and labelled it. They then formulated the mathematical representation of the area and equated it to 54. Since it was given that the length was longer than the width by 3 metres, learners were able to represent the length as $l = b + 3$ where *l* represented the length and *b* the breadth or *w* the width.

$$A = l \times b$$
$$54 = (b+3)b$$
$$54 = b^2 + 3b$$
$$0 = b^2 + 3b - 54$$

Upon conducting data analysis, I realised that while multiple representation was deemed necessary, not every learner employed it when answering the test questions.

5.7 Findings

From the word problems used in the test, learners were required to identify the specific questions they needed to answer. The next step involved assigning variables, typically letters representing unknown quantities in a mathematical expression or equation. However, in this study, none of the participants explicitly stated what each variable represented. For example, in Question 1, one would expect statements such as "Let *l*represent the length and *b*the breadth." The assignment of variables should then have been followed by the formulation of a mathematical model, either in the form of an equation or an algebraic expression involving these variables. Thereafter, learners were expected to solve the equations to determine the values of the variables. However, most learners left their answers in terms of variables. For instance, in Question 3, some learners presented final answers such as $x = 18$and $y = 36$without interpreting them in the context of the problem. The expected response would have been a contextual statement such as: "The dimensions that produce the maximum area are 18 metres by 36 metres." The final step, which involved verification or validation of the results, was not undertaken by any of the participants.

The analysis indicated that most participants were partially competent in mathematical modelling. This was also confirmed by the rubric analysis, which showed that no learner successfully completed all steps of the modelling process. Even those who were able to formulate and solve correct equations failed to make appropriate assumptions, interpret their solutions, or verify their answers. A small number of learners were unable to complete even a single step correctly and were therefore classified as not competent in mathematical modelling. These findings are consistent with Chan et al. (2012), who similarly reported that none of their participants were fully competent, with learners being classified as either partially competent or not competent.

To be considered competent in mathematical modelling, a learner is expected to complete all stages of the modelling process. However, learners often wrote down variables without defining what they represented. For instance, if $l$was intended to represent length and $w$width, this should have been explicitly stated as: "Let $l$represent the length and $w$the width." This would ensure clarity and allow any reader to understand the meaning of each variable.

Another critical step that was largely omitted was the formulation of assumptions. This finding is consistent with studies by Seino (2005) and Krawitz et al. (2022), which also report that learners struggle to formulate assumptions, despite their importance in solving real-life mathematical problems. At the modelling stage, learners should explicitly state assumptions that simplify the problem and make it solvable. For example, in Question 1, when determining dimensions for a garden with an area of 54 m², assumptions should have been made about the nature of the physical space. If, for instance, the area contained obstacles such as a large rock or a permanent pond, the usable area would be reduced, and the dimensions would no longer yield 54 m². Clearly stating such assumptions is essential in modelling.

Some learners in Question 1 obtained a final answer such as –9 metres. This indicates a lack of understanding of the context of the problem. At this level, negative lengths are not meaningful, as length represents magnitude rather than direction and should therefore be interpreted in absolute terms. This suggests that learners did not fully grasp the distinction between mathematical results and their real-world interpretation.

Language may also have posed a barrier for some learners, particularly in Questions 3 and 4. Some learners appeared not to have read and interpreted the questions carefully, as reflected in their sketches and equations. Question 3 required learners to determine the lengths of the three sides to be fenced, given that one side was already enclosed by a rock. However, some learners incorrectly included all four sides, indicating a lack of comprehension of the problem statement. Since all participants came from isiZulu-speaking backgrounds and used English as a second language, it is possible that language comprehension challenges contributed to these misunderstandings.

Difficulties in comprehending word problems, particularly in Questions 3 and 4, are consistent with findings by Zerafa (2016), Stillman (2000) as cited in Leiß et al. (2010), Leiß et al. (2010), Göksen-Zayim et al. (2019), Govender and Machingura (2023), and English (2003). Zerafa (2016), for instance, found that language plays a significant role in learners' performance in arithmetic word problems and that learners perform better when problems are presented in their first language rather than in English. Collectively, these studies highlight that word problem solving is not only mathematically demanding but also linguistically challenging.

The analysis of learners' scripts was conducted to address the research questions. During the process, recurring patterns and themes emerged, some of which were already evident during marking. The systematic categorisation of data into themes is referred to as qualitative coding. Following a detailed analysis, the findings were organised into themes presented in Table 5.18.

Table 5.18: Themes emerged from data analysis

Theme 1: Failure to make assumptions
• Never mentioned what variables they have used represent • No mentioning of conditions that would make their solutions to be always true
Theme 2: Incapability to obtain correct solutions
• Formulated incorrect mathematical model • Failed to work the solution fully (incomplete working) • Working carelessly ending up omitting essential information
Theme 3: Unfamiliarity with the verification of the results
• After obtaining solutions they did not validate them • Did not prove the correctness of their solutions with respect to the given situations
Theme 4: Inability to interpret the results
• Left their answers still equated to variables • Did not connect the solutions back to the problem situations • In Question 1 some learners did not explain why -9 was an invalid solution
Theme 5: Inability to understand given scenarios or English (Question 3)
• Read the scenario without understanding • Did not realise that although the garden is rectangular, only three sides had to be fenced
Theme 6: Failing to comprehend the given diagram(Question 4)
• Some learners could not consider the interior sides that divide the pasture into camps • Only the perimeter (sides right round the figure was considered).

5.8 Discussions

Most participants demonstrated that they understood the requirements of all four questions in the study. Typically, mathematical modelling problems are presented in textual or word format. As evidence of their comprehension, learners were able to read the problems and construct appropriate mathematical models. Some learners also produced diagrams or sketches of the scenarios, such as rectangles representing the layout of a garden or camp. However, although such diagrams were useful, some learners did not consider them necessary, as they were able to visualise the problems mentally. For this reason, no marks were awarded for diagrams. Van Garderen (2006), as cited in van Garderen and Scheuermann (2015), notes that diagrams are often recommended for representing word problems, particularly in the middle and upper school grades. Diagrams serve as cognitive tools that support understanding and facilitate problem-solving.

Although learners were able to comprehend the problems and formulate correct mathematical models, a key limitation was their failure to make realistic assumptions. Assumptions are essential in modelling real-life situations because they define the boundaries of the problem and clarify the conditions under which solutions are valid. For example, in Question 2, learners were required to determine the dimensions of a vegetable garden that would yield the maximum area given 32 metres of fencing. A solution such as 8 metres by 8 metres would only be valid under assumptions such as the entire enclosed area being usable land and the absence of physical obstacles such as rocks. According to Krawitz et al. (2022), assumptions are necessary in open-ended problems because important information is often missing, and assumptions help to fill these gaps by defining the conditions required for a solution.

Another important finding of the study was that learners did not interpret their results. If a mathematical solution is not interpreted in the context of the original problem, it lacks meaning. For instance, in Question 3, most learners provided final answers such as $x = 18$and $y = 36$. However, they were expected to interpret these values in context, for example by stating that the width of the garden is 18 metres and the length is 36 metres. Interpretation is a critical step in the modelling process because it translates mathematical results back into real-world meaning, linking the solution to the original problem situation.

Furthermore, the study revealed that learners did not validate their solutions, which is the final stage of the modelling process. In mathematical modelling, learners are expected to critically evaluate their answers to determine whether they are reasonable and appropriate for the given context. Validation ensures that the solution makes sense in real-life terms. If the solution is incorrect or unrealistic, learners are expected to revisit and refine their modelling process. In this study, learners did not attempt to verify their answers, which supports the findings of Blum and Ferri (2009), who observed that validation is one of the most challenging stages of modelling. In many cases, learners do not check the validity of their solutions and instead rely on the teacher to confirm correctness.

The findings further indicated that none of the learners were able to complete all stages of the modelling process successfully. Most learners were competent in the initial stages—namely, understanding the problem, formulating a mathematical model, and solving the model—but struggled with interpretation and validation. This finding is consistent with Kaygısız and Senel (2023), who also reported that learners are generally able to perform the early stages of modelling but encounter difficulties in interpreting and verifying solutions. A small minority of learners were unable to complete even a single stage correctly and were therefore considered not competent. Overall, the results point to partial or fragmented competence in mathematical modelling.

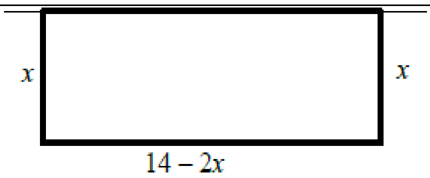
This incomplete competence may be attributed to limited exposure to mathematical modelling processes, as noted by Asempapa (2015), who argues that modelling is not sufficiently emphasised in classroom practice globally. His study shows that many school-based word problems do not adequately engage learners in full modelling cycles, and modelling activities often play a marginal role in everyday Mathematics teaching. Furthermore, the structure of many assessment questions, including those in Grade 12 NSC examinations, often does not require full modelling processes. Many tasks focus on procedures such as “solve for x” or “factorise the expression,” which learners can complete without engaging in modelling. Even when real-life contexts are included, they may not fully require the application of all modelling stages, resulting in limited emphasis on these skills in classroom practice.

To illustrate this point, I examined a question and its marking guideline from a Grade 11 end-of-year national examination paper (2016). The question and one of its alternative solutions are presented below.

QUESTION 10 (DBE Grade Eleven 2016 Paper 1)

Bongani wants to start a small vegetable garden at his house. He wants to use an existing wall and 14m of fencing to enclose a rectangular area for the garden. Calculate the dimensions of the largest rectangular area that he can enclose.

QUESTION/*VRAAG* 10

10	 Let one of the equal sides $= x$ the other side $= 14 - 2x$ Area $= (14 - 2x)x$ $= -2x^2 + 14x$ $x = \frac{-14}{2(-2)}$ $= \frac{7}{2} m$ $y = 7m$		✓area formula/*oppervl.for.* ✓ $x = \frac{-14}{2(-2)}$ ✓ answer for x ✓ answer for y (4)

From the extract of the marking guideline, I identified several key observations:

- The variables were not explicitly defined. Although variables $x$and $y$were used, there was no indication of what each represented in terms of breadth and length.
- No assumptions were stated, even though assumptions are essential when solving real-life mathematical problems.
- Correct mathematical models were formulated and solved accurately.
- There was no emphasis on interpretation of solutions, as the marking guideline did not require a contextual statement explaining what the variables represented, and no marks were allocated for interpretation. A contextual statement was expected, for example: "The breadth (width) of the garden is 3.5 metres, and the length is 7 metres." Such a statement should clearly connect the mathematical answer to the problem context.
- There was no verification of solutions. Verification involves checking whether the obtained solutions satisfy the original mathematical model, for example by substituting values into the equation. In this case, learners could have checked whether the dimensions added up to the total fence length of 14 metres. It is also important to assess whether the solutions are reasonable in context; for instance, any total exceeding 14 metres would be inconsistent with the given information.

For learners to fully engage in mathematical modelling, assessment practices should allocate marks for all modelling stages, including defining variables, making assumptions, interpreting results, and validating solutions. However, the findings of this study suggest that most learners focused primarily on formulating and solving mathematical models correctly. This may indicate that learners tend to prioritise aspects of problem-solving that are explicitly rewarded in assessments, rather than engaging with the full modelling process.

5.9 Conclusion

Data extracted from learners' scripts were presented and analysed in this chapter. The purpose of the analysis was to determine learners' competencies in mathematical modelling. A five-step modelling framework was used as the basis for evaluating learners' performance. The findings indicate that none of the learners were fully competent in mathematical modelling, as none successfully completed all five stages of the modelling process. Overall, the results show that most learners demonstrated partial competence in mathematical modelling. In addition, the chapter examined learners' mathematical proficiencies. This analysis was based on four of the five strands of mathematical proficiency, namely conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning. The findings similarly revealed that learners were only partially proficient, as none demonstrated competence across all four strands. All learners performed poorly in adaptive reasoning, which was evident in their failure to verify or justify their solutions.

CHAPTER SIX: CONCLUSIONS, LIMITATIONS AND RECOMMENDATIONS

6.1 Introduction

The concluding chapter presents the findings, conclusions, and recommendations of the study based on the data collected and analysed. It also outlines the limitations of the study and provides detailed recommendations directed to the Department of Basic Education (DBE) to raise awareness of learners' levels of competence in mathematical modelling. These conclusions and recommendations are also intended to be of value to researchers interested in mathematical modelling and the teaching and learning of Mathematics.

The aim of the study was to investigate Grade 11 learners' mathematical modelling through problem-solving. The findings revealed that learners were only partially competent in mathematical modelling, as they were unable to complete all stages of the modelling process. The main challenges identified included the failure to make assumptions, the lack of clear definition of variables, the absence of interpretation of results, and the failure to validate solutions. Despite these limitations, most participants demonstrated an understanding of the problems, were able to formulate correct mathematical models, and solved them accurately.

6.2 Summary of the Findings

In this section, the findings of the study derived from the analysis of learners' scripts are presented. Tables were used to record each learner's performance in relation to understanding the problem. This was assessed by examining whether learners established relationships between variables and whether they made appropriate assumptions to support the modelling process. The next step involved determining whether learners were able to translate the given situation into a mathematical model or equation and solve it correctly. The final stages focused on whether learners interpreted their results, provided a contextual statement explaining their solutions, and verified their answers.

For a learner to be considered competent in mathematical modelling, all five criteria had to be met. If any one of the criteria was not achieved, the learner was classified as partially competent in mathematical modelling. These criteria were derived from mathematical modelling competencies identified by Maaß (2006) and Kaiser and Stender (2013) in Vorhölter et al. (2019). Given the qualitative nature of the study, attention was also given to emerging patterns observed across the tables for each school. The analysis indicated that none of the participants from the three schools were able to meet all the criteria successfully. As a result, all learners were classified as partially competent in mathematical modelling, based on the questions they attempted.

Furthermore, none of the participants made or attempted to make assumptions, interpret their solutions in the context of the given problem, or verify their answers. A major concern in this regard was the consistent failure to interpret results. Result interpretation is crucial, particularly in word or context-based problems that represent real-world situations. Learners are required to relate mathematical solutions back to the original context to ensure that the results are meaningful and applicable. Without proper interpretation, numerical or computational outputs remain abstract and cannot be meaningfully applied to the real-world situations they are intended to model.

6.3 Limitations of the Study

There were several factors that posed challenges to the conduct of this study. The first limitation was the impact of the COVID-19 pandemic, which made it impossible for learners to work in groups. To comply with health and safety regulations and prevent the spread of the virus, learners completed the test individually. Maintaining the required social distancing of 1.5 metres would have been difficult in a group-work setting. In the application submitted to the Head of the KwaZulu-Natal Department of Education, it was indicated that learners would not be exposed to COVID-19 risk, as they would complete the tasks individually. Consequently, this study did not utilise group work, which distinguishes it from many other studies on mathematical modelling. Modelling tasks are often implemented through collaborative learning or group work, as noted by Hernández-Martínez and Harth (2015), who emphasise the importance of collaboration in mathematical modelling and problem-solving.

The second limitation was the inability to conduct interviews. Access to learners at the scheduled interview times could not be secured, as they were engaged in examinations. On some days, learners wrote two examinations in one day, while on other days, teachers used available time for revision in preparation for upcoming papers. In addition, some teachers were still addressing learning gaps from Grade 10, as learner attendance had been inconsistent during the COVID-19 period. Interviews were therefore not possible, despite their importance in gaining deeper insight into learners' understanding of mathematical modelling.

The third limitation was related to time constraints. In the initial request to schools, it was indicated that the test would take 30 minutes to complete. However, after administering the test at the first school, it became evident that this duration was insufficient, as most learners were unable to progress beyond Question 3 or complete the test. This affected both the quantity and quality of data collected from the first school. As a result, additional time of 15 minutes was negotiated for the remaining two schools, allowing participants there 45 minutes to complete the test.

The fourth limitation may have been the language used in the test, which was English. Some learners may not have fully understood the requirements of the questions, suggesting that language could have acted as a barrier, particularly as the tasks required careful reading and comprehension. This was evident in Question 3, where learners were informed that a rectangular area was to be enclosed, with one side already formed by an existing brick wall. This implied that only three sides required fencing. However, some learners incorrectly calculated the perimeter of all four sides, indicating a lack of full comprehension of the problem statement.

The fifth limitation relates to the geographical context of the selected schools. All three schools were in a rural area within a radius of approximately 7 kilometres. As a result, the contextual conditions across the schools were likely to be similar. This may have influenced the findings, as the results might have differed had the schools been drawn from more diverse settings, such as rural, township, and urban contexts. This assumption is informed by general performance trends, which suggest that urban schools, particularly former Model C schools, often perform better than many township and rural schools.

6.4 Some Implications of the Study

This study was conducted to establish learners' competencies in mathematical modelling. Since mathematical modelling is a relatively new teaching and learning approach emphasised in the current South African Mathematics curriculum, it was necessary to assess learners' abilities in this area. The findings of the study revealed that most learners were not fully competent in mathematical modelling, but rather partially competent. This is evident from their inability to complete all five stages of the modelling process used in the study. No learner successfully completed all the steps across any of the four questions in the test. Most learners were, however, able to complete the first three stages, namely understanding the problem, formulating a mathematical model, and solving the model.

The study further revealed that many learners experienced difficulties in interpreting and verifying their solutions. After obtaining answers, learners generally did not explain the meaning of their results in relation to the original context. For example, in Question 3, an appropriate interpretation would have been: “The dimensions that give the maximum area are 18 metres and 36 metres.” However, learners often left their answers as $x = 18$and $y = 36$, without contextual interpretation.

Validation of solutions was another stage that was almost entirely omitted. This suggests that learners are not routinely encouraged to check the validity or reasonableness of their answers. For instance, in Question 1, some learners obtained a solution such as –9 metres for a length. Had they verified their solutions, they would have recognised that a negative length is not meaningful in a real-world context. The findings therefore suggest that although mathematical modelling is emphasised in the curriculum, it is not being implemented effectively in classroom practice. The failure to interpret and verify results indicates that these aspects are not sufficiently emphasised in learners’ day-to-day mathematical activities. If these stages were consistently reinforced, a greater number of learners would likely attempt to interpret and validate their answers. It is concerning that Grade 11 learners demonstrated limited awareness of the need to verify their solutions, suggesting that without targeted intervention, they may progress through schooling without fully developing this critical aspect of mathematical thinking.

Furthermore, the study showed that all participants failed to make assumptions based on the given information. The absence of assumptions suggests that learners are not accustomed to incorporating simplifying conditions when solving real-life problems. Instead, they were able to obtain solutions without considering contextual factors that may influence the correctness of their answers. The role and importance of assumptions were discussed in Chapter Two.

The failure to make assumptions in mathematical modelling has significant implications. It suggests that learners ignore physical constraints such as the availability of arable land, the presence of rocks or obstacles, and variations in terrain such as slope. As a result, they perform calculations that are logically inconsistent in real-world contexts. In addition, the absence of assumptions indicates that learners do not recognise the open-ended nature of modelling problems and instead treat them as closed, routine exercises with complete information. This undermines the fundamental purpose of mathematical modelling, which is to connect Mathematics to real-world situations.

Neglecting assumptions can lead to incorrect models, inaccurate solutions, and misleading generalisations, as important contextual factors are excluded. While the mathematical procedures may appear correct, the resulting solutions may be unrealistic or meaningless in practice. This weakens learners' ability to integrate mathematical reasoning with real-world applications and limits meaningful problem-solving.

The Department of Basic Education (DBE) therefore has an important role to play in ensuring that mathematical modelling is effectively implemented in classrooms, with real-life problems integrated across the curriculum. One of the specific aims of the CAPS document emphasises that contextual problems should be incorporated into the teaching and learning of Mathematics. These contexts should include areas such as economics, health, science, politics, and everyday life (DBE, 2011). It is therefore essential that the Mathematics curriculum is implemented as intended in official curriculum documents.

Considering these findings, and in answering the research question:

What is the nature of competencies of Grade Eleven learners in mathematical modelling when resolving real world problems?

I found that all learners who participated in this study were partially competent in mathematical modelling. Most learners were able to complete only two stages of the modelling process, namely formulating a mathematical model and solving it. However, they were unable to complete the full modelling cycle. In particular, the interpretation and verification of results were not undertaken. The initial stage, understanding the problem, was only partially achieved, as learners were able to comprehend the task and construct mathematical models, but they failed to make appropriate assumptions. In summary, none of the learners were able to make assumptions, interpret their solutions in context, or verify their answers. The inability to perform these key stages is a major concern, as they distinguish mathematical modelling from routine problem-solving exercises.

Mathematical modelling is a cyclical process that requires, among other things, the formulation and evaluation of assumptions, as well as the interpretation and validation of solutions. Unlike traditional textbook problems, which often have a single correct answer, modelling tasks are rooted in real-world contexts and therefore require the solver to engage in the following processes:

- **Make assumptions:** The modeller must first identify and explicitly state assumptions relevant to the real-world situation. These assumptions define the boundaries, conditions, and constraints of the problem. For example, in one of the questions used in this study, learners were asked to determine the maximum area that could be enclosed by a fixed length of fencing. In such a case, reasonable assumptions would include that the ground is flat and that there are no physical obstructions such as a river or gully. As discussed in Section 2.6 (see Figures 2.7 and 2.8), the length of fencing required for a flat surface differs from that required for uneven terrain.

- **Interpret results:** Once a mathematical solution has been obtained, it must be translated back into the original context of the problem. This stage determines whether the results are reasonable, meaningful, and applicable to the real-world situation. In Question 3, for example, several learners provided final answers still expressed in terms of variables. Where variables were not clearly defined, such responses had little meaning. Learners failed to relate their solutions back to the context of the problem. A correct response should have been expressed in a meaningful statement such as: "The length of the garden is 36 metres," where 36 metres represents the computed solution.

- **Validate results:** Learners are expected to verify whether their solutions are correct and reasonable within the context of the problem. In this study, learners did not attempt to check their answers. This suggests that they did not evaluate the correctness or logical consistency of their solutions. Validation would have enabled learners to revisit and revise their modelling process in cases where their answers were unrealistic or incorrect.

The inability of learners to interpret, validate, and formulate assumptions is a serious concern. One possible explanation is limited exposure to mathematical modelling practices. Given that 75 learners participated in the study, it would be expected that at least some would demonstrate attempts at assumptions, interpretation, or validation if they had previously engaged with modelling activities. Another possible contributing factor is the nature of tasks found in textbooks, which may not consistently emphasise full modelling cycles. Furthermore, although mathematical modelling is an important component of the Mathematics curriculum, assessment practices in national examinations—such as Grade 10 and 11 end-of-year examinations and even the Grade 12 National Senior Certificate marking guidelines—do not consistently foreground the modelling process.

The findings of this study therefore suggest that learners have had limited exposure to authentic mathematical modelling. Essential stages such as defining variables, making assumptions, interpreting results, and verifying solutions were largely absent in their responses.

6.5 Recommendations to the Department of Education and Teachers

This research explored learners' mathematical modelling competencies when solving mathematical problems grounded in real-life contexts. The tasks focused on the area and perimeter of rectangular figures. The findings revealed that learners lacked adequate competence in applying mathematical modelling processes. The procedures they followed suggested that they had not been sufficiently exposed to mathematical modelling as an instructional approach. None of the learners made any attempt to formulate assumptions. In addition, only a very small number of learners attempted to verify or validate their solutions to determine whether they were correct or appropriate within the given context. Furthermore, all learners introduced variables without specifying what each represented. In some cases, particularly in Questions 3 and 4, learners produced incorrect formulae, indicating a lack of clear understanding or misinterpretation of the questions.

Based on the findings, the following recommendations are made:

- Mathematics teachers should expose learners to mathematical modelling processes so that they are able to understand and apply all stages correctly. To improve and strengthen learners' competencies in mathematical modelling, they should be given regular opportunities to engage with tasks that require full modelling processes. This is essential, as the current South African CAPS curriculum clearly states that mathematical modelling should be a central feature of Mathematics teaching and learning.

- The DBE should actively promote mathematical modelling within the curriculum and assessment practices. In the Grade 12 Mathematics end-of-year NSC examinations, there are currently very few questions set in real-life contexts. Furthermore, although such questions are limited, the marking guidelines (memoranda) suggest that they are often approached using traditional procedural methods rather than authentic modelling processes.

- Learners should be explicitly taught the essential component of mathematical modelling, namely the formulation of assumptions. Assumptions account for real-world factors that may influence or distort the solution if they are not taken into consideration.

➢ Learners must be taught that once they come out with their own variables, they must specify what each variable stands for or represents. In most cases, when solving word problems, one must develop an equation. Therefore, it is essential to specify what each variable represents in a real-life situation so that the known relationships can be expressed in equation form. For example, if in Question 1, it was necessary to specify the variables such as: let l represent length and b the breadth, it would be easy for anyone that knows the formula of a rectangle to express the area (A) in terms of l and b. Area of rectangle is then defined as $A = l \times b$

- Learners should be encouraged, where appropriate, to begin by drawing sketches or diagrams before developing mathematical formulae. Drawing and labelling diagrams is an important step that supports deeper understanding of the problem situation.
- Learners must be taught to interpret their solutions. After obtaining an answer, they should be able to translate and explain it in the context of the original problem. In this study, most learners left their solutions in symbolic form, without contextual interpretation.
- Teachers should insist that learners verify their solutions in order to check their validity. Verification or validation is the final stage of the modelling process and is essential for confirming whether the obtained solution is reasonable within the given context.
- School textbooks should be designed in a way that deliberately promotes the teaching and learning of mathematical modelling processes. The questions and examples included should support the development of full modelling competencies.
- Tertiary institutions should offer short courses on mathematical modelling to support both practising and prospective teachers. The content of such courses should be aligned with the specific phase or grade levels that teachers are qualified to teach.

- Teachers should expose learners to a wide range of word problems to provide regular practice in reading with understanding. Learners should also be guided in developing correct mathematical expressions and formulae independently from word problem statements.
- Learners should be introduced to problems requiring assumptions from the lower grades. Failure to make assumptions may result in incorrect or unrealistic solutions. For example, consider the Grade 9-level question below; it cannot be solved meaningfully unless appropriate assumptions are made:

It takes 4 builders 32 hours to build a wall, how long will it take 6 builders?

This question looks simple but without assumption, it becomes difficult. Adding a statement that says "assuming that the 6 builders will work at the same rate as the 4 builders" would make this problem to be solved easily. It is also possible that the 6 builders may take longer hours than the 4, if they can decide to work slower than the 4 builders.

It is hoped that, if all the above recommendations are implemented, learners' competencies in mathematical modelling will be significantly enhanced.

6.6 Recommendations for Future Researchers

1. To enhance future research on Grade 11 learners' competencies in mathematical modelling when solving real-life contextual problems, the following recommendations are made:
2. Researchers should first investigate teachers' competencies in mathematical modelling and determine the extent to which they integrate mathematical modelling into their classroom practice before focusing on learners' competencies.
3. Studies should be conducted across schools of different categories and geographical locations to determine whether findings differ between urban, rural, former Model C, and farm schools.
4. The assessment tasks used in future studies should not be limited to a single topic, as was the case in this study where all questions focused on measurement (area and perimeter). Instead, tests should include a variety of topics such as speed, time, and other relevant areas to provide a broader assessment of modelling competencies.

5. Unlike the present study, where learners completed tasks individually, future researchers should consider using group work. Collaborative learning allows learners to engage in meaningful discussions and share ideas. As noted by Asempapa (2015), mathematical modelling promotes communication and teamwork through social interaction. Group work may also support weaker learners through peer assistance.
6. Future studies should employ multiple data collection methods, such as interviews and classroom observations, in addition to written tests. This would enrich the quality and depth of the data collected and provide a more comprehensive understanding of learners' modelling competencies.

6.7 Conclusion

This chapter presented the conclusions and recommendations derived from the findings of the study. Overall, the study satisfactorily addressed the research question by identifying learners' competencies in mathematical modelling when solving real-life problems. The findings revealed that none of the learners were able to correctly follow all the stages of the mathematical modelling process. Consequently, learners were found to be inadequately competent in applying the modelling approach. Based on the criteria used in this study to assess modelling competence, most learners were classified as partially competent. The most significant gaps were observed in learners' failure to make assumptions during problem simplification, difficulties in model formulation, and the inability to justify or validate their solutions in the final stage of the modelling process. Furthermore, during the formulation of equations or expressions, learners did not clearly define the variables used, resulting in mathematical representations where variables were not meaningfully interpreted. Even in their final answers, learners did not provide complete contextual statements that directly responded to the questions posed. In addition, learners from all three schools were unable to verify their solutions, even when analysed through the lens of mathematical proficiency.

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APPENDICES

Appendix A: Test used in the study

Grade: Eleven **Task: Test**

Duration: 30 minutes **Total Marks: 25**

INSTRUCTIONS:

(a) Write your **code** name assigned to you on your script. Do not write your real name

(b) This question paper consists of **4** questions. Answer **ALL** questions.

(c) Clearly show all calculations. Where possible and time permits, try different approaches

(d) Diagrams are not necessarily drawn to scale

1. Mr Nene wants to plant seedlings in his rectangular garden with an area of 54 m^2. The length of the garden is longer than its width by 3m. What are the dimensions of his garden? (6)

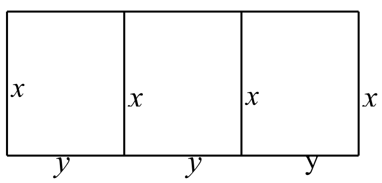
2. In order to protect his vegetables, a farmer buys 32m of fencing for his rectangular garden. Determine the dimensions of the vegetable garden if it should cover a maximum area (6)

3. A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 72m of wire fencing:

   3.1. Determine the dimensions that would create a garden of maximum area. (5)

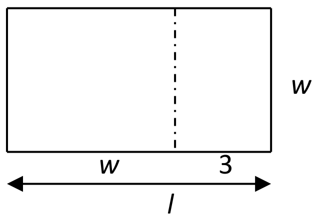
   3.2. What is the maximum area? (1)

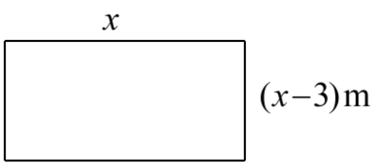
4. A farmer has 600m of fence to create a rectangular pasture which has to be divided into three camps all with the same dimensions. What is the maximum area that can be enclosed with this fence? (7)

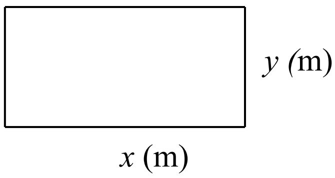


TOTAL: 25 MARKS

Appendix B: Marking guidelines

QUESTION 1	Descriptors
1. Understanding the problem By making and label the sketch correctly is an indication that a learner has understood the problem. Forming of a relationship between selected variables is also a way of showing the understanding of the problem  **2.** **3. Formulating the mathematical model** Let the width be w and length be l Given that the length is 3m longer than the width $\therefore\ l = w + 3$ ....(1) $l\,w = 54$ $(w+3)w = 54$ (substituting l) **4. Solving the mathematical model** $w^2 + 3w - 54 = 0$ $(w-6)(w+9) = 0$ $\therefore\ w = 6$ OR $w = -9$ $w = 6$ is the only solution and $w = -9$ is invalid since the length cannot be negative. From equation (1) substitute w by 6 to find the value of l $l = w + 3$ $= 6 + 3$ $= 9$	Possible assumptions: - All the available land is arable - No large boulders of rocks Defining the variables. For example, let the l be the length and w the width and express the length in terms of width ✓ $l = w + 3$ ✓ equating (w + 3)w to 54 ✓ factors ✓ explanation of why 6 is the only solution

5. Interpreting the results The dimensions of this garden are as follow: width is 6m and the length is 9m	✓ Interpreting the results: width is 6m and length is 9m
6. Validating the results 9m – 6m = 3m (this satisfies that the length is 3m longer than the width). $Area = l \times b$ $= 9m \times 6m$ $= 54m^2$ The two dimensions are also correct since they give the area of $54m^2$	✓ verification of the solutions (6)
Alternative Response	**OR**
1. Understanding the problem  	Let the length be x and then write the width in terms of length
2. Formulating the mathematical model Let the length be x m $\therefore$ the width = $(x - 3)$ m $\therefore$ Area $= x(x-3) = x^2 - 3x$ $\therefore x^2 - 3x = 54$	✓ width = x -3 ✓ equating $x(x-3)$ to 54
3. Solving the mathematical model $x^2 - 3x - 54 = 0$ $(x-9)(x+6) = 0$ $\therefore x = 9$ or $x = -6$ $x = 9$ is the only solution and $x = -6$ is invalid since the length cannot be negative. Width is calculated by deducting 3m from the length $\therefore$ Width= $x - 3 = 9 - 3 = 6$m	✓ factorising the quadratic equation ✓ explanation of why 9 is the only solution ✓ Interpretation of the results: width is 6m and length is 9m

4. Interpreting the results The dimensions of the garden are the length of 9 metres and width of 6 metres **5. Validation of results** Verification of the solution: 9m – 6m = 3m Also 9m × 6m = $54m^2$ This satisfies the conditions stated in the problem that the length is 3m longer than the width and that the area of the garden is $54m^2$	✓ verification of the solutions (6)
	6 marks
QUESTION 2	Descriptors
1. Understanding the problem  **2. Formulating the mathematical model** Let the length be x and width be y. Then $P = 2x + 2y$ $32 = 2x + 2y$ ....(1) **Solving the mathematical model** Making y the subject of the formulas in equation (1) will give: $32 = 2x + 2y$ $16 = x + y$ $y = 16 - x$ ....(2) The area of this can be calculated as $A = xy$ ....(3). Then substituting y in equation (3) will give:	Possible assumptions: - No space between the edge of the garden and the fence - All covered land is arable - No large rocks in the area to be enclosed ✓Defining variables and write the equation 2x + 2y = 32 ✓ y = 16 - x

$A = xy$ $A = x(16 - x)$ $A = 16x - x^2$ $A = -x^2 + 16x$ ......(4) This is the quadratic equation with the maximum turning point. The x-coordinate of the maximum turning point is given by: $x = \frac{-b}{2a}$ $= \frac{-(16)}{2(-1)}$ $x = 8$ The value of y-coordinate can be calculated by substituting x = 8 in equation (2). $y = 16 - x$ $y = 16 - 8$ $y = 8$ The coordinates of the maximum turning point is $(8;8)$. Area $= 8m \times 8m = 64m^2$ **3. Interpretations of results** The dimensions of the garden that will give maximum area are the length of 8m and the width of 8m. The maximum area is $64m^2$. **4. Validation of the results** The perimeter of this figure= 8m + 8m + 8m + 8m = 32m. This make the solution to be considered correct since its perimeter is 32m, which has been stated in the problem **Alternative response** An investigative table can be used to find solutions to problems of optimisation **1. Understanding the problem** Learners have to choose the combinations that give the perimeter of 32m	✓ $-x^2 + 16x$ ✓ $x = \frac{-(16)}{2(-1)}$ ✓Interpreting the results: the length is 8m and the width is also 8m ✓ verification of the solution (6) **OR**

2. Formulating the mathematical model

Compiling a table containing all the combinations that give the perimeter of 32m

Length(m)	Breadth(m)	Area(m^2)
15	1	$15 \times 1 = 15$
14	2	$14 \times 2 = 28$
13	3	$13 \times 3 = 39$
12	4	$12 \times 4 = 48$
11	5	$11 \times 5 = 55$
10	6	$10 \times 6 = 60$
9	7	$9 \times 7 = 63$
8	8	$8 \times 8 = 64$
7	9	$7 \times 9 = 63$
6	10	$6 \times 10 = 60$
5	11	$5 \times 11 = 55$
4	12	$4 \times 12 = 48$

✓ multiplication of two numbers that give a perimeter of 32 (at least five)

✓ multiplication of two numbers that give a perimeter of 32 (at least ten)

✓ multiplication of two numbers that give a perimeter of 32 (up to the point where the values of areas start to decrease)

✓ identification of the maximum area

3. Solving the mathematical model

Finding the areas of different combinations as illustrated in the third column. They have to try lot of combinations up to the point where the values of their products start to decrease.

From the table, they have to establish which combination gives the maximum area. None of the values of the areas are above $64m^2$.

Maximum area is obtained when both the length and width are equal to 8. **That is to say maximum area is obtained when the rectangle has equal sides (a square)**

4. Interpreting the results

The dimensions of the garden that will give maximum area are the length of 8m and the width of 8m. The maximum area is $64m^2$.

✓stating of both dimensions that give the maximum value

5. Validation of results

The perimeter of this figure= 8m + 8m + 8m + 8m = 32m. This make the solution to be considered correct since its perimeter is 32m, which has been stated in the problem

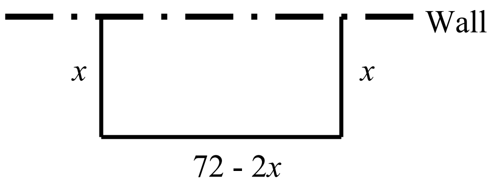
✓verification of the solutions
(6)

6 marks

QUESTION 3

3.1

1. Understanding the problem



Possible assumptions:

- No space between the edge of the garden and the fence
- All covered land is arable
- The land is flat, not undulating

2. Formulating the mathematical model

Let the length of each side perpendicular to the brick wall be x.

$\therefore$ the length of the side parallel to the wall = $72 - 2x$

✓Specifying the variables and write the formula $72 - 2x$

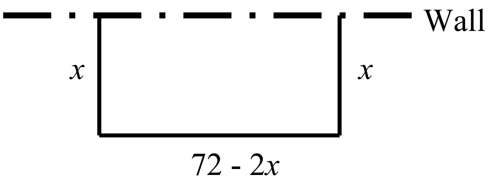
3. Solving the mathematical model

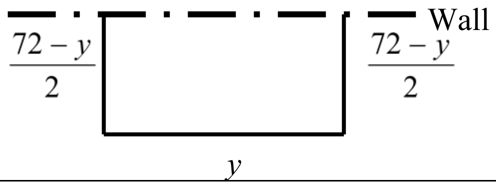
Then the area of the enclosed field:

$$\text{Area of a rectangle} = l \times b$$
$$= (72 - 2x)x$$
$$= -2x^2 + 72x$$

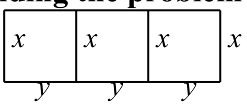
The area is a quadratic expression with the maximum turning point. Then the x coordinate of the turning point is given by:

✓ $-2x^2 + 72x$

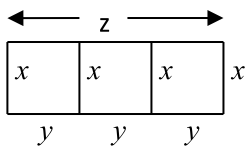
$x = \frac{-b}{2a}$ $= \frac{-72}{2(-2)}$ $x = 18$ Substituting x by 18 in y = 72 – 2x gives the value of y as: y = 72 – 2x y = 72 – 2(18) y = 36	✓ $x = \frac{-72}{2(-2)}$
4. Interpretation of results The dimensions that gives the maximum area are the length of 36m and width of 18m	✓ The width is 18m and the length is 36m
5. Validating the results From the sketch, the sum of the three sides $= x + (72 - 2x) + x$ = 18m + [72m – 2(18m)] + 18m = 72m This agrees with the statement which said you are given the wire that is 72m long	✓verification of the solution (5)
ALTERNATIVE RESPONSE	**OR**
1. Understanding of the problem 	✓Specifying the variables and write the formula 72 – 2x
2. Formulation of the mathematical model The same procedure as in the first response can be followed up to the point of the quadratic function A = $-2x^2 + 72x$.	

	✓ $-2x^2 + 72x$
3. Solving the mathematical model From here the completion of the square has to be done. $A=-2((x^2-36x+(36)^2-(36)^2)$ $=-2[(x-18)^2-18^2]$ $=-2(x-18)^2+648$	✓ $-2(x-18)^2+648$
$\therefore$ When $x=18$, the area of the enclosed portion of the car park will have a maximum area. Substituting 18 in y = 72 – 2x $y=72-2x$ $=72-2(18)$ $y=36$ **4. Interpreting of the results** The side parallel to the wall is 36m while the other two sides perpendicular to the wall are 18m each.	✓ The width is 18m and the length is 36m
5. Validating the results Adding the length of the side parallel to the rock and the other two equal sides gives $y+x+x$ = 36m +18m +18m = 72m This shows that the obtained solutions are correct since their total gives the length of the fence	✓verification of the solutions (5)
ALTERNATIVE RESPONSE 1. **Understanding the problem** 	

2. Formulating the mathematical model Let the length be y. $\therefore$ the breadth or width $=\frac{72-y}{2}$. **3. Solving the mathematical model** The area $=y\left(\frac{72-y}{2}\right)=y\left(36-\frac{1}{2}y\right)$ $y=-\frac{1}{2}y^2+36y$ $\therefore$ The function has a maximum turning point at: $y=\frac{-(36)}{2\left(-\frac{1}{2}\right)}=36m$ $\therefore$ The width $=\frac{72-y}{2}=\frac{72-36}{2}=18m$ **4. Interpretation of results** The dimensions of the vegetable garden that would give a maximum area are the length of 36 metres and breadth of 18 metres	✓Defining the variables and write the other two sides as $\frac{72-y}{2}$ ✓ $-\frac{1}{2}y^2+36y$ ✓ $y=\frac{-(36)}{2\left(-\frac{1}{2}\right)}$
5. Validation of the results Like in the other alternative methods, the solutions can be verified by adding the obtained three values to see that they add up to 72metres. There should be a pair of equal sides. **3.2** The maximum area is given as: $A=l\times b$ $=36m\times 18m$ $=648m^2$ The maximum area of the car park is 648m^2	✓ The width is 18m and the length is 36m ✓verification of the solution (5) ✓ 648m^2

ALTERNATIVE RESPONSE In the option that used the completion of the square, the maximum area can be extracted from the expression $A = -2(x-18)^2 + 648$ It can be seen that the maximum area is 648m^2	(1) **OR** ✓ 648m^2 (1)
	6 marks
QUESTION 4	
1. Understanding the problem  Total length of fence = 4x + 6y 6004x + 6y …… (1) **2. Formulating a mathematical model** Making y the subject of the formula from equation (1) 600 = 4x + 6y $600 = 4x + 6y$ $6y = 600 - 4x$ $y = 100 - \frac{2}{3}x$ .....(2) **6. Solving the mathematical model** $6y = 600 - 4x$ Make y the subject of the formula $y = 100 - \frac{2}{3}x$ .....(2	Possible assumptions: - No boulders of rocks or trees with large trunks along the path where fence is to be erected - The land is flat, not undulating ✓ Producing the formula 600= 6y + 4x ✓ $y = 100 - \frac{2}{3}x$

Area of the pasture is given as: $A = xy$ $A = x(100 - \frac{2}{3}x)$ $= 100x - \frac{2}{3}x^2$ $= -\frac{2}{3}x^2 + 100x$	✓ $-\frac{2}{3}x^2 + 100x$
This is a quadratic function with the maximum turning point with $a = -\frac{2}{3}$ and b = 100. The value of the x-coordinate of the turning point is then calculated as: $x = \frac{-b}{2(a)}$ $= \frac{-100}{2(-\frac{2}{3})}$ $x = 75$	✓ $x = \frac{-100}{2(-\frac{2}{3})}$
The value of y can be calculated by substituting x = 75 in equation (2) $y = 100 - \frac{2}{3}x$ $y = 100 - \frac{2}{3}(75)$ $y = 100 - 50$ $y = 50$ From the diagram the length = $y + y + y = 3y$ and breadth = x $\therefore Area = l \times b$ $= 3y \times x$ $= 3(50m) \times 75m$ $= 11250\, m^2$	✓The length of each camp is 75m and width of each camp is 50m
7. Interpretation of results ∴ The maximum area that can be enclosed by this fence is 11250m^2	✓The maximum area is 11250m^2

8. Validation of results	
To check the validity of the answer, you can add all the dimensions to see if they add up to 600m’	✓ verifying the solution
Length of wire = $x + x + x + x + y + y + y + y + y + y$ $= 4(75\text{m}) + 6(50\text{m})$ $= 300\text{m} + 300\text{m}$ $= 600\text{m}$	(7)
ALTERNATIVE RESPONSE	**OR**
1. Understanding the problem  	Adding all sides and equate it to 600 is an indication that the problem has been understood
2. Formulating a mathematical model	
Let the total length of the pasture be z. The total length of the fence can be given as $P = 2z + 4x$	✓ 600= 2z + 4x
Equating 600m to the perimeter gives 2z + 4x = 600	
3. Solving the mathematical model	
Making z the subject of the formula gives $2z + 4x = 600$ $2z = 600 - 4x$ $z = 300 - 2x$	✓ 300 – 2x
Area of the pasture can be calculated using, A= xz $= x(300 - 2x)$	

$= 300x - 2x^2$ This is a quadratic function with the maximum turning point. The value of the x-coordinate is then calculated as: $x = \frac{-b}{2(a)}$ $x = \frac{-300}{2(-2)} = 75$ We can then substitute x in equation 3 to find the value of z: $z = 300 - 2x$ $= 300 - 2(75)$ $z = 150$ The maximum area will be found if the length is 150m and the width is 75m. ∴ Maximum Area = 75m × 150m = 11 250m^2 **4. Interpretation of results** ∴ The maximum area that can be enclosed by this fence is 11250m^2 **5. Validation of results** To check the validity of the answer, you can add all the dimensions to see if they add up to 600m' Length of wire = $x + x + x + x + y + y$ = 4(75m) + 2(150m) = 300m + 300m = 600m	✓ $-2x^2 + 300x$ ✓ $x = \frac{-300}{2(-2)}$ ✓The length of the pasture is 150 m and the width is 75m ✓The maximum area is 11250m^2 ✓ verifying the solution (7)
7 marks	
TOTAL MARKS; 25	

Appendix C: Marking Rubric

QUESTION 1		
Level	**Definition**	**Score/ Marks**
	Understanding the Problem	
Level 1	Includes the expressions showing that s/he did not understand the problem, and did not make an assumption, did not form a relationship between variables	0
Level 2	Includes the expressions showing that s/he understood the problem to some extent, did not make an assumption, but formed a relationship between variables	1
Level 3	Includes the expressions showing that s/he understood the problem completely, made a correct assumption, formed a relationship between variables	2
	Mathematise (formulating a model)	
Level 1	Fails to select appropriate mathematical symbols, fails to develop a mathematical model (equation or expression), makes no attempt to simplify the relevant quantities and their relationships	0
Level 2	Selects appropriate mathematical symbols, develops a mathematical problem (equation or expression) that is partial correct and simplifies the quantities and their relations to some extent	1
Level 3	Selects appropriate mathematical symbols, develops a mathematical problem (equation or expression) that is correct to represent the real model simplified the quantities and their relations	2
	Mathematical work (Solving the mathematical problem)	
Level 1	Makes no utilisation of mathematical knowledge to solve the problem, does not apply proper mathematical methods to find solutions, does not solve the constructed model to get solutions	0

Level 2	Makes deficiencies/mistakes in the solution of the correctly constructed mathematical models.	1
Level 3	Obtains the correct mathematical solution by solving the correctly constructed mathematical models using proper methods or strategies	2
Interpreting the results		
Level 1	Misinterprets, or does not interpret the obtained mathematical solution in a real-life context. Gives no general statement regarding solution(s) obtained	0
Level 2	Partially interprets the obtained correct mathematical solution in a real-life context, makes attempts to make a general statements regarding solutions to certain extent	1
Level 3	Interprets clearly the obtained correct mathematical solution in a real-life context. Gives a clear general statements regarding results	2
Validate the results		
Level 1	Not validating or making an invalid validation.	0
Level 2	Validates partially by not giving the reason as well as not correcting the determined mistakes. States that the obtained solution is invalid but no reason given	1
Level 3	Validates completely, giving the correct reason why the solutions is acceptable/not acceptable and correcting the determined mistakes, verifies that the multiplication of the two obtained dimensions gives an area of 54m Considers alternative ways of solving the problem	2
	Question 1 subtotal	/10
	QUESTION2	
	Definition	**Score/ Marks**
Understanding the Problem		
Level 1	Includes the expressions showing that s/he did not understand the problem, and did not make an assumption, did not form a relationship between variables	0

Level 2	Includes the expressions showing that s/he understood the problem to some extent, did not make an assumption, but formed a relationship between variables	1
Level 3	Includes the expressions showing that s/he understood the problem completely, made a realistic assumption, formed a relationship between variables	2
Mathematise (formulating a model)		
Level 1	Fails to select appropriate mathematical symbols, fails to develop a mathematical model (equation or expression), makes no attempt to simplify the relevant quantities and their relationships	0
Level 2	Selects appropriate mathematical symbols, develops a mathematical problem (equation or expression) that is partial correct and simplifies the quantities and their relations to some extent	1
Level 3	Selects appropriate mathematical symbols, develops a mathematical problem (equation or expression) that is correct to represent the real model simplified the quantities and their relations	2
Mathematical work (Solving the mathematical problem)		
Level 1	Makes no utilisation of mathematical knowledge to solve the problem, does not apply proper mathematical methods to find solutions, does not solve the constructed model to get solutions	0
Level 2	Makes deficiencies/mistakes in the solution of the correctly constructed mathematical models.	1
Level 3	Obtains the correct mathematical solution by solving the correctly constructed mathematical models using proper methods or strategies	2
Interpreting the results		
Level 1	Misinterprets, or does not interpret the obtained mathematical solution in a real-life context. Gives no general statement regarding solution(s) obtained	0

Level 2	Partially interprets the obtained correct mathematical solution in a real-life context, makes attempts to make a general statements regarding solutions to certain extent	1
Level 3	Interprets clearly the obtained correct mathematical solution in a real-life context. Gives a clear general statements regarding results	2
Validate the results		
Level 1	Makes no effort to validate the solutions or making an invalid validation.	0
Level 2	Validates partially and not correcting the determined mistakes.	1
Level 3	Validates completely, giving the correct reason why the solutions is acceptable/not acceptable and correcting the determined mistakes, verifies that the obtained solutions give a perimeter of 32m	2
	Question 2 subtotal	/10
QUESTION 3		
	Definition	**Score/ Marks**
Understanding the Problem		
Level 1	Includes the expressions showing that s/he did not understand the problem, and did not make an assumption, did not form a relationship between variables	0
Level 2	Includes the expressions showing that s/he understood the problem to some extent, did not make an assumption, but formed a relationship between variables	1
Level 3	Includes the expressions showing that s/he understood the problem completely, made a realistic assumption, formed a relationship between variables	2
Mathematise (formulating a model)		
Level 1	Fails to select appropriate mathematical symbols, Could not develop a mathematical model (equation or expression), not simplifying the relevant quantities and their relationships	0

Level 2	Selects appropriate mathematical symbols, developed a mathematical problem (equation or expression) that is partial correct and simplified the quantities and their relations to some extent	1
Level 3	Select appropriate mathematical symbols, developed a mathematical problem (equation or expression) that is correct to represent the real model simplified the quantities and their relations	2
Mathematical work (Solving the mathematical problem)		
Level 1	Not presenting a mathematical solution, wrongly solving the constructed models or solving the wrong mathematical model	0
Level 2	Including deficiencies/mistakes in the solution of the correctly constructed mathematical models.	1
Level 3	Achieving the correct mathematical solution by solving the correctly constructed mathematical models.	2
Interpreting the results		
Level 1	Misinterprets, or does not interpret the obtained mathematical solution in a real-life context. Gives no general statement regarding solution(s) obtained	0
Level 2	Partially interprets the obtained correct mathematical solution in a real-life context, makes attempts to make a general statements regarding solutions to certain extent	1
Level 3	Interprets clearly the obtained correct mathematical solution in a real-life context. Gives a clear general statements regarding results	2
Validate the results		
Level 1	Makes no effort to validate the solutions or making an invalid validation.	0
Level 2	Validates partially and not correcting the determined mistakes.	1
Level 3	Validates completely, giving the correct reason why the solutions is acceptable/not acceptable and correcting the determined mistakes, verifies that the obtained solutions for the three sides of the garden give a total length of 72m	2
Question 3 subtotal		/10

QUESTION 4		
Level	**Definition**	**Score/ Marks**
	Understanding the Problem	
Level 1	Includes the expressions showing that s/he did not understand the problem, and did not make an assumption, did not form a relationship between variables	0
Level 2	Includes the expressions showing that s/he understood the problem to some extent, did not make an assumption, but formed a relationship between variables	1
Level 3	Includes the expressions showing that s/he understood the problem completely, made a realistic assumption, formed a relationship between variables	2
	Mathematise (formulating a model)	
Level 1	Could not select appropriate mathematical symbols, Could not develop a mathematical model (equation or expression), not simplifying the relevant quantities and their relationships	0
Level 2	Select appropriate mathematical symbols, developed a mathematical problem (equation or expression) that is partial correct and simplified the quantities and their relations to some extent	1
Level 3	Select appropriate mathematical symbols, developed a mathematical problem (equation or expression) that is correct to represent the real model simplified the quantities and their relations	2
	Mathematical work (Solving the mathematical problem)	
Level 1	Not presenting a mathematical solution, wrongly solving the constructed models or solving the wrong mathematical model	0
Level 2	Including deficiencies/mistakes in the solution of the correctly constructed mathematical models.	1
Level 3	Achieving the correct mathematical solution by solving the correctly constructed mathematical models.	2

Interpreting the results		
Level 1	Misinterprets, or does not interpret the obtained mathematical solution in a real-life context. Gives no general statement regarding solution(s) obtained	0
Level 2	Partially interprets the obtained correct mathematical solution in a real-life context, makes attempts to make a general statements regarding solutions to certain extent	1
Level 3	Interprets clearly the obtained correct mathematical solution in a real-life context. Gives a clear general statements regarding results	2
Validate the results		
Level 1	Makes no effort to validate the solutions or making an invalid validation.	0
Level 2	Validates partially and not correcting the determined mistakes. Verifies the obtained but does not try to find the mistakes	1
Level 3	Validates completely, giving the correct reason why the solutions is acceptable/not acceptable and correcting the determined mistakes, verifies that the obtained solutions for four sides that are equal to x and six sides equal to y add up to 600m	`2
Question 4 subtotal		/10
TOTAL MARKS		/40

Appendix D: Simplified or reduced data analysis per question

QUESTION 1 ANALYSIS
Mr Nene wants to plant seedlings in his rectangular garden with an area of 54 m^2. The length of the garden is longer than its width by 3m. What are the dimensions of his garden?

Learner	Defined variables YES/NO	Made assumption YES/NO	Comprehended the scenario correctly YES/NO	Formulated a correct mathematical model YES/NO	Solve the formulated model correctly YES/NO	Verified the obtained solutions YES/NO	Interpreted the results YES/NO
L1	NO	NO	YES	YES	YES	NO	NO
L2	NO	NO	YES	YES	YES	NO	NO
L3	NO	NO	YES	YES	YES	NO	NO
L4	NO	NO	YES	YES	YES	NO	NO
L5	NO	NO	NO	NO	NO	NO	NO
L6	NO	NO	YES	YES	YES	NO	NO
L7	NO	NO	YES	YES	YES	NO	NO
L8	NO	NO	YES	YES	YES	NO	NO
L9	NO	NO	YES	YES	NO	NO	NO
L10	NO	NO	NO	NO	NO	NO	NO
L11	NO	NO	YES	YES	YES	NO	NO
L12	NO	NO	YES	YES	YES	NO	NO
L13	NO	NO	YES	YES	YES	NO	NO
L14	NO	NO	YES	YES	YES	NO	NO
L15	NO	NO	YES	YES	NO	NO	NO
L16	NO	NO	YES	YES	YES	NO	NO
L17	NO	NO	YES	YES	YES	NO	NO
L18	NO	NO	YES	YES	YES	NO	NO
L19	NO	NO	YES	YES	YES	NO	NO
L20	NO	NO	YES	YES	YES	NO	NO
L21	NO	NO	YES	YES	YES	NO	NO
L22	NO	NO	YES	YES	YES	NO	NO
L23	NO	NO	YES	YES	YES	NO	NO
L24	NO	NO	YES	YES	YES	NO	NO
L25	NO	NO	YES	YES	YES	NO	NO
L26	NO	NO	YES	YES	YES	NO	NO
L27	NO	NO	YES	YES	YES	NO	NO
L28	NO	NO	YES	YES	YES	NO	NO

L29	NO	NO	YES	YES	YES	NO	NO
L30	NO	NO	YES	YES	YES	NO	NO
L31	NO	NO	YES	YES	YES	NO	NO
L32	NO	NO	YES	YES	YES	NO	NO
L33	NO	NO	YES	YES	YES	NO	NO
L34	NO	NO	YES	YES	YES	NO	NO
L35	NO	NO	YES	YES	YES	NO	NO
L36	NO	NO	YES	YES	YES	NO	NO
L37	NO	NO	YES	YES	YES	NO	NO
L38	NO	NO	YES	YES	YES	NO	NO
L39	NO	NO	YES	YES	YES	NO	NO
L40	NO	NO	YES	YES	YES	NO	NO
L41	NO	NO	YES	YES	YES	NO	NO
L42	NO	NO	YES	YES	YES	NO	NO
L43	NO	NO	YES	YES	YES	NO	NO
L44	NO	NO	YES	YES	YES	NO	NO
L45	NO	NO	YES	YES	YES	NO	NO
L46	NO	NO	YES	YES	YES	NO	NO
L47	NO	NO	YES	YES	YES	NO	NO
L48	NO	NO	YES	YES	YES	NO	NO
L49	NO	NO	YES	YES	YES	NO	NO
L50	NO	NO	YES	YES	NO	NO	NO
L51	NO	NO	YES	YES	YES	NO	NO
L52	NO	NO	YES	YES	YES	NO	NO
L53	NO	NO	YES	YES	YES	NO	NO
L54	NO	NO	YES	YES	YES	NO	NO
L55	NO	NO	YES	YES	YES	NO	NO
L56	NO	NO	YES	YES	YES	NO	NO
L57	NO	NO	YES	YES	YES	NO	NO
L58	NO	NO	YES	YES	YES	NO	NO
L59	NO	NO	YES	YES	YES	NO	NO
L60	NO	NO	YES	YES	YES	NO	NO

L61	NO	NO	YES	YES	YES	NO	NO
L62	NO	NO	YES	YES	YES	NO	NO
L63	NO	NO	YES	YES	YES	NO	NO
L64	NO	NO	YES	YES	YES	NO	NO
L65	NO	NO	YES	YES	YES	NO	NO
L66	NO	NO	YES	YES	YES	NO	NO
L67	NO	NO	YES	YES	YES	NO	NO
L68	NO	NO	YES	YES	YES	NO	NO
L69	NO	NO	YES	YES	YES	NO	NO
L70	NO	NO	YES	YES	YES	NO	NO
L71	NO	NO	YES	YES	YES	NO	NO
L72	NO	NO	YES	YES	YES	NO	NO
L73	NO	NO	YES	YES	YES	NO	NO
L74	NO	NO	YES	YES	YES	NO	NO
L75	NO	NO	YES	YES	YES	NO	NO
L	NO	NO	YES	YES	YES	NO	NO

Question 2

In order to protect his vegetables, a farmer buys 32m of fencing for his rectangular garden. Determine the dimensions of the vegetable garden if it should cover a maximum area

Learner	Defined variables YES/NO	Made assumption YES/NO	Comprehended the scenario correctly YES/NO	Formulated a correct mathematical model YES/NO	Solve the formulated model correctly YES/NO	Verified the obtained solutions YES/NO	Interpreted the results YES/NO
L1	NO	NO	NO	NO	NO	NO	NO
L2	NO	NO	NO	NO	NO	NO	NO
L3	NO	NO	NO	NO	NO	NO	NO
L4	NO	NO	NO	NO	NO	NO	NO
L5	NO	NO	NO	NO	NO	NO	NO
L6	NO	NO	NO	NO	NO	NO	NO
L7	NO	NO	YES	YES	YES	NO	NO
L8	NO	NO	YES	YES	YES	NO	NO
L9	NO	NO	YES	YES	YES	NO	NO
L10	NO	NO	YES	YES	YES	NO	NO

L11	NO	NO	YES	YES	YES	NO	NO
L12	NO	NO	YES	YES	YES	NO	NO
L13	NO	NO	YES	YES	YES	NO	NO
L14	NO	NO	YES	YES	YES	NO	NO
L15	NO	NO	NO	NO	NO	NO	NO
L16	NO	NO	NO	NO	NO	NO	NO
L17	NO	NO	YES	YES	YES	NO	NO
L18	NO	NO	YES	YES	YES	NO	NO
L19	NO	NO	YES	YES	YES	NO	NO
L20	NO	NO	YES	YES	YES	NO	NO
L21	NO	NO	YES	YES	YES	NO	NO
L22	NO	NO	YES	YES	YES	NO	NO
L23	NO	NO	YES	YES	YES	NO	NO
L24	NO	NO	NO	NO	NO	NO	NO
L25	NO	NO	YES	YES	YES	NO	NO
L26	NO	NO	YES	YES	YES	NO	NO
L27	NO	NO	YES	YES	YES	NO	NO
L28	NO	NO	YES	YES	YES	NO	NO
L29	NO	NO	YES	YES	YES	NO	NO
L30	NO	NO	YES	YES	YES	NO	NO
L31	NO	NO	YES	YES	YES	NO	NO
L32	NO	NO	YES	YES	YES	NO	NO
L33	NO	NO	YES	YES	YES	NO	NO
L34	NO	NO	YES	YES	YES	NO	NO
L35	NO	NO	YES	YES	YES	NO	NO
L36	NO	NO	YES	YES	YES	NO	NO
L37	NO	NO	YES	YES	NO	NO	NO
L38	NO	NO	YES	YES	YES	NO	NO
L39	NO	NO	YES	YES	YES	NO	NO
L40	NO	NO	YES	YES	YES	NO	NO
L41	NO	NO	YES	YES	YES	NO	NO
L42	NO	NO	YES	YES	YES	NO	NO

L43	NO	NO	YES	YES	NO	NO	NO
L44	NO	NO	YES	YES	NO	NO	NO
L45	NO	NO	YES	YES	NO	NO	NO
L46	NO	NO	YES	YES	NO	NO	NO
L47	NO	NO	YES	YES	YES	NO	NO
L48	NO	NO	YES	YES	NO	NO	NO
L49	NO	NO	NO	NO	NO	NO	NO
L50	NO	NO	YES	YES	YES	NO	NO
L51	NO	NO	YES	YES	YES	NO	NO
L52	NO	NO	NO	NO	YES	NO	NO
L53	NO	NO	NO	NO	YES	NO	NO
L54	NO	NO	YES	YES	YES	NO	NO
L55	NO	NO	YES	YES	YES	NO	NO
L56	NO	NO	YES	YES	YES	NO	NO
L57	NO	NO	YES	YES	NO	NO	NO
L58	NO	NO	YES	YES	YES	NO	NO
L59	NO	NO	YES	YES	YES	NO	NO
L60	NO	NO	YES	YES	YES	NO	NO
L61	NO	NO	YES	YES	YES	NO	NO
L62	NO	NO	YES	YES	NO	NO	NO
L63	NO	NO	YES	YES	YES	NO	NO
L64	NO	NO	YES	YES	NO	NO	NO
L65	NO	NO	YES	YES	YES	NO	NO
L66	NO	NO	YES	YES	YES	NO	NO
L67	NO	NO	YES	YES	YES	NO	NO
L68	NO	NO	YES	YES	YES	NO	NO
L69	NO	NO	YES	YES	YES	NO	NO
L70	NO	NO	YES	YES	YES	NO	NO
L71	NO	NO	YES	YES	YES	NO	NO
L72	NO	NO	YES	YES	YES	NO	NO
L73	NO	NO	YES	YES	YES	NO	NO
L74	NO	NO	YES	YES	NO	NO	NO

L75	NO	NO	YES	YES	NO	NO	NO

Question 3

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 72m **of wire fencing:**

- **Determine the dimensions that would create a garden of maximum area.**
- **What is the maximum area?**

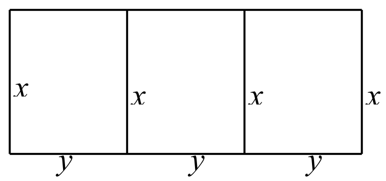
Learner	Defined variables YES/NO	Made assumption YES/NO	Comprehended the scenario correctly YES/NO	Formulated a correct mathematical model YES/NO	Solve the formulated model correctly YES/NO	Verified the obtained solutions YES/NO	Interpreted the results YES/NO
L1	NO	NO	NO	NO	NO	NO	NO
L2	NO	NO	NO	NO	NO	NO	NO
L3	Learner did not attempt the question						
L4	NO	NO	NO	NO	NO	NO	NO
L5	NO	NO	NO	NO	NO	NO	NO
L6	NO	NO	NO	NO	NO	NO	NO
L7	Learner did not attempt the question						
L8	Learner did not attempt the question						
L9	NO	NO	NO	NO	NO	NO	NO
L10	Learner did not attempt the question						
L11	Learner did not attempt the question						
L12	NO	NO	NO	NO	NO	NO	NO
L13	Learner did not attempt the question						
L14	Learner did not attempt the question						
L15	NO	NO	NO	NO	NO	NO	NO
L16	NO	NO	NO	NO	NO	NO	NO
L17	Learner did not attempt the question						
L18	NO	NO	NO	NO	NO	NO	NO
L19	Learner did not attempt the question						
L20	Learner did not attempt the question						
L21	Learner did not attempt the question						
L22	NO	NO	NO	NO	NO	NO	NO
L23	Learner did not attempt the question						

L24	NO	NO	NO	NO	NO	NO	NO
L25	Learner did not attempt the question						
L26	NO	NO	NO	NO	NO	NO	NO
L27	Learner did not attempt the question						
L28	Learner did not attempt the question						
L29	NO	NO	NO	NO	NO	NO	NO
L30	Learner did not attempt the question						
L31	NO	NO	YES	YES	YES	NO	NO
L32	NO	NO	YES	YES	YES	NO	NO
L33	NO	NO	YES	YES	YES	NO	NO
L34	NO	NO	YES	YES	YES	NO	NO
L35	NO	NO	YES	YES	YES	NO	NO
L36	NO	NO	YES	YES	YES	NO	NO
L37	NO	NO	YES	YES	NO	NO	NO
L38	NO	NO	YES	YES	NO	NO	NO
L39	NO	NO	YES	YES	YES	NO	NO
L40	NO	NO	YES	YES	YES	NO	NO
L41	NO	NO	YES	YES	YES	NO	NO
L42	NO	NO	YES	YES	YES	NO	NO
L43	NO	NO	YES	YES	YES	NO	NO
L44	NO	NO	YES	YES	YES	NO	NO
L45	NO	NO	YES	YES	NO	NO	NO
L46	NO	NO	YES	YES	YES	NO	NO
L47	NO	NO	YES	YES	YES	NO	NO
L48	Learner did not attempt the question						
L49	NO	NO	YES	YES	NO	NO	NO
L50	NO	NO	YES	YES	YES	NO	NO
L51	NO	NO	YES	YES	YES	NO	NO
L52	NO	NO	YES	YES	YES	NO	NO
L53	NO	NO	YES	YES	YES	NO	NO
L54	NO	NO	YES	YES	YES	NO	NO
L55	NO	NO	YES	YES	YES	NO	NO

L56	NO	NO	YES	YES	YES	NO	NO
L57	NO	NO	YES	YES	YES	NO	NO
L58	Learner did not attempt the question						
L59	NO	NO	NO	NO	NO	NO	NO
L60	NO	NO	YES	YES	YES	NO	NO
L61	Learner did not attempt the question						
L62	NO	NO	YES	YES	YES	NO	NO
L63	Learner did not attempt the question						
L64	NO	NO	YES	YES	YES	NO	NO
L65	NO	NO	YES	YES	YES	NO	NO
L66	NO	NO	YES	YES	YES	NO	NO
L67	NO	NO	YES	YES	YES	NO	NO
L68	NO	NO	YES	YES	YES	NO	NO
L69	Learner did not attempt the question						
L70	NO	NO	YES	YES	YES	NO	NO
L71	NO	NO	YES	YES	YES	NO	NO
L72	NO	NO	YES	YES	YES	NO	NO
L73	NO	NO	YES	YES	NO	NO	NO
L74	NO	NO	YES	YES	NO	NO	NO
L75	NO	NO	YES	YES	YES	NO	NO

Question 4

A farmer has 600m of fence to create a rectangular pasture which has to be divided into three camps all with the same dimensions. What is the maximum area that can be enclosed with this fence?



Learner	Defined variables YES/NO	Made assumption YES/NO	Comprehended the scenario correctly YES/NO	Formulated a correct mathematical model YES/NO	Solve the formulated model correctly YES/NO	Verified the obtained solutions YES/NO	Interpreted the results YES/NO
L1	NO	NO	NO	NO	NO	NO	NO
13	Learner did not attempt the question						
L4	Learner did not attempt the question						
L5	Learner did not attempt the question						
L6	NO	NO	NO	NO	NO	NO	NO
L7	Learner did not attempt the question						
L8	Learner did not attempt the question						
L9	NO	NO	NO	NO	NO	NO	NO
L10	Learner did not attempt the question						
L11	Learner did not attempt the question						
L12	Learner did not attempt the question						
L13	Learner did not attempt the question						
L14	Learner did not attempt the question						
L15	NO	NO	NO	NO	NO	NO	NO
L16	Learner did not attempt the question						
L17	Learner did not attempt the question						
L18	Learner did not attempt the question						
L19	Learner did not attempt the question						
L20	Learner did not attempt the question						
L21	Learner did not attempt the question						
L22	Learner did not attempt the question						
L23	Learner did not attempt the question						
L24	NO	NO	NO	NO	NO	NO	NO
L25	Learner did not attempt the question						
L26	Learner did not attempt the question						
L27	Learner did not attempt the question						
L28	Learner did not attempt the question						
L29	Learner did not attempt the question						

L30	Learner did not attempt the question						
L31	NO	NO	YES	YES	YES	NO	NO
L32	NO	NO	NO	NO	NO	NO	NO
L33	NO	NO	NO	NO	NO	NO	NO
L34	NO	NO	NO	NO	NO	NO	NO
L35	NO	NO	YES	YES	YES	NO	NO
L36	NO	NO	YES	YES	YES	NO	NO
L37	NO	NO	NO	NO	NO	NO	NO
L38	NO	NO	YES	YES	YES	NO	NO
L39	Learner did not attempt the question						
L40	NO	NO	YES	YES	YES	NO	NO
L41	NO	NO	YES	YES	YES	NO	NO
L42	NO	NO	YES	YES	YES	NO	NO
L43	Learner did not attempt the question						
L44	NO	NO	NO	NO	NO	NO	NO
L45	Learner did not attempt the question						
L46	NO	NO	YES	YES	YES	NO	NO
L47	NO	NO	NO	NO	NO	NO	NO
L48	Learner did not attempt the question						
L49	NO	NO	YES	YES	YES	NO	NO
L50	NO	NO	YES	YES	YES	NO	NO
L51	NO	NO	NO	NO	NO	NO	NO
L52	NO	NO	NO	NO	NO	NO	NO
L53	NO	NO	NO	NO	NO	NO	NO
L54	NO	NO	YES	YES	YES	NO	NO
L55	NO	NO	YES	YES	YES	NO	NO
L56	Learner did not attempt the question						
L57	Learner did not attempt the question						
L58	Learner did not attempt the question						
L59	NO	NO	YES	YES	YES	NO	NO
L60	NO	NO	YES	YES	YES	NO	NO
L61	Learner did not attempt the question						

L62	NO	NO	YES	YES	YES	NO	NO
L63	NO	NO	YES	YES	YES	NO	NO
L64	NO	NO	YES	YES	YES	NO	NO
L65	NO	NO	YES	YES	YES	NO	NO
L66	NO	NO	YES	YES	YES	NO	NO
L67	NO	NO	YES	YES	YES	NO	NO
L68	NO	NO	YES	YES	YES	NO	NO
L69	Learner did not attempt the question						
L70	NO	NO	YES	YES	YES	NO	NO
L71	NO	NO	YES	YES	YES	NO	NO
L72	NO	NO	YES	YES	YES	NO	NO
L73	NO	NO	YES	YES	NO	NO	NO
L74	NO	NO	YES	YES	NO	NO	NO
L75	NO	NO	YES	YES	NO	NO	NO

Appendix E: Ethical Clearance Certificate

UNISA university of south africa

UNISA COLLEGE OF EDUCATION ETHICS REVIEW COMMITTEE
Date: 2020/09/09

Ref: **2020/09/09/30721709/07/AM**
Name: Mr RB Dlamini Student
No.: 30721709

Dear Mr RB

Dlamini

Decision: Ethics Approval from 2020/09/09 to 2025/09/09

Researcher(s): Name: Mr RB Dlamini
E-mail address: rbdlamini@gmail.com
Telephone: 078 241 3651

Supervisor(s): Name: Prof. D. Brijlall
E-mail address: deonaraib@dut.ac.za
Telephone: 083 555 2390

Title of research:
Investigating Grade Eleven Learners Mathematical Modelling Competencies through Problem Solving

Qualification: PhD Mathematics Education

Thank you for the application for research ethics clearance by the UNISA College of Education Ethics Review Committee for the above mentioned research. Ethics approval is granted for the period 2020/09/09 to 2025/09/09

*The **low risk** application was reviewed by the Ethics Review Committee on 2020/09/09 in compliance with the UNISA Policy on Research Ethics and the Standard Operating Procedure on Research Ethics Risk Assessment.*

The proposed research may now commence with the provisions that:

1. The researcher will ensure that the research project adheres to the relevant guidelines set out in the Unisa Covid-19 position statement on research ethics attached.
2. The researcher(s) will ensure that the research project adheres to the values and principles expressed in the UNISA Policy on Research Ethics.
3. Any adverse circumstance arising in the undertaking of the research project that is relevant to the ethicality of the study should be communicated in writing to the UNISA College of Education Ethics Review Committee.
4. The researcher(s) will conduct the study according to the methods and procedures set out in the approved application.
5. Any changes that can affect the study-related risks for the research participants, particularly in terms of assurances made with regards to the protection of participants' privacy and the confidentiality of the data, should be reported to the Committee in writing.
6. The researcher will ensure that the research project adheres to any applicable national legislation, professional codes of conduct, institutional guidelines and scientific standards relevant to the specific field of study. Adherence to the following South African legislation is important, if applicable: Protection of Personal Information Act, no 4 of 2013; Children's act no 38 of 2005 and the National Health Act, no 61 of 2003.
7. Only de-identified research data may be used for secondary research purposes in future on condition that the research objectives are similar to those of the original research. Secondary use of identifiable human research data requires additional ethics clearance.
8. No field work activities may continue after the expiry date **2025/09/09**. Submission of a completed research ethics progress report will constitute an application for renewal of Ethics Research Committee approval.

Note:

The reference number **2020/09/09/30721709/07/AM** *should be clearly indicated on all forms of communication with the intended research participants, as well as with the Committee.*

Kind regards,

Prof AT Motlhabane
CHAIRPERSON: CEDU RERC
motlhat@unisa.ac.za

Prof PM Sebate
EXECUTIVE DEAN
Sebatpm@unisa.ac.za

Appendix F: APPLICATION FOR PERMISSION TO CONDUCT RESEARCH

education
Department:
Education
PROVINCE OF KWAZULU-NATAL

Application for Permission to Conduct Research in KwaZulu Natal Department of Education Institutions

1. Applicants Details

Title: Mr.

Surname: DLAMINI

Name(s) Of Applicant(s): REUBEN BAFANA

Email: rbdlamini@gmail.com

Tel No: 017 826 5703 Fax: None

Cell: 078 241 3651

Postal Address: P O BOX 1269
PIET RETIEF
2380

2. Proposed Research Title: Investigating Grade Eleven Learners Mathematical Modelling Competencies through Problem

3. Have you applied for permission to conduct this research or any other research within the KZN DoE institutions? Yes [X] No

If "yes", please state reference Number: there was no reply but my application letter was dated 16 October 2020

4. Is the proposed research part of a tertiary qualification? Yes [X] No

If "yes"
Name of tertiary institution: University of South Africa (UNISA)

Faculty and or School: College of Education

Qualification: PhD Mathematics Education

Name of Supervisor: Prof. D. Brijlall **Supervisors Signature** Brijlall

If "no", state purpose of research: Not applicable

5. **Briefly state the Research Background**

BACKGROUND

Learners' achievement in Mathematics in South Africa has been very disappointing when compared to other developing countries. This has been revealed by learners' performance in Grade Twelve every year. Although learners performed poorly in all topics, their performance differs from one topic to another. Diagnostic reports have revealed that the worst performed topic every year in Paper One is Application of Differential Calculus, usually consisting of practical and realistic problems. In this topic learners are expected to solve problems involving real life situations and usually rely on Calculus procedures taught to them by their teachers. Table 1 below shows the average performance in each year for questions on optimisation. It can be seen that learners' performances in these problems have never surpassed 40%. The worst performance was in 2017.

Table 1: Average achievement in sub topics of Calculus

Average learners' achievement in percentage for the past 6 years in questions involving Application of Calculus						
YEAR	2014	2015	2016	2017	2018	2019
Average Performance in %	32	22	38	9	18	39

(Source: DBE; 2014, 2015, 2016, 2017, 2018 & 2019)

One of the specific aims of teaching Mathematics in South Africa is that mathematical modelling is the theme in which didactic of mathematics is delivered. Implementing mathematical modelling has to be done through the use of real life problems, if possible, in every topic of the curriculum. Illustrations employed in mathematical modelling must be pragmatic or lifelike (DBE 2011). Learners exposed to this curriculum must be able to represent mathematical information and concepts using mathematical expressions, symbols, equations, graphs, tables and pictures

Blum (1993) said there have been criticisms about teaching and learning of mathematics as some people do not see the reasons of studying mathematics. Most of the time learners are of the opinion that mathematics is just a manipulation of numbers. In other words, these learners do not see the how mathematics is useful and relevant to their lives. Therefore there is a need to change the way the subject is taught. He continued to say that if the developments of meaning are important goals of mathematics teaching and learning, therefore there is a need to incorporate modelling in teaching and learning of mathematics (Blum 1993)

It is for this reason that the researcher intends to conduct this study with the aim of improving learners' performance in mathematics deep-rooted in real life problems so that by the time they reach Grade 12 they have more than one strategy to resolve such problems. It has been indicated in Table 1 that learners' performance was poor in mathematics questions that involved real life problems, and therefore the researcher is hopeful that if learners' performance can improve in these types of questions, there may be an improvement in the overall performance in mathematics. Learners need to be equipped with several ways of solving mathematics questions in Grade 12 that involve optimisation.

Modeling involves formulating the real-life situations or converting the problems in mathematical explanations to a real situation. Therefore this study is aimed at determining learners' competencies in this approach where they can solve real life problems by applying the knowledge they learnt in topics such as Algebra and Functions. Learners have to realise that real life problems can be resolved by using other approaches other than the application of Calculus.

6. **What is the main research question(s) :**
   What are the competencies of Grade Eleven learners in mathematical modelling when solving real life problems?

7. **Methodology including sampling procedures and the people to be included in the sample:**

Grade 11 learners doing Mathematics in the three selected schools will be eligible to participate in the study. The schools have been selected using purposive sampling. This means the schools were selected based on the knowledge that they are having learners in Grade eleven who are doing Mathematics. Also these schools have the three year average performance that is above 50%. This made the researcher to believe that these schools can provide the data he is interested in. Bernard (2002) described purposive sampling technique as the intentional choice of participants due to the qualities they possess. The researcher takes a decision on what he or she needs to be known and specify people who can and are willing to give information because of their knowledge and experience

Learners from three selected secondary who will volunteer to partake in the study will now form the population for the study. If it happened that the population size is too large or is above 30, the researcher will then use stratified sampling to select his sample for the study. Since this is a qualitative research, it is not necessary for the sample size to be large. Mugo (2002) defined a sample as a restricted and definable portion or section of a population whose attributes or features are investigated solely to obtain information about the entire population

8. What contribution will the proposed study make to the education, health, safety, welfare of the learners and to the education system as a whole

This study will contribute to the improvement of performance in mathematics, especially on problems involving the real life situations. Learners will be exposed to new methods or approaches of learning. That is to say this research will provide learners opportunities of solving real life problems using knowledge gained from other topics such as Algebra and Functions, and at the same time to use their own strategies other than following procedures laid down by their teachers. Learners who will partake in this study are going to gain knowledge and skills that will improve their performances in mathematics. Research has shown that mathematical modelling improves learners' performance in mathematics. For example, a study by Doruk and Umay (2011) in Karaci Yasa and Karatas (2018), revealed that mathematical modelling is influencing learners' achievement in mathematics positively.

The emphasis of mathematical modelling is new in the South African school mathematics content or curriculum. It has been stated in the Curriculum and Assessment Policy Statements (CAPS) documents that mathematical modelling is essential for the coordination of mathematics curriculum. Therefore, the Department of Education is also going to benefit from this study because the results will inform the Department what are the levels of learners' competencies in mathematical modelling. On the other hand the study will prepare learners while still in Grade Eleven to develop conceptual understanding of working out these problems. Therefore the Department of Education can suggest or come up with correct interventions to assist the learners while still in Grade Eleven so that they are fully prepared once they reach Grade Twelve

Mathematical modelling is also done at the institutions of higher learning as a course/module. For example, at the University of South Africa, a student doing a Bachelor of Science degree and taking Mathematics as one of his/her major has to take Mathematical Modelling - APM1514 at first year. This module assists students to develop better comprehension of simple optimisation and applications. Learners who participated in this study gained deep understanding of mathematical modelling while still at school. By the time they have to go to institutions of higher learning they would be better prepared to do mathematical modelling.

KZN Department of Education Schools or Institutions from which sample will be drawn – If the list is long please attach at the end of the form		
1. Bambanani High School		
2. Langa High School		
3. Sigqamise High School		

9. **Research data collection instruments**: *(Note: a list and only a brief description is required here - the actual instruments must be attached)*:

A short test designed by the researcher will be used to collect data for this study. Participants will be required to write this test that will only take 30minutes. I will then collect learners' scripts in order to analyse and determine their competencies in modelling. (A copy of the test has been attached to this application

10. Procedure for obtaining consent of participants and where appropriate parents or guardians:

Since the study involves learners who will be participating in this sudy are minors (below the age of 18) consent from parents has to be obtained. As a researcher I will write letters to the parents of the learners to request for their permission to involve their children. The letters will indicate the following information; the title of the research, the purpose of the study, the institution where the researcher is studying, the name of the supervisor, contact details of both the supervisor and the researcher, and possible benefits of the study. The letters will also state clearly to the parents that participation of their child in the study is voluntary and his or her child can withdraw his/ her participation at any time when she or he feels uncomfortable to continue. The letter will also state the role of child in the study, confidentiality of the data collected and the guarantee that the child will never be exposed to any form harm. The parent or guardian of the child has to sign the declaration form if he/she agree that his/her child may participate in the study. If the consent form is not signed, the researcher will not involve the child in the study.

Letters will also be given to learners to ask for their assent or permission to be part of the study. In the letter the researcher will explain the purpose of the study, how the child will participate in the study. The letters will also explain that their participation is voluntary and they are free to withdraw at any time when they feel like. Learners will be requested to sign the assent form and a witness who is above 18 years need also to sign. Learners will be requested to also sign the declaration form to indicate they have read the contents of the letter and he/she agrees that he/she will participate in the writing the test and that his/her script can be used in the study.

11. Procedure to maintain confidentiality (if applicable):

The anonymity as well as confidentiality of the informants in this study is going to be protected by covering or obscuring their identities during the accumulation of data, analysis as well as during reporting of the results. Learners' real names will not appear in their scripts; instead they will use code names that will be issued to them by the researcher. Even the names of their schools will not appear in learners' scripts. In short, any form of information that may reveal participants' identification, such as their names or any significant feature of identity will be removed during data transcription.

All documentation that will be utilised for the purpose of this probe will be kept safe. These include learners' scripts for the test, consent letters or any document which bears the participant's personal detail will be retained in an enclosed lockable cabinet with no having access to it except me. The stored information is going to be eradicated or shattered in line with the University of South Africa Ethic Research Review Committee methods. After 5 years these scripts will be destroyed or burnt. Any form of data that will be stored as a soft copy will be kept under a secret password until is deleted after 5 years. Even that data that will be stored as a soft copy it is not going to have real names or identities of participants.

12. Questions or issues with the potential to be intrusive, upsetting or incriminating to participants (if applicable):

No questions or issues with the potential to be intrusive, upsetting or incriminating to participants. This is because the questions asked in the test are relevant to the grade and already covered in the First Term.

13. Additional support available to participants in the event of disturbance resulting from intrusive questions or issues (if applicable):

After the study has been conducted, teachers will be provided with the marking guidelines so that they can assist learners with remedial work. This will be helpful to the learners as they will learn other approaches different from the one they have used in resolving questions asked in the test

14. Research Timelines :

Letters to the HOD in KZN Department of Education, Circuit Manager, schools, parents and learners were sent last year (2020). However since I could not get a response from the HOD, I could not proceed with the collection of data. Schools and circuit could not grant permit without the approval from the HOD. The plan was to complete the study in March 2021. I have revised the research timelines as follow:

27 - 31 August 2021 – Writing and sending of the letter to the HOD in KZN Department of Education
01- 03 September 2021– Writing and sending of the letter to the Circuit Manager, Principals and Teachers to ask for the permission to carry the study

13 – 17 September 2021 – Sending of the letters to parents requesting for their consent and to the learners asking for their assent

27 – 30 September 2021 – Administering of Test in the selected school

01October to 15 November 2021 to 31 May – Analysis of learners' scripts and completion of Chapter 4

16 November to 15 December 2021– Interpretation of data and analysis of findings as well as finalising Chapter 5

16 December 2021 to 28 February 2022 – Writing of the final report and conclusions based on the results of the study

31March 2022 – Submission of the research to the University Of South Africa

15. Declaration

I hereby agree to comply with the relevant ethical conduct to ensure that participants' privacy and the confidentiality of records and other critical information.

I, REUBEN BAFANA DLAMINI declare that the above information is true and correct

[signature] 27 - 08 - 2021

Signature of Applicant **Date**

16. Agreement to provide and to grant the KwaZulu Natal Department of Education the right to publish a summary of the report.

I/We agree to provide the KwaZulu Natal Department of Education with a copy of any report or dissertation written on the basis of information gained through the research activities described in this application.

I/We grant the KwaZulu Natal Department of Education the right to publish an edited summary of this report or dissertation using the print or electronic media.

[signature] 31 – 08 – 2021

Signature of Applicant(s) **Date**

Return a completed form to:
Phindile Duma – Tel: 033 392 1063
Office of the HOD; KwaZulu Natal Department of Education
Hand Delivered:
Office 318; 247 Burger Street; Anton Lembede House; Pietermaritzburg; 3201
Or
Ordinary Mail
Private Bag X9137; Pietermaritzburg; 3200
Or
Email
Phindile.Duma@kzndoe.gov.za
Or
Fax
033 392 1203

Appendix G: Permission to conduct Research in the KZN Department of Education schools

KWAZULU-NATAL PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

Private Bag X9137, PIETERMARITZBURG, 3200
Anton Lembede Building, 247 Burger Street, Pietermaritzburg, 3201

Email: Phindile.duma@kzndoe.gov.za

Enquiries: Mrs B.T. Ntuli

Ref.:2/4/8/7281

M Reuben Bafana Dlamini
P O BOX 1269
PIET RETIEF
2380

Dear Ms Dlamini

PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: **"INVESTIGATING GRADE ELEVEN LEARNERS MATHEMATICAL MODELLING COMPETENCIES THROUGH PROBLEM SOLVING:",** in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the Intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from **28 April 2022 to 31 March 2025**.

7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental

   Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Miss Phindile Duma at the contact numbers above.
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report/dissertation/thesis must be submitted to the research office of the Department. Please address it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to schools and institutions in KwaZulu-Natal Department of Education.

Mr GN Ngcobo
Head of
Department:
Education Date:
28 April 202

Appendix H: Permission letter to principals

Enquiries: Dlamini RB

E-mail: 30721709 @mylife.unisa.ac.za

The Principal of ……………………………………High School

Subject: Request for permission to conduct research at your school

Dear Principal,

I am a registered Doctorate student in the Department of Education at the University of South Africa (UNISA). My supervisor is Prof Deonarain Brijlall, a professor in the Department of Mathematics, Statistics and Physics at Durban University of Technology (DUT). The proposed topic of my research is: **Investigating Grade Eleven Learners Mathematical Modelling Competencies through Problem Solving**

The objectives of the study are:

(a) To establish learners' performance in Mathematics incorporated in pragmatic and real life problems
(b) To identify successes and challenges experienced by learners when solving mathematical problems using mathematical modelling approach
(c) Making contribution on improving learners' performance in solving mathematical questions that represent real life situations through mathematical modelling approach

I am hereby seeking your consent to conduct a research in your school. To assist you in reaching a decision, I have attached to this letter a copy of an ethical clearance certificate issued by the University

The benefit of this study will be its contribution in improving learners' performance in Mathematics, especially in solving real life problems that are part of application of differential calculus. This is because once learners learn to solve such problems while in

Grade Eleven and without using derivatives, they will have other options to solve these problems if they experience challenges when solving optimisation problems in Grade Twelve

Should you require any further information, please do not hesitate to contact me or my supervisor. Our contact details are as follows:

Person to contact	Cell	Telephone	Email address
Mr. R B Dlamini (Student)	078 241 3651	017 826 5703	30721709 @mylife.unisa.ac.za
Prof. D Brijlall (Supervisor)	083 555 2390	031 573 2126	deonarainb@dut.ac.za

The current Mathematics content in South Africa requires that Mathematical Modelling should be a focal point of the curriculum and that real life problems should be integrated into all sections whenever appropriate or suitable. Therefore, this study is aimed at determining learners' competencies in mathematical modelling because it has to be emphasised in teaching and learning of Mathematics.

Participants will be required to write a short test designed by myself which will only take 30 minutes. Once the test has been written, the researcher will analyse learners' scripts to find out how do learners respond to these questions. The study is qualitative in nature. This means it does not require all the learners in Grade Eleven but a few that will volunteer to participate.

There is no potential risk that may cause harm or injury to the participants as they will write the assessment under the normal classroom settings. However, due to the outbreak of COVID –19, the researcher must ensure that all safety measure to fight the spread of the disease are in place. This may also include maintaining the correct social distancing of more than 1, 5 metres, ensuring that all participants have put on their face masks and that they sanitise their hands before the session starts.

There will be no reimbursement or any incentives for participation in the research. However, upon completion of the study, I will visit the school to give feedback to you, teachers and learners in form of a briefing about some of the helpful and interesting things I found in my study. This will include what teachers can do in order to assist learners improve their weaknesses that would be revealed by the study. I also undertake to provide your school with a bound copy of the thesis as part of the feedback.

Once given the permission I will then write letters to the parents to seek for permission their consent to involve their children in the study. The researcher will also write letters to the learners to ask for their assent to be part of this study. To assist you in reaching a decision, I have attached to this letter a copy of an ethical clearance certificate issued by the University of South Africa (UNISA) College of Education Ethics Review Committee as well as the letter from the KZN Head of Department that grants me the permission to conduct the research in the selected schools.

Attached to this letter is the consent form that may be used by the principal to respond to my request.

Your permission to conduct this study will be greatly appreciated.

Yours sincerely,
Reuben Bafana Dlamini (Mr)

Appendix I: Principals' consent form

(RETURN SLIP)

I, __________________________, (Principals' name), the Principal of _________________ _________________ School confirm that I have received the application letter from the person intending to conduct a research in my school. The letter stated clearly the nature, procedures, potential benefits and anticipated risk of participation. It also stated the conditions under which the study will be conducted as the COVID 19 Regulations are still in place.

I have read the letter in addition to the information the researcher has explained to me in person and understood the study. I have then taken my decision based on the following information tabulated below:

Criteria for making my decision	Tick YES or NO (where applicable) YES or NO	
1. Had sufficient opportunity to ask questions	YES	NO
2. Participation of my school is voluntary	YES	NO
3. Teachers and learners are free to withdraw at any time	YES	NO
4. Findings of this study will be processed into a research report,	YES	NO
5. My school's participation will be kept confidential	YES	NO
6. Agree to the collection of data through the writing of the Test	YES	NO

Based on the above information I am ***prepared to*** / ***not prepared to*** (delete one not applicable to you) allow the researcher to conduct the research in my school.

Principal's Name & Surname (please print) _______________________________

Principal's Signature________________ Date:_____________

Researcher's Name & Surname (please print) _________________ ___________

Researcher's signature______________ Date_____________

Appendix J: Letter requesting permission from the Grade 11 Mathematics Teachers

Enquiries: Dlamini RB

30721709 @mylife.unisa.ac.za

Grade 11 Mathematics Teacher

Name of School: ………………………….

Dear Teacher

I, Reuben Bafana Dlamini, am a registered Doctorate student in the Department of Education at the University of South Africa (UNISA). My supervisor is Prof. Deonarain Brijlall, a professor in the Department of Mathematics, Statistics and Physics at Durban University of Technology (DUT). The proposed topic of my research is: **Investigating Grade Eleven Learners Mathematical Modelling Competencies through Problem Solving.**

The objectives of the study are:

(a) To establish learners' performance in Mathematics incorporated in pragmatic and real life problems
(b) To identify successes and challenges experienced by learners when solving mathematical problems using mathematical modelling approach
(c) Making contribution on improving learners' performance in solving mathematical questions that represent real life situations through mathematical modelling approach

I have written a letter to your school principal asking for permission to conduct a research in your school. I am hereby seeking your consent to conduct a research in your school because if I am granted the permission your normal programmes will be affected. Also, I will need your help in terms of arranging participants (learners), monitoring and invigilation of the tests. To assist your principal in making a decision, I need your help to explain to the principal the difficulties experienced by learners in Grade Twelve in solving mathematical problems involving real life problems, as well as how this study is going to assist your learners.

Should you require any further information, please do not hesitate to contact me or my supervisor. Our contact details are as follows:

Person to contact	Cell	Telephone	Email address
Mr. R B Dlamini (Student)	078 241 3651	017 826 5703	30721709 @mylife.unisa.ac.za
Prof. D Brijlall (Supervisor)	083 555 2390	031 573 2126	deonarainb@dut.ac.za

The benefit of this study will be its contribution in improving learners' performance in Mathematics, especially in solving real life problems that are part of application of differential calculus. This is because once learners learn to solve such problems while in Grade Eleven and without using derivatives, they will have other options to solve these problems if they experience challenges when solving optimisation problems in Grade Twelve.

Participants will be required to write a short test designed by myself which will only take 30 minutes. Once the test has been written, the researcher will analyse learners' scripts to find out how do learners respond to these questions. The study is qualitative in nature. This means it does not require all the learners in Grade Eleven but a few that will volunteer to participate.

There is no potential risk that may cause harm or injury to the participants as they will write the assessment under the normal classroom settings. The possible risk may be discomfort which may arise during the interview session. In such a case, the researcher will try by all means to create a friendly atmosphere which will enable learners to provide the necessary information freely. Another possible risk is the recent outbreak of COVID – 19. To reduce the widespread of the disease, the researcher will ensure that all safety precautions are followed. Also, because the strict regulations regarding COVID – 19, the researcher is compelled to let learners work as individuals instead of groups.

There will be no reimbursement or any incentives for participation in the research. However, upon completion of the study, I will visit the school to give feedback to all Mathematics teachers, principal and learners in form of a briefing about some of the helpful and interesting things I found in my study. This will include what teachers can do in order to assist learners improve their weaknesses that would be revealed by the study. I also undertake to provide your school with a bound copy of the thesis as part of the feedback.

Your permission to conduct this study will be greatly appreciated. If you agree to assist in this study, kindly sign the attached consent form and send it back to me.

Yours sincerely

Reuben Bafana Dlamini (Mr)

Appendix K: Consent/assent by teachers to participate in the study

(RETURN SLIP)

I, ________________________________ (participant name), I attest that the person requesting my permission to participate in this study has informed me of its nature, methodology, possible advantages, and expected drawbacks. I have read the explanation regarding the study and understand it as described in the letter requesting my permission,

I have had enough time to ask questions, and I am ready to take part in the research. I am aware that participation is completely voluntary and that there are no consequences if I decide to withdraw at any moment.

I understand that the results of this study may be used in a research report, journal articles, conference proceedings, and/or other publications, but that, unless otherwise noted, information about my participation will remain confidential.

I am aware that the findings of this study will be processed into a research report, journal publications and/or conference proceedings, but that my participation will be kept confidential unless otherwise specified.

I have received a signed copy of the informed consent agreement.

Participant Name & Surname (please print) ______________________________

Participant Signature _______________ Date__________________

Researcher's Name & Surname (please print_______________________________

Researcher's signature________________ Date:_________________

Appendix L: Parent/Guardian Consent

To: Parents/Guardians

RESEARCH PROJECT: **Investigating Grade Eleven Learners Mathematical Modelling Competencies through Problem Solving**

YEAR: 2022

I, Mr R B Dlamini am doing a study through the College of Education at the University of South Africa (UNISA) with Professor. D. Brijlall as his supervisor. Prof Brijlall's contacts are 0315732125 or 0835552390.

We want to research the competency level of Grade Eleven learners in Mathematical modelling when solving real life problems

Learners are asked to help by taking part in this research as it would be of benefit to interested educationists, learners and teachers. However, the participation is completely voluntary and has no impact or bearing on evaluation or assessment of the learner in any studies or course while at school

Learners who will volunteer to participate in the study will be expected to write a short test designed by the researcher. The type of questions to be used in the test will prepare learners in advance to be able to solve problems examined under Calculus in Grade 12. Once the test has been written, the researcher will analyse learners' scripts to find out how do learners respond to these questions

The test will be analysed in order to establish their competence level in Mathematical Modelling. The identities of the participants will be kept strictly confidential as no copies will bear the names of the learners. Their scripts will only have a code name assigned to them by the researcher. All data collected and used in this study will be kept in a safe place and not been used for any other purpose except for the research.

Participants may leave the study at any time by notifying the researcher. Participants may review and comment on any part of the researcher's written report. No risk anticipated as learners will write the test under normal classroom conditions. The researcher will follow all COVID – 19 safety measures followed at the school

If you agree or disagree that your child can participate in this study, please sign the declaration attached to this letter and indicate whether you agree or disagree by making a tick on the appropriate box. The signed declaration must be returned to the school on or before ………………….. 2022.

_____________________ ____________

Researcher's Signature Date

DECLARATION

I, ________________________ (Parent's name) the parent of __________________ (Learner's name), agree/disagree

________________ (Signature) ___________ (Date)

☐ Agree

NB. Tick ONE

☐

Appendix M: Letter requesting assent from learners

TITLE OF THE RESEARCH: **Investigating Grade Eleven Learners Mathematical Modelling Competencies through Problem Solving**

Dear Learner

I am doing a study that determines the competency level of Grade Eleven learners in Mathematical Modelling as part of my studies at the University of South Africa. Your principal has given me permission to do this study in your school. I would like to invite you to be a very special participant of my study. I am doing this study so that I can find ways that your Mathematics teachers can use to make teaching and learning of Mathematics better. This may help you and many other learners of your age in different schools to have a better understanding of Mathematics and then improve your performance in the subject.

This letter is to explain to you what I would like you to do. There may be some words you do not know in this letter. You may ask me or any other adult to explain any of these words that you do not know or understand. You may take a copy of this letter home to think about my invitation and talk to your parents about this before you decide if you want to be in this study.

If you decide to be part of the study you will be requested to write a short test that will has duration of 30 minutes. The test consists of questions covering subtopics of Measurement like Area and Perimeter of rectangular structures.

This test has no affect your promotion but will strengthen your problem solving skills. You are not going to write your real name on the scripts or answer sheet but you will use code name which will be given to you by the researcher.

I will write a report on the study but I will not use your name in the report or say anything that will let other people know who you are. Participation is voluntary and you do not have to be part of this study if you do not want to take part. If you choose to be in the study, you are free to withdraw at any time you feel uncomfortable and no one will penalise you. You may tell me if you do not wish to answer any of my questions. No one will blame or criticise you. When I am finished with my study, I shall return to your school to give a short talk about some of the helpful and interesting things I found out in my study. I shall invite you to come and listen to my talk.

The benefit of this study will be its contribution in improving learners' performance in questions involving real life problems that are part of application of differential calculus done in Grade Twelve. It will be an advantage for learners to learn a number of different approaches other than relying on calculus. By the time they reach Grade Twelve, learners will be aware and have better knowledge of optimisation problems.

There are no risks anticipated in participating in the study since everything you will be requested to do will all be done under normal classroom conditions. All the regulations that are followed at present regarding COVID – 19 will be followed during the writing of the test. Information collected in this study will remain confidential. There will be no presents, gifts or any form of incentives that will be given to participants of the study.

If you decide to be part of my study, you will be asked to sign the form on the next page. If you have any other questions about this study, you can talk to me or you can have your parent or another adult call me at 078 241 3651 or call my supervisor Prof. Brijlall at 031 573 2125 or 083 555 2390. Do not sign the form until you have all your questions answered and understand what I would like you to do.

Researcher: **Reuben Bafana Dlamini** Phone number: **078 241 3651**

Do not sign the written assent form if you have any questions. Ask your questions first and ensure that someone answers those questions. Signed written assent must be returned to the school on or before ………………………………2022.

Appendix N: Learner written assent

I have read this letter which asks me to be part of a study at my school. I have understood the information about the study and I know what I will be asked to do. I am willing to be in the study.

________________________ _____________________ _______________

Learner's name (print) Learner's signature Date

__________________ ____________________ _____________

Witness's name (print) Witness's signature Date

(The witness is over 18 years old and present when signed.)

____________________ __________________ __________

Parent/guardian's name (print) Parent/guardian's signature: Date:

Dlamini Reuben Bafana _________________ _____________

Researcher's name (print) Researcher's signature: Date:

Appendix O: Editing Certificate

Shiraz 16

50 Quail Avenue

Thatchfield Close

Centurion, Pretoria

0157

To whom it may concern

Date: 7 April 2026

This letter confirms that the Thesis entitled "**INVESTIGATING GRADE ELEVEN LEARNERS' MATHEMATICAL MODELLING COMPETENCIES IN ALGEBRAIC PROBLEM SOLVING"** written by **REUBEN BAFANA DLAMINI** has been edited by Sam Ramaila.

Sincerely,

Sam Ramaila (PhD)

Cell: 0646566387

Appendix P: Turnitin Report

PAPER NAME	AUTHOR
Completed thesis.docx	REUBEN BAFANA DLAMINI
WORD COUNT	CHARACTER COUNT
102184 Words	551793 Characters
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